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MODELS FOR SYNTHETIC SUPERGEOMETRY

by David N. YETTER

RÉSUMÉ. Les notions de base de la Géométrie Différentielle Synthétique sont modifiées pour englober la "géométrie différentielle avec paramètres commutants et anti-commutants" nécessaire pour les théories de supergravité.
Des modèles analogues aux topos de Dubuc et Stein sont construits, et leurs rapports avec la théorie de Kostant des variétés graduées sont explicités. Plusieurs façons de retrouver des espaces "bosniques" sont explorées.
Finalement des résultats élémentaires de GDS sont étendus au cas supersymétrique: en particulier on montre que la fibre (synthétique) tangente à l'unité d'un objet groupe assez régulier a une structure d'algèbre de Lie graduée. L'intégration de superchamp est considérée brièvement dans le cadre synthétique.

0. INTRODUCTION.

The enterprise of Synthetic Differential Geometry (SDG), begun in Lawvere's 1967 lecture on "Categorical Dynamics", may be seen as an attempt to axiomatize (hence the name synthetic), and to provide a model theory for the way in which physicists work with smooth phenomena - for example, in SDG vector fields really *are* infinitesimal flows, or, equivalently, infinitesimal deformations of the identity map, on a manifold.

Seen in this light, it is reasonable to attempt to bring the tools of SDG to bear on the construction of mathematical models for supergravity in which a "differential geometry with both commuting and anti-commuting parameteres" is needed. The algebraic-geometric flavor of both Kostant's theory of graded manifolds and the model theory for SDG as developed by Dubuc and others further suggests the possibility of fruitful interaction.

It is the purpose of this paper to begin that work, but only to begin: all results contained herein may be seen either "super" versions of results contained in Kock's book on SDG [6], or as "synthetic" versions of results contained in the Proceedings of the NATO Workshop on Mathematical Aspects of Superspace [9]. We refer readers unfamiliar with either of the subjects considered to those works as good introductions, and in particular to Kock's monograph, since many of the proofs of results contained herein are more or less routine generalizations of Kock's proofs to the "super" case. Proofs of this sort will in general be sketched briefly, as the reader, armed with Kock's book, can easily fill in the details.

Our main result is the construction of "super" analogues of the Dubuc topos and the Stein topos (cf. [4; 5]). We go on to consider some properties of these categories, both as models for supergeometry in their own right, and in comparison to Kostant's theory of graded manifolds [7] and standard models of SDG (cf. Dubuc [4] and Hoskin [5]).

In the presence of these results, it is hoped that any sufficiently general synthetic proof of a result in differential geometry will carry the "super" result (provided, of course, that some care is taken in how definitions are "superfied" and how integration is handled - the view of Batchelor that Berezin integration *is* really odd-variable differentiation is undoubtedly correct in the synthetic setting).

1. "SUPERFICATION" OF THEORIES.

Had the synthetic approach been considered in Batchelor [1], it would have been classed among the "geometric" approaches in that we begin with an algebra with anti-commuting elements. Rather than living in the category of **Sets**, and being endowed with a topology, this algebra will lie in a Grothendieck topos constructed along with it, and it will be the very structure of the underlying topos that will carry the "geometric" data. The actual construction of that algebra will, however, have to wait until after some algebraic preliminaries:

DEFINITION 1.1. A *differentially closed theory*, $\mathbf{T}$, over $K = \mathbf{R}$ or $\mathbf{C}$ is an equational theory (in the sense of Lawvere [8]), extending the theory of commutative K-algebras, whose n-ary operations are named by infinitely differentiable functions $K^n \to K$, and satisfying

DCT1. Any generalized composition of operations in **T** names an operation in **T**, and any equation holding among the functions holds among the operations they name in **T**.

DCT2. If $\phi: K^n \to K$ names an operation in **T**, then so does $\phi_i: K^n \to K$, the i^{th} partial derivative for $i = 1,\dots,n$.

Examples include the theories of polynomials over **R** or **C** (the usual notion of algebra over the base field), the theory of C^∞-functions, and the theory of analytic functions (real or complex).

Note that a model of any DCT over K is *a fortiori* a commutative K-algebra. To introduce anti-commuting elements, we modify the theory:

DEFINTIION 1.2. The *superfication* **S(T)** of a DCT **T** is the theory whose operations are named by all *formal* composites of operations in **T** and unary operations, B and F. Two names of operations, ϕ and ψ, name the same operation if their values, $\|\phi\|_v$ and $\|\psi\|_v$, in every *Grassmann instantiation* **v** are equal.

A Grassman instantiation **v** for an expression in n-variables $X_1,\dots,X_n$ is a choice of a finitely generated Grassmann algebra (over K), Λ, and a vector $\mathbf{v} = (v_1,\dots,v_n) \in \Lambda^n$. The value $\|\phi\|_v$ of an expression ϕ in the instantiation **v** is defined inductively as follows:

$$\|X_i\|_v = v_i,$$
$$\|s\|_v = s \text{ for any } s \in K \text{ (as a 0-ary operation in } \mathbf{T}),$$
$$\|B(\phi)\|_v = \text{even part of } \|\phi\|_v, \quad \|F(\phi)\|_v = \text{odd part of } \|\phi\|_v,$$

and if $f: K^n \to K$ is an operation of **T**, then

$$\|f(\phi_1,\dots,\phi_r)\|_v = f(\|\phi_1\|_{v,o},\dots,\|\phi_r\|_{v,o}) + \Sigma_I f_I(\|\phi_1\|_{v,o},\dots,\|\phi_r\|_{v,o}).\|\phi_I\|_{v,+}$$

where I ranges over all ordered multiindices $1 \leqslant i_1 \leqslant \dots \leqslant i_s \leqslant r$ (for all $s \geqslant 1$), and where

$$f_I = \partial^s f/\partial X_{i_1}\dots\partial X_{i_s},$$
$$\|\phi_I\|_{v,+} = \|\phi_{i_1}\|_{v,+}\cdots\cdots\|\phi_{i_s}\|_{v,+}, \quad \|\phi_I\|_{v,o} = \|\phi_{i_1}\|_{v,o}\cdots\cdots\|\phi_{i_s}\|_{v,o},$$

and
$$\|\phi_i\|_{v,+} = \|\phi_i\|_v - \|\phi_i\|_{v,o}.$$

The following theorem makes precise the way in which we have replaced commutativity with ($\mathbb{Z}/2$-)graded commutativity. Throughout the following we use the subscript B to denote the even grade (which we call the *bosonic* grade) and F denote the odd grade (which we call *fermionic*).

THEOREM 1.3. *If* **T** *is any DCT over* K, *then any* **S(T)**-*model is a* ($\mathbb{Z}/2$-)*graded commutative* K-*algebra; moreover, if* **T** *is the theory of commutative* K-*algebras, then* **S(T)** *is the theory of* ($\mathbb{Z}/2$-)*graded commutative* K-*algebras.*

PROOF. Observe that **S(T)** has among its operations: for each element s of K, a constant; two binary operations named by multiplication and addition as operations in **T**; and two unary operations B "bosonic part" and F "fermionic part".

Letting

$$A_B = \{x \mid x = B(x)\} \quad \text{and} \quad A_F = \{x \mid x = F(x)\},$$

it is easy to show that A is a graded commutative K-algebra with these as grades, and B and F as projections onto the grades: all the relevant equations follow from the fact that they hold in any Grassmann instantiation, since Grassmann algebras are themselves graded commutative algebras.

Conversely, letting

$$B(x) = \text{degree 0 part of } x, \quad \text{and} \quad F(x) = \text{degree 1 part of } x$$

any graded commutative K-algebra becomes an **S**(K-**alg**) model. To verify any equation of **S**(K-**alg**), it suffices to verify it in the free graded commutative algebra over K on n-generators (for n the number of variables in the equation), but this is the Grassmann algebra on n-generators $F(x_i)$, $i = 1,\ldots,n$ over the polynomial algebra $K[B(x_i) \mid i=1,\ldots,n]$.

To verify an equation in this algebra, it suffices to verify it at a sufficiently large finite number of instantiation of the variables $B(x_i)$ by field elements (how many depends on the degree of the equation). But these are simply Grassmann instantiations in the sense above, and the equation must hold in *all* such.

The next few propositions give a wealth of **S(T)**-models (in **Sets**) for any DCT **T**.

PROPOSITION 1.4. *For any* K-DCT **T**, *any* **T**-*model* A *becomes an* **S(T)**-*model when equipped with the operations* $B(x) = x$ *and* $F(x) = 0$.

PROOF. Given any equation in **S(T)**, its even part in any Grassmann instantiation is an instantiation of an equation of **T** in a Weil algebra over K. (It is easy to show that any Weil algebra - i.e., finite dimensional algebra of the form K⊕I, for I a nilpotent ideal -

can be given a **T**-model structure for any K-DCT, cf. Kock [6].) Thus the even parts of the two sides of the equation must hold since A is a **T**-model, while the odd parts are equal trivially.

PROPOSITION 1.5. *For any* K-DCT **T**, *any Grassmann algebra over* K *is an* **S(T)**-*model, as is any* $\mathbb{Z}/2$-*graded sub-quotient of a Grassmann algebra over* K.

PROOF. All operations of **S(T)** are defined on any Grassmann algebra over K (or any subquotient) by the formulas used in defining Grassmann instantiations. Likewise any equation must hold in any Grassmann algebra (and hence in any subquotient) because the equations of **S(T)** were taken to be precisely those which hold in all Grassmann instantiations!

DEFINITION 1.6. A *graded Weil algebra over* K is a finite dimensional unital graded-commutative algebra A, of the form $(K \oplus A_{B'}) \oplus A_F$, where the part in parentheses is the bosonic grade, and $A_{B'} \oplus A_F$ is a nilpotent ideal. A *Weil algebra over* K is a finite dimensional unital commutative algebra A of the form $K \oplus I_A$ where I_A is a nilpotent ideal and K is spanned by the multiplicative identity. We identify Weil algebras with those graded Weil algebras with trivial fermionic grade.

We then have

PROPOSITION 1.7. *Any graded Weil algebra has an* **S(T)**-*model structure for any* K-DCT **T**.

PROOF. All graded Weil algebras are isomorphic to subquotients of finitely generated Grassmann algebras.

As a hint to the reader where all these preliminaries are leading, we could right now define a "topos of superspaces" for any DCT **T**: namely, $\mathbf{Sets}^{\mathbf{fgS(T)\text{-}mod}}$, where **fgS(T)-mod** is the category of finitely generated **S(T)**-models in **Sets**, and the algebra from which "supermanifolds" will be built inside the topos is the object named by the forgetful functor U. While the algebra U has many of the good properties we are seeking, this topos lacks some of the geometric flavor we want, so some more preliminaries are in order.

2. GEOMETRIC NOTIONS ASSOCIATED TO DCT'S AND THEIR SUPERFICATIONS.

In this section we shall at first consider DCT's and their superfications on an equal footing, and denote theories of either type by **T** unless otherwise clear from context. The next three definitions are extensions of notions found in Kock [6].

DEFINITION 2.1. A *point* of a **T**-model A is an algebra homomorphism p: A $\to$ K. We denote the set of points of A by pts(A). A model is *point determined* if

$$[\forall\ p \in \mathrm{pts}(A)\ p(a) = p(b)] \Rightarrow a = b.$$

DEFINITION 2.2. If A and B are **T**-models, I a (2-sided) ideal in A, and Σ a subset of A, then by A/I (resp. $A\{\Sigma^{-1}\}$, $A \otimes_T B$, $A\{x\}$) we mean a **T**-model equipped with a **T**-model homomorphism A $\to$ A/I such that I is mapped to 0, and universal among such (resp. a **T**-model equipped with a **T**-model homomorphism A $\to$ $A\{\Sigma^{-1}\}$ such that all elements of Σ are mapped to invertible elements and universal among such; the coproduct in **T-mod** of A and B; the coproduct of A and the free **T**-model on 1 generator (x)).

Note that all the above must exist by standard exactness properties of algebraic theories. For completeness we introduce the notion of germ-determined algebras, although we use it little in the sequel.

DEFINITION 2.3. If p: A $\to$ K is a point of A, let

$$\Sigma_p = \{a \in A \mid \neg\, (p(a) = 0)\},$$

then $A_p = A\{\Sigma_p^{-1}\}$ is called the *algebra of germs at p*. Let $(-)_p$: A $\to$ A_p be the canonical map. An ideal I is *germ-determined* if

$$[\forall\ p \in \mathrm{pts}(A)\ a_p \in I_p] \Rightarrow a \in I.$$

The *germ-radical*, $I^{\wedge}$, of I is the smallest germ-determined ideal containing I. A **T**-model is *germ-determined* if its 0-ideal is germ-determined.

NOTE. The full subcategory of germ-determined **T**-models is a reflective subcategory, the reflection being given by $A \to A/0^{\wedge}$.

It is claimed (Kock [6]) that germ-determinedness is a rigorous version of "geometrically interesting".

We can now construct a large number of interesting models for our superfied theories. The first observation to be made is that for any DCT **T** there are many point-determined **T**-models: for the theory of **C**-algebras, the coordinate ring on any algebraic variety; **R**-algebras, the coordinate ring of any algebraic variety whose real-locus is Zariski dense in its complex-locus; for C^{∞}-algebras, the ring of C^{∞}-functions on any smooth (paracompact, Hausdorff) manifold; and for the theory of homorphic functions, the ring of holomorphic functions on any Stein space, are all points determined models of their respective theories. Moreover in each case, the algebraic points correspond to the geometric points of the undrlying space.

Armed with this, the following proposition provides a wealth of (germ-determined) **S(T)**-models:

PROPOSITION 2.4. *If* A *is a point-determined* **T**-*algebra (for* **T** *a* K-DCT), *then* A *with the trivial grading of Proposition 1.4 is a point-determined (and hence a fortiori germ-determined)* **S(T)**-*model. If, moreover,* W *is a graded Weil algebra over* K, *then* $A \otimes_K W$ *is a germ-determined* **S(T)**-*model (and is in fact the coproduct of* A *with the trivial grading and* W).

PROOF. That A with the trivial grading is point determined as a **S(T)**-model follows immediately from the observation that **S(T)**-model homomorphisms between **T**-models equipped with the trivial grading are precisely **T**-model homomorphisms.

That $A \otimes_K W$ has an **S(T)**-model structure follows from the fact that to verify the equations it suffices to verify them pointwise (i.e., after passing along $p \otimes_K W$), and in W they hold by Corollary 1.6. That $A \otimes_K W$ is a coproduct in **S(T)-mod** follows from a standard result: if a (co)limit, in the category of models for a weaker theory, of models for a stronger theory is a model of the stronger theory, then it is a (co)limit in the category of models for the stronger theory. In light of this, we drop the subscript on the tensor product.

To see that $A \otimes W$ is germ-determined, note first that A is *a fortiori* germ-determined. Now given any point $p: A \otimes W \to K$, this factors through the map $A \otimes W \to A$ with kernel $A \otimes I_W$ where I_W is the unique nilpotent ideal of W, giving a point $\underline{p}: A \to K$. Now analysing this factorization shows that Σ_p is in fact

$$\{a \otimes 1 + \Sigma_{w \in I_W} a_w \otimes w \mid a \in \Sigma \underline{p}\}.$$

But since I_W is nilpotent, $A \otimes I_W$ is also, and thus inverting Σ_p is equivalent to inverting $\Sigma \underline{p} \otimes 1$. Thus if

$$[\alpha \otimes 1 + \Sigma_{w \in I_W} \alpha_w \otimes w]_p = 0$$

for all points p, this is equivalent to

$$\alpha_{\underline{p}} \otimes 1 + \Sigma_{w \in I_W} (\alpha_w)_{\underline{p}} \otimes w = 0$$

for all points p. Thus this is equivalent to

$$\alpha_{\underline{p}} = (\alpha_w)_{\underline{p}} = 0 \quad \text{for all} \quad w \in I_W,$$

and hence, since A is germ-determined, to $\alpha = \alpha_w = 0$ for all $w \in I_W$, and thus

$$\alpha \otimes I + \Sigma_{w \in I_W} \alpha_w \otimes w = 0.$$

Thus $A \otimes W$ is germ-determined.

COROLLARY 2.5. *If* M *is a smooth* (C^∞) *manifold (resp. a Stein manifold), and* W *is any graded Weil algebra over* $\mathbb{R}$ *(resp.* $\mathbb{C}$*), then* $C^\infty(M) \otimes W$ *(resp.* $\mathrm{Hol}(M) \otimes W$*) is an* SC^∞*-model (resp.* $S\mathrm{Hol}$*-model).*

The reader will note that, in particular, W may be taken to be a finitely generated Grassmann algebra, in which case these algebras will be useful in our consideration of the relationship between Kostant's graded manifolds and the topoi we will construct. The reader will also note that the restriction to Stein manifolds in the super-holomorphic case is related to the failure of Batchelor's Theorem for graded holomorphic manifolds. (Although it is outside the scope of this paper, we conjecture that Batchelor's Theorem is restored in the holomorphic case if one restricts one's attention to graded manifolds whose body is a Stein manifold.)

Finally we define the category of manifolds which are "good" with respect to a DCT, $\mathbb{T}$.

DEFINITION 2.6. A (paracompact) manifold is a T-*manifold* if it is equipped with an atlas such that the transition functions between the charts are restrictions of T-operations to the coordinate chart. A continuous map between T-manifolds is a T-*manifold map* if its restriction to the intersection of any chart in the source, and the inverse image of a chart in the target, is the restriction of a T-operation. Given a T-manifold M, the *coordinate* T-*algebra* T(M) is the algebra of global sections of the sheaf of T-algebras associated to the presheaf of T-algbras whose sections on coordinate charts are the T-operations restricted to the chart. (Note T( ) is a contravariant functor from **T-mf** to **T-alg**). By abuse of notation, we also denote the sheaf of T-algebras described above by T(M), it being clear from context whether an algebra or sheaf is meant.

A T-manifold M is (T-)*complete* if the "evaluation map" $|M| \to \mathrm{Pts}(T(M))$ is epi. M is (T-)*separated* if the evaluation map is monic.

A T-manifold M is *good* if it is complete and separated. M is *locally good* if every cover by open sub-T-manifolds admits a refinement by good open sub-T-manifolds.

The reader will note, for example, that all C^∞-manifolds are C^∞-good, while for complex analytic manifolds, only Stein spaces are good, but every analytic manifold is locally good.

3. TOPOI OF SUPERSPACES.

3.1. Construction and General Properties.

For any DCT, we consider the topos $\mathbf{E}_{S(T)}$ given as Shv(**G**, **J**), where **G** is the full subcategory of S(T)-alg consisting of all S(T)-algebras of the form $T(M)\otimes W$ where M is a good T-manifold, and W is a graded Weil algebra; and **J** is the Grothendieck topology induced by T( ) of all open coverings of T-manifolds. In the case where T is C^∞-**alg**, we call this the "super-Dubuc topos"; in the case where T is the theory of holomorphic functions, we call this the "super-Stein topos".

DEFINITION 3.1.1. For a commutative ring k in a fixed base topos **S**, a *superlined topos*/k (resp. *lined topos*/k) is an **S**-topos **E** equipped with a graded commutative ring object (resp. commutative ring object) R satisfying

L1. For any graded Weil algebra/k (resp. Weil algebra/k) W, the canonical map $R\otimes W \to R^{Spec(W)}$ transpose to

$$ev: (R\otimes W)\times Spec(W) \to R$$

(where Spec(W) is $Hom_{R\text{-}alg}(R\otimes W, R)$, interpreted internally), is an isomorphism.
and L2. Spec(W) is tiny for every graded Weil algebra (resp. Weil algebra) W (i.e., $(\ )^W$ has a *right* adjoint, cf. Yetter [11]).

THEOREM 3.1.2. *For any* k-DCT $\mathbf{E}_{S\langle T\rangle}$ *is a superlined topos over* k *when equipped with* R, *the sheafification of the forgetful functor.*

PROOF. L1 follows by the proof of Kock [6], Theorem III.1.2, when that proof is taken at its full generality. To see this, it is necessary to verify that the tensor product of graded k-algebras is in fact the coproduct in the category of k-algebras, and (for DCT's other than the theory of k-algebras) the observation concerning colimits for models of different theories made in the proof of 2.4.

For L2, note that R is representable, and hence by a result of Bunge [3] the representable presheaf is tiny (since the site has coproducts) in $\mathbf{Sets}^{C^{op}}$. The result then follows from the sufficient condition in Yetter [11] for sheafification to preserve tininess.

We now turn to a way of recovering purely bosonic spaces which is intrinsic in the sense that it can be done in any superlined topos without regard to how that topos was constructed. Recall from Yetter [11]:

DEFINITION 3.1.3. An object X is *A-discrete* whenever for all objects Y and all maps *f*: Y×A → X, *f* factors through the projection onto Y (i.e., "Maps from A to X are all constant", interpreted internally).

DEFINITION 3.1.4. An object in $\mathbf{E}_{S\langle T\rangle}$ is *pure bosonic* if it is Spec(W)-discrete for all graded Weil algebras W generated by their fermionic grade.

PROPOSITION 3.1.5. *The full subcategory of purely bosonic objects is a reflective, coreflective subtopos of* $\mathbf{E}_{S\langle T\rangle}$, which we denote $\mathbf{BOS}_{S\langle T\rangle}$. We denote the reflection by body( ), and the coreflection by cobody( ).

PROOF. Immediate by results in Yetter [11].

Intuitively, these functors correspond to the two ways to pass from a graded commmutative algebra to a commutative algebra: body( ) is quotienting by the ideal generated by the fermionic grade; cobody( ) is cutting down the bosonic part. Care is required in interpreting this, since "bosonic part" means here not the bosonic grade, but the part of the algebra to which no odd element can be mapped under any morphism in the topos (internally!). Regretably, the cobody is the more interesting topos theoretically, and is as yet too little understood to be of use in applications. As an example of its interest, we prove:

PROPOSITION 3.1.6. cobody(R) *is a line in* $\mathbf{BOS}_{S\langle T\rangle}$.

PROOF. Recall from Yetter [11] that the discrete reflection is an adjoint to the inclusion of discretes as functors enriched over the topos of discretes. Thus for any purely bosonic Weil algebra W we have

$$\mathrm{cobody}(R)^{\mathrm{cobody}(\mathrm{Spec}(W))} \simeq \mathrm{cobody}(R^{\mathrm{cobody}(\mathrm{Spec}(W))})$$
$$\simeq \mathrm{cobody}(R\otimes W) \simeq \mathrm{cobody}(R)\otimes W.$$

Note in the middle isomorphism that cobody is idempotent. The last isomorphism in the sequence follows from the fact that $R\otimes W$ (resp. $\mathrm{cobody}(R)\otimes W$) is isomorphic to R^n (resp. $\mathrm{cobody}(R)^n$) for $n = \dim(W)$, while cobody( ) is limit preserving. (Warning: cobody( ) does not in general preserve colimits (e.g. $\otimes$) - it does so in this case only because these instances of $\otimes$ can be canonically re-expressed as limits, which are preserved.)

Although the intrinsic nature of these constructions suggests that their study is fruitful, the cobody construction depends upon the little understood, but powerful, properties of tiny objects (see Yetter [11]), so that some fundamental work is required before this construction can be properly applied. We turn therefore to a construction of a subtopos of "bosonic" objects, which is extrinsic in the sense that it is carried out at the level of defining sites:

DEFINITION 3.1.7. The subtopos of *bosonic sheaves*, $\mathbf{BSh}_{S\langle T\rangle}$, in the topos $\mathbf{E}_{S\langle T\rangle}$ is the topos $\mathrm{Shv}(\mathbf{G},\mathbf{K})$, where $\mathbf{G}$ is as in the definition of $\mathbf{E}_{S(T)}$, and K is the topology generated by $\mathbf{J}$ and all one object covers of the form $A_B \rightarrowtail A$ (where A_B is the bosonic grade of A, considered as a trivially graded $\mathbf{S(T)}$-algebra).

The following proposition then establishes the relationship between our topoi of superspaces and standard models for SDG:

PROPOSTIION 3.1.8. *If $\mathbf{E}_{S(T)}$ is the super-Dubuc topos (resp. super-Stein topos), then $\mathbf{BSh}_{S(T)}$ is equivalent to the Dubuc topos (resp. the Stein topos), and if R_B is the sheafification of R, and is the usual line in the Dubuc topos (resp. the Stein topos).*

PROOF. The sites of definition are equivalent. (The defining site for the latter topos is included in the defining site for the former, and every object in the larger site is canonically covered by an object in the smaller.) For the conclusion about the superline, observe that R and R_B are representable, and that R's representing object in the site is covered by the representing object for R_B. Moreover, it is easy to see that R_B is carried to the usual line, in the Dubuc (resp. Stein) topos by the equivalence of sites.

3.2. Graded manifolds and topoi of superspaces.

Adapting Kostant's definition [7] of graded manifolds to T-manifolds, we make:

DEFINITION 3.2.1. A ($\mathbf{Z}$/2-)graded T-manifold is a pair (X,A), where X is a T-manifold, and A is a sheaf of graded algebras over X such that there is an open cover of X by T-manifolds, $\{U_i\}_{i\in I}$ such that $A(U_i) \simeq \mathbf{T}(A)\otimes\Delta$, for Δ some finitely generated Grassmann algebra. Maps are defined in the obvious way.

We let $\mathbf{GT\text{-}Mf}$ denote the category of graded T-manifolds with X locally good, and let $\mathbf{GT\text{-}Mf}_0$ denote the category of graded manifolds with X good and A $=\mathbf{T}(X)\otimes\Delta$ ("good trivially graded manifolds").

We can now state and prove a comparison theorem showing the relation between graded manifolds and our topoi of superspaces:

THEOREM 3.2.2. *There is a functor i: $\mathbf{GT\text{-}Mf} \to \mathbf{E}_{S(T)}$ extending the composite functor* ΓyL: $\mathbf{GT\text{-}Mf}_0 \to \mathbf{E}_{S(T)}$ (Γ being the global section functor, y *the Yoneda embedding into the presheaf topos, and* L *the sheafification functor) and satisfying:*

0. i is full and faithful.

1. i preserves all pullbacks which are transversal pullbacks when restricted to the body.

2. i carries open covers to epimorphic families.

and 3. $i(k \otimes \Delta(i)) = R$ is a superline.

PROOF. We begin by noting that if i extends $\Gamma y L$, then we have already shown 3, since $(k, \mathbf{T}(k) \otimes \Delta(i))$ is in $\mathbf{GT\text{-}Mf}_0$.

We next note that $\Gamma y L$ satisfies 2, by construction of the topology in the site of definition, while $\Gamma y L$ satisfies 1 by applying results of Kock [6] once it is noted that $(M, \mathbf{T}(M) \otimes \Delta)$ is isomorphic to the product $(M, \mathbf{T}(M)) \times (*, \Delta)$ and that Γ, y, and L all preserve products.

To see that $\Gamma y L$ satisfies 0, it suffices to examine Γ, since y is full and faithful and the topology in question is subcanonical. For Γ, fullness and faithfulness follow from the product decomposition, together with the observations that on $(*, \Delta)$ a map of graded $\mathbf{T}$-manifolds is entirely determined by its behaviour on the ring of functions, while "goodness" allows us to imitate the classical proof that $C^\infty(\ )$ is full and faithful for any $\mathbf{T}$.

To extend $\Gamma y L$ to all of $\mathbf{GT\text{-}Mf}$, note that any locally good graded $\mathbf{T}$-manifold is canonically the colimit of its good trivializations, that is of a canonically chosen diagram in $\mathbf{GT\text{-}Mf}_0$. We let i be the result of applying $\Gamma y L$ to this diagram, then taking the colimit in $\mathbf{E}_{S\langle T\rangle}$. Note that this extends $\Gamma y L$, since it agrees with $\Gamma y L$ on $\mathbf{GT\text{-}Mf}_0$, since here the diagram of good trivializations has a terminal object.

Now since 1 and 2 are local in nature, the colimiting construction will preserve them. For 0 note that the image of $\mathbf{GT\text{-}Mf}_0$ generates $\mathbf{E}_{S\langle T\rangle}$, and thus i must be faithful, while fullness follows from 2 by passing to a good trivialization of the target, and then to a good trivialization of the source which refines its preimage.

Thus the "super-Dubuc topos" plays the same role in the "super" theory as the Dubuc topos does for classical differential geometry.

3.3. Formal spermanifolds.

Although all objects in the topoi $\mathbf{E}_{S\langle T\rangle}$ can be regarded as "superspaces", they do not all possess manifold-like properties. Two approaches may be taken to isolating "formal supermanifolds". The first is essentially classical: choose model objects and define manifolds as those objects which "look locally like the models". The second is purely synthetic: determine what properties of manifolds are essential to the problem at hand and consider those objects which satisfy them (having shown that those objects which intuitively "should" be manifolds satisfy the properties). We begin with the former:

The obvious notion of supermanifold arises by taking as model objects all objects of the form $R_B{}^p \times R_F{}^q$, where

$$R_B = \{x \in R \mid x = B(x)\} \quad \text{and} \quad R_F = \{x \in R \mid x = F(x)\},$$

then considering all objects X such that there is a formal etale cover $\{U_i\}_{i \in I}$ by formal etale subobjects of the model objects, where:

DEFINITION 3.3.1. A map f: X → Y is *formal etale* if for any tiny subobject A of $\neg\neg\{0\} \subset R^n$, containing 0, for any n, the diagram

$$\begin{array}{ccc} X^A & \xrightarrow{f^A} & Y^A \\ {\scriptstyle (0)}\downarrow & & \downarrow{\scriptstyle (0)} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pullback.

This notion of "supermanifold" is sufficient to include the internal versions of graded manifolds, but fails to capture the "superfunction spaces" which have good local behaviour and are one of the points of the synthetic approach.

For the purely synthetic approach, we wish to distinguish some particularly interesting Weil algebra spectra:

DEFINITION 3.3.2. Let

$$D(p,q) = \{(x_1,\dots,x_p,\theta_1,\dots\theta_q) \mid x_i \text{ bosonic, } \theta_k \text{ fermionic, } x_i x_j = x_i\theta_k = \theta_k\theta_l = 0\} \subset R^{p+q},$$

$$D_k(p,q) = \{(x_1,\dots,x_p,\theta_1,\dots\theta_q) \mid x_i \text{ bosonic, } \theta_k \text{ fermionic, any } (k+1)\text{-fold product of the } x_i\text{'s and } \theta_k\text{'s is } 0\} \subset R^{p+q}.$$

Note that $D(0,1) = R_F$ since all fermionic elements are 2-nilpotents.

We can now formulate "super" versions of infinitesimal linearity and "Property W_n" (see Kock [6]). Both of the following definitions are to be read internally, so that maps are to be taken as generalized elements of the appropriate function objects.

DEFINITION 3.3.3. An object in a superlined topos is *infinitesimally linear* if given any family of maps

$$f_i\colon D(i,0) \to X, \quad i = 1,\dots,p, \qquad \phi_j\colon D(0,1) \to X, \quad j = 1,\dots,q$$

such that

$$f_1(0) = \dots = f_p(0) = \phi_1(0) = \dots = \phi_p(0),$$

there exist uniquely

$$F\colon D(p,q) \to X \quad \text{such that} \quad i_{(i,0)}F = f_i \quad \text{and} \quad i_{(0,j)}F = \phi_j,$$

where $i_{(i,0)}$ (resp. $i_{(0,j)}$) is inclusion by setting all coordinates except the i^{th} bosonic (resp. j^{th} fermionic) to 0.

DEFINITION 3.3.4. An object *satisfies Property $W(p,q)$* if for all maps $\tau\colon D(1,0)^p{\times}D(0,1)^q \to X$ such that

$$\tau(0,x_2,\dots x_p,\theta_1,\dots,\theta_q) = \dots = \tau(x_1,\dots,x_{p-1},0,\theta_1,\dots,\theta_q) = \dots$$
$$\dots = \tau(x_1,\dots,x_p,\theta_1,\dots,\theta_{q-1},0)$$

there exists uniquely $t\colon D(\epsilon,\lambda) \to X$ such that

$$\tau(x_1,\dots,x_p,\theta_1,\dots,\theta_q) = t(x_1\dots x_p\theta_1\dots\theta_q),$$

where

$$(\epsilon,\lambda) = (1,0) \text{ if } q \text{ is even and } (\epsilon,\lambda) = (0,1) \text{ if } q \text{ is odd.}$$

PROPOSITION 3.3.5. R *is infinitesimally linear and satisfies Property* $W(p,q)$.

PROOF. As in the ungraded case (in Kock [6]) this follows readily from L1.

THEOREM 3.3.6. *The class of infinitesimally linear objects (resp. objects satisfying Property* W(p,q)*) is closed under:*

C1. formal etale subobjects,

C2. limits,

C3. exponentiation by arbitrary objects,

C4. passing to factors of products,

and C5. arbitrary coproducts.

PROOF. C3 is immediate from the internal nature of the conditions involved, while C2 is immediate from the universal property of limits and the uniqueness in the conclusions of the definitions. For C1, note that every map involved factors through any formal etale neighborhood of the image of 0 in the objects involved.

For C4, note that the maps to A×B in the hypotheses of the definitions are uniquely expressed as a pair of maps, one to A, one to B, *which each satisfy the hypotheses;* the unique pair, each of which is given by the existential part of the definitions, defines a map to A×B which has the same property.

C5 follows from the tininess (and hence connectedness) of the object involved. (Note: tininess gives the preservation of the existential conditions in the definitions by arbitrary colimits, but will not in general give uniqueness.)

COROLLARY 3.3.7. *Formal supermanifolds (in the sense above) are infinitesimally linear and satisfy Property* W(p,q) *for all* (p,q).

It is in fact these two properties: infinitesimal linearity and Property W(p,q) (for certain p and q) which give most of the "classical" properties of the tangent bundle once the correct definition of that notion is introduced. Two reasonable notions present themselves.

DEFINITION 3.3.8. The *total tangent bundle* of X is the object over X given by

$$(0): X^{D(1,1)} \to X.$$

The *bosonic tangent bundle* is the object over X given by

$$(0): X^{D(1,0)} \to X.$$

The latter of these corresponds more or less to the tangent module for DeWitt supermanifolds, and has similar properties. We concentrate our attention on the more genuinely "super" notion of tangent bundle, the total tangent bundle:

THEOREM 3.3.9. *If* X *is infinitesimally linear, then the total tangent bundle is a bundle of* R,R-*bimodules over* X, *satisfying moreover*

$$ax = (-1)^{|a||x|}xa$$

where $a \in R$, $x \in X^{D(1,1)}{}_p$, *and* $|\ |$ *denotes the* 0–1 *valued grading in each case.*

PROOF. By infinitesimal linearity we have an isomorphism

$$X^{D(1,1)} \times_X X^{D(1,1)} \to X^{D(2,2)}.$$

Composing this with the map X^{Δ}: $X^{D(2,2)} \to X^{D(1,1)}$ gives the addition on the total tangent bundle. Verification that this gives a fibrewise abelian group structure is essentially as in Kock [6].

The bimodule structure is given by

$$R \times X^{D(1,1)} \to X^{D(1,1)} \quad \text{by} \quad (r,x) \to [d \to x(rd)],$$
$$X^{D(1,1)} \times R \to X^{D(1,1)} \quad \text{by} \quad (x,r) \to [d \to x(dr)].$$

Both distributivity and associativity are easy to verify, while graded commutativity follows from the graded commutativity of R (D(1,1) being as subobject of R).

COROLLARY 3.3.10. *For* M *infinitesimally linear, the object of vector fields on* M,

$$\mathrm{Vect}(M) = \Pi_M(X^{D(1,1)} \to X),$$

is a graded commutative R^M-*module.*

Property $W(p,q)$ can now be used to provide the additional structure existing classically on Vect(M): a (graded) Lie algebra structure.

THEOREM 3.3.11. *If* M *is infinitesimally linear and satisfies Properties* W(2,0), W(1,1), W(0,2), *then*

$$\mathrm{Vect}(M) = \Pi_M(X^{D(1,1)} \to X) \simeq \Pi_M(X^{D(1,0)} \to X) \oplus \Pi_M(X^{D(0,1)} \to X)$$

is a graded Lie algebra over R, *when equipped with the operation given gradewise by*

$$[X,Y]\colon D(\epsilon,\lambda) \to M^M$$

which is the unique map given by Property W(p,q) *such that*

$$[X,Y](d\delta) = X(d)Y(\delta)X(-d)Y(-d): D(a,b))\times D(\alpha,\beta) \to M^M,$$

where (a,b) *and* (α,β) *are each one of* $(1,0)$ *or* $(0,1)$, *and* (ϵ,λ) *is* $(1,0)$ *if* $b+\beta$ *is even and* $(0,1)$ *if* $b+\beta$ *is odd.*

PROOF. An imitation of the argument due to Reyes and Wraith (see Kock [6]) suffices when the graded commutativity of R is taken into account.

COROLLARY 3.3.12. *If* G *is a group object, which is infinitesimally linear and satisfies Properties* W(2,0), W(1,1), W(0,2), *then*

$$T_\bullet(G) = \{v: D(1,1) \to G \mid v(0) = e\}$$

is a graded Lie algebra/R.

PROOF. Identify $T_\bullet(G)$ with the object of left invariant vector fields and restrict the Lie algebra structure of Theorem 3.3.11.

Note that besides internal versions of finite dimensional supergroups, such exotic but physically interesting objects as the internal versions of "super-loop groups" satisfy the hypotheses of the Corollary, and thus are included in the same synthetic constructions as the finite dimensional cases.

3.4. Order and Intgration in the super-Dubuc topos.

Finally, we turn to superspace integration in the context of our models.

Recall that superspace integration in other models of superspace (cf. Rogers [10] or Berezin [2]) is carried out by treating bosonic and fermionic coordinates differently: bosonic variables are integrated classically, while fermionic coordinates are integrated according to the Berezin prescription:

$$\int d\theta = 0 \quad \text{and} \quad \int \theta \, d\theta = 1.$$

As noted in the introduction, it is the view of Batchelor that superspace integration is really a hybrid: integration in bosonic coordinates, differentiation in fermionic coordinates.

We restrict our attention now to the theory **S** (C$^\infty$-alg), and the super-Dubuc topos, which we now denote **E**. This will be necessary only to consider integration in bosonic parameters. For fermionic ones in any superlined topos we have:

DEFINITION 3.4.1. The *Berezin integral map* $\int: R^{R_F} \to R$ is the composite $\alpha^{-1}p_2$, where α: $R\times R \to R^{R_F}$ is given by

$$\alpha(a,b) = [\theta \to a+b\theta],$$

and is invertible by L1 in the definition of superline.

Note that this definition is internal, and hence "smooth in parameters". It is also precisely the fermionic parameter version of synthetic differentiation: in the view of Batchelor, "Berezin integration *is* odd parameter differentiation".

Proceeding on to bosonic parameter integration, note that Theorem 3.1.8 allows us to lift the order structure and "classical" integration structure on the Dubuc topos to the super-Dubuc topos. To be precise:

THEOREM 3.4.2. R *(resp.* R_B*) has two preorderings,* $<$ *and* $\leqslant$ *satisfying:*

01. $<$ *and* $\leqslant$ *are transitive.*
02. $\leqslant$ *is reflexive;* $<$ *is irreflexive.*
03. $x \leqslant y \Rightarrow x+z \leqslant y+z$; $x<y \Rightarrow x+z < x+z$.
04. $[x \leqslant y \wedge 0 \geqslant t] \Rightarrow xt \leqslant yt$; $[x < y \wedge 0 < t] \Rightarrow xt < yt$.
05. $0 < 1$.
06. $x < 0 \Rightarrow x \leqslant 0$.
07. d nilpotent $\Rightarrow (0 \leqslant d \wedge d \leqslant 0)$.
08. $x < 0 \Rightarrow x$ *invertible.*
09. $\neg(x < 0) \equiv 0 \leqslant x$.
010. x *invertible* $\Rightarrow [x < 0 \vee 0 < x]$.
011. $[0 < x \wedge x \leqslant y] \Rightarrow 0 < y$.

PROOF. For R_B this is a result of Kock [6] for the Dubuc topos. To extend the orderings of R, note that any element of R is of the form $B(x)+F(x)$ for $B(x) \in R_B$ and $F(x) \in R_F$. Let

$$x \leqslant y \text{ iff } B(x) \leqslant B(y) \quad\text{and}\quad x < y \text{ iff } B(x) < B(y).$$

It is then easy to verify that 01-011 are preserved by this extension (the crucial thing is to note that fermionic elements are always nilpotent).

We denote by $[0,1]$ the subobject of R_B, $\{x \mid 0 \leq x \leq 1\}$, and by $\langle 0,1\rangle$ the subobject of R given by the same formula. Note that

$$\langle 0,1\rangle \simeq [0,1]\times R_F.$$

Except for the difficulty that we need our result to hold "smoothly in fermionic parameters" (i.e., for generalized elements given by fermionic objects), we could now just lift the integration from the Dubuc topos to give our superspace integration in bosonic parameters. Instead we must imitate the proof of the "Integration Axiom" for the Dubuc topos, and check that the resulting bosonic parameter integration commutes with Berezin integration in fermionic parameters.

Before proceeding further, we must note:

PROPOSITION 3.4.3. *The functor i: $\mathbf{GC^\infty\text{-}Mf} \to \mathbf{E}$ extends to a functor from the category of graded (C^∞) manifolds with boundary, so as to agree with the extension of the functor from smooth manifolds to the Dubuc topos to smooth manifolds with boundary.*

We continue to denote this extension by i.

COROLLARY 3.4.4.
$$[0,1] \simeq i([0,1],C^\infty[0,1]),$$
$$\langle 0,1\rangle \simeq i([0,1],C^\infty[0,1])\otimes\Delta^1).$$

THEOREM 3.4.5. *For any $f \in R^{[0,1]}$ in $\mathbf{E}$, there is a unique $g \in R^{[0,1]}$ such that $g(0) = 0$ and $g' = f$, where $(\ )'$ denotes synthetic differentiation in one bosonic parameter (i.e., $\alpha^{-1}p_2$, where α: $R\times R \to R^{D(1,0)}$ is given by*

$$\alpha(a,b) = [d \to a+bd]$$

and α is invertible by L1 in the definition of superline).

We denote $g(x)$ by $\int f(x)\, dx$.

PROOF. Consider generalized elements $f \in R^{[0,1]}$ of type $i(M)\times\mathrm{Spec}(W)$ for M a C^∞-manifold, and W a graded Weil algbra. We then have a sequence of natural correspondences:

$$\frac{\dfrac{\dfrac{\text{maps } i(M)\times\mathrm{Spec}(W) \to R^{[0,1]} \text{ in } \mathbf{E}}{\text{maps } i(M\times[0,1]) \to R^{\mathrm{Spec}(W)} \text{ in } \mathbf{E}}}{\text{maps } i(M\times[0,1]) \to R^{\dim(W)} \text{ in } \mathbf{E}}}{\dim(W)\text{-tuples of maps } i(M\times[0,1]) \to R \text{ in } \mathbf{E}}$$

$$\frac{\dim(W)\text{-tuples of maps } \imath(M\times[0,1]) \to R_B \text{ in } \mathbf{E} \text{ (equiv. in Dubuc topos)}}{\dim(W)\text{-tuples of maps } M\times[0,1] \to \mathbf{R} \text{ in the category of smooth manifolds with boundary.}}$$

The passage to the Dubuc topos requires us to note that there are not global non-zero maps from any bosonic object to R_F (equiv. the bosonic sheafification of R_F is 1).

We now integrate classically in each coordinate and reverse the sequence of natural equivalences to obtain the (generalized) element g, noting that each equivalence "preserves (bosonic) differentiation" in the evident sense.

It is then an easy consequence of cartesian closedness that:

PROPOSITION 3.4.6. *The value of iterated integrals in several parameters (of possibly mixed types) is independent of the order of integration.*

A final note on integration: the synthetic approach makes clear why the "differentiatiuon backwards" aspect of integration must be lost in notions of integration applicable to superfields.

Consider the definition of differentiation in a lined topos:

f': R → R is the unique function such that

$$\forall\, x \in R\ \forall\, d \in D \quad f(x+d) - f(x) = df'(x).$$

When we pass to a superlined topos, and replace D by D(1,1) (the object of 2-nilpotents in the super setting), no such function exists in general: instead there is a unique function f': R → M(1,1), where M(1,1) is the object of (1,1)-square supermatrices. It is thus impossible to identify functions with vector-fields on the superline by any "superEuclidean metric" and thus to identify integration of superfields with genuine anti-differentiation.

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