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LIMITS AND COLIMITS OF CONVEXITY SPACES

BY Robert J. MacG. DAWSON

RÉSUMÉ. Une *préconvexité [convexité]* sur un ensemble consiste en une famille de sous-ensembles (*ensembles convexes*), fermée par intersections arbitraires [et unions filtrantes]. On examine deux catégories principales d'espaces à préconvexité, dont les morphismes sont, d'une part, les *fonctions de Darboux* (celles qui préservent les ensembles convexes) et, d'autre part, les *fonctions monotones* (celles qui reflètent les ensembles convexes). On compare les limites et les colimites dans ces catégories, et dans certaines sous-catégories importantes. Finalement, on identifie une sous-catégorie réflexive de la catégorie d'espaces à convexité comme étant la complétion inductive de la catégorie connue des complexes simpliciaux finis.

There has recently been considerable interest in the properties of "convexity spaces" or "aligned spaces", structures which generalise the linear structure of Euclidean space in the same way that topological spaces generalise its metric structure. Like a topological space, a convexity space consists of a set with a distinguished family of subsets (usually called *convex sets*); they obey different axioms of union and intersection, however. Generalised convexity theory has been described as "rigid sheet" geometry, in contrast to the topologists' "rubber sheet".

Many important constructions in topology, such as the product of topological spaces, depend on the nature of the continuous functions between topological spaces. In the case of convexity spaces, there are two obvious analogues. Most authors so far have dealt mainly with maps such that the inverse images of convex sets are convex; Jamison-Waldner [6] in particular has considered the category of convex aligned spaces (in the terminology of this paper). Here, we will extend that investigation, and also consider a different categorical structure on the class of convexity spaces, using maps which map convex sets to convex sets. We will see that an important subcategory of this category is the completion under directed colimits (ind-completion) of the familiar category of finite simplicial complexes with simplicial maps.

1. DEFINITION.

A *preconvexity* K on a set X is a collection of subsets of X such that the intersection of any (non-empty) subcollection of K is again a member of K. If K is also closed under nested unions (or directed unions; the apparently more general case gives rise to the same closure system), we will call it a *convexity* on X. We will call the pair (X,K) a *(pre)convexity space*, and the elements of K *convex sets*. (We do not call the elements of a preconvexity "preconvex", because that would suggest that the sets themselves neeeded to be completed in some way to become convex; whereas we will see that the most natural way to make a convexity out of a preconvexity is to add new sets, rather than to change the existing ones.) If X is itself an element of K, we will call (X,K) a *convex* (pre)convexity space. Several authors have taken this as an axiom for all convexity spaces (see, e.g., [6, 8, 10, 11, 12]). While this is useful in many cases, we will keep the greater generality here.

Various authors, such as Jamison-Waldner [6] and van de Vel [12], have considered functions between (pre)convexity spaces. Usually, these have been taken to be the functions which reflect convex sets; if A is convex in the codomain of f, $f^{-1}(A)$ is convex in the domain of f. Perhaps a little confusingly, these are referred to in [12] and elsewhere in the literature as "convexity-preserving" or "CP" functions - a term which might be applied more appropriately to a function under which the image of a convex set is convex. Here, we will consider (separately) both types of function. In order to avoid confusion, we will call a function which reflects convex sets *monotone*, and one which preserves them *Darboux* - in each case, by analogy with the special case of the real line. (It should be noted, however, that other authors have generalised the concept of a Darboux function topologically as a function that preserves *connected* sets.)

In an established categorical tradition, we will distinguish between objects in categories with Darboux maps and objects in categories with monotone maps. We will refer to a *(pre)convexity* space in the first instance, and to a *(pre)aligned* space in the second. (The term "alignment" was introduced by Jamison-Waldner [6], who considered the category of convex aligned spaces. I will take the liberty here of extending his term to include the nonconvex case; it will be seen that many of his results remain true in the more general setting.) The justification for this is that, while every alignment is a convexity, and *vice versa*, the coproduct alignment (for instance) is not the same as the coproduct convexity. We will thus use such categories as:

Precxy: preconvexity spaces with Darboux maps;
Cxy: convexity spaces with Darboux maps;
Prealn: prealigned spaces with monotone maps;
Aln: aligned spaces with monotone maps.

A preconvexity space (or prealigned space) will be said to be S_1 if every point is a convex set [6]. If, for any two distinct points x,y, there is a chain of convex sets $\mathbf{A}_0$, $\mathbf{A}_1$, $\mathbf{A}_2$, ..., $\mathbf{A}_n$ such that

$$\mathbf{A}_i \cap \mathbf{A}_{i+1} \neq \emptyset, \quad x \in \mathbf{A}_0, \quad y \in \mathbf{A}_n,$$

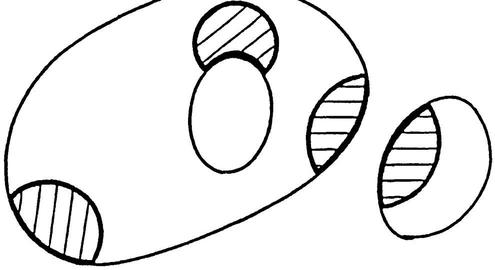
we will call the space *connected*. Note in particular that any convex space is connected.

2. EXAMPLES.

In the examples that follow, $P(\mathbf{X})$ is the power set of $\mathbf{X}$, and $\mathbf{E}$ is the Euclidean convexity on $\mathbf{R}^n$, consisting of all sets which are convex in the usual sense.

EXAMPLE 2.1. Let $\mathbf{X}$ be a subset of $\mathbf{R}^n$, and let $\mathbf{K} = \mathbf{E} \cap P(\mathbf{X})$. Then $(\mathbf{X},\mathbf{K})$ is a convexity space. Its embedding into $(\mathbf{R}^n,\mathbf{E})$ is Darboux. $(\mathbf{X},\mathbf{K})$ is convex iff $\mathbf{X} \in \mathbf{E}$; in such a case, and only then, the embedding is also monotone. $(\mathbf{X},\mathbf{K})$ is S_1, and connected iff it is polygon-connected. This is an example of a *subspace convexity*.

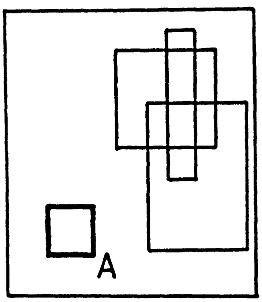
EXAMPLE 2.2. Let $\mathbf{X}$ be a subset of $\mathbf{R}^n$, and let $\mathbf{K} = \{\mathbf{A}\cap\mathbf{X} \mid \mathbf{A} \in \mathbf{E}\}$ (Figure 1).

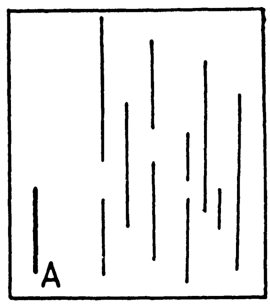


The embedding of this space into (R^n, E) is monotone, but only Darboux if $X \in E$; (X, K) is always convex and S_1. This is an example of a *subspace alignment*.

EXAMPLE 2.3. Lassak [9] considers the family K_A of sets in R^n generated from a given set **A** by intersections, homotheties, and directed unions. If **A** is the unit ball, $K_A = E$. However, the unit cube yields the *box convexity*, whose convex sets are cartesian products of intervals. We shall see later that this is a special case of *product alignment*.

The properties of (R^n, K_A) depend on **A**. The space is convex iff the affine hull of **A** is all of R^n; otherwise, it is not even connected.





It is S_1 iff **A** is not the union of a parallel family of semiinfinite rays.

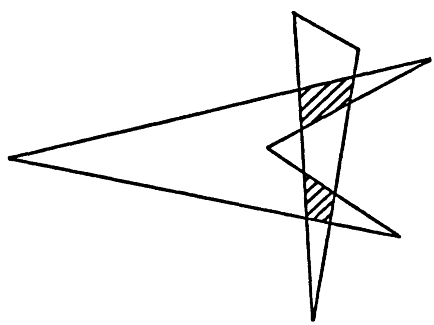
EXAMPLE 2.4. The compact sets in R^n form a preconvexity but not a convexity. It is S_1, connected, but not convex.

EXAMPLE 2.5. The subspaces of a vector space form a convexity which is convex but not S_1.

If we define an *algebra* to be a set equipped with a family, possibly infinite, of finitary operations, and a *subalgebra* to be any subset closed under all of those operations, we may consider Example 2.5 to be a special case of the next example. (Note that a vector space is an algebra with one nullary operation [the origin], one binary operation [addition], and one unary operation for every scalar in the underlying field [scalar multiplication].)

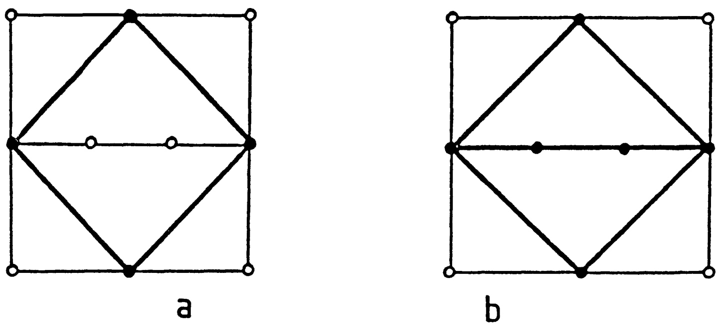
EXAMPLE 2.6. The subalgebras of any algebra form a convexity. This was proven by Birkhoff and Frink [1] in 1948, although they did not emphasize the connection with geometry, or call the structure a convexity. Such a convexity space is always convex; it was shown in [1] that any convex convexity space may be so obtained, for a suitable algebra over the underlying set. It may be shown also that such a convexity space is S_1 iff for every operation f and for every element a, $f(a,a,\dots,a) = a$ [3]. (Such an algebra is sometimes called *affine*.)

EXAMPLE 2.7. The sets in $\mathbf{R}^n$ which are starshaped about the origin form a convexity which is convex but not S_1. (Note that the sets which are starshaped about an unspecified point do not even form a preconvexity, as the intersection of two of them may not be starshaped about any point.)



EXAMPLE 2.8. In a differentiable manifold $\mathbf{M}$, let $\mathbf{K}_g$ consist of those sets such that there is a unique minimal geodesic arc joining every pair of their points, and which contain that arc. (If $\mathbf{M}$ is $\mathbf{R}^n$ with the usual manifold structure, this reduces to the usual convexity.) This is a convexity on $\mathbf{M}$. Its properties are described in [2], where it is shown that any sufficiently small ϵ-ball about a given point in $\mathbf{M}$ is in $\mathbf{K}_g$. $(\mathbf{M},\mathbf{K}_g)$ is not generally convex; it may be shown that it is connected iff $\mathbf{M}$ is topologically connected.

EXAMPLE 2.9. Let G be a graph. A set of vertices of G is *geodesically convex* if it contains, with each pair of its members, all vertices of all shortest paths between them; and *monophonically convex* if it contains, with each pair of its vertices, all vertices of all chordless paths between them [4, 6]. (A path is *chordless* if it is the induced graph on its vertices.) Both of these definitions give rise to convexities on graphs. Figure 4a shows a geodesically convex set. Figure 4b a monophonically convex set.



EXAMPLE 2.10. Let G be a graph; the *cliques* (complete induced subgraphs) form a convexity on G. Note that every induced subgraph of a clique is itself a clique; thus in this convexity, every subset of a convex set is itself convex. Such a convexity space is called *downclosed*.

EXAMPLE 2.11. Let K be a finite simplicial complex; then the simplexes of K form a convexity on K. Like the previous example, this convexity space is downclosed. Furthermore, any simplicial map from K to another finite simplicial complex L corresponds to a Darboux map between the corresponding convexity spaces. The implications of this will be considered in the final section of this paper.

3. FUNCTORIAL CONSTRUCTIONS.

DEFINITION 3.1. Let X be a set. The following are convexity spaces:

3.1.1. $X_P = (X, \mathcal{P}(X))$ *(power set convexity).*

3.1.2. $X_T = (X, \emptyset)$ *(trivial convexity).*

3.1.3. $X_D = (X, \{\{x\} \mid x \in X\})$ *(discrete convexity).*

3.1.4. $X_I = (X, \{X\})$ *(indiscrete convexity).*

3.1.5. $X_S = (X, \{\{x\} \mid x \in X\} \cup \{X\})$ *(singleton convexity).*

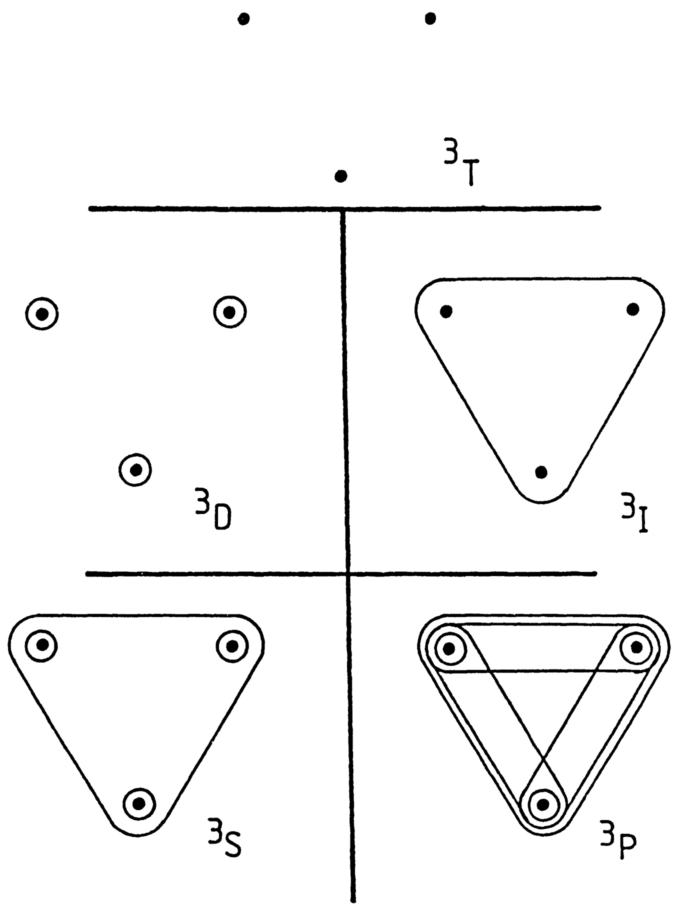


Figure 5 illustrates these for a three-point set. It is clear that any function $X \to Y$ induces a Darboux and monotone function $X_P \to Y_P$. Thus, there is a functor P: *Set* $\to$ *Cxy* and a functor P: *Set* $\to$ *Aln*, each of which assigns to each set the power set convexity (alignment). Similarly, there are functors D, T: *Set* $\to$ *Cxy* and functors I, T: *Set* $\to$ *Aln*. If $f\colon X \to Y$ is not mono, $f\colon X_D \to Y_D$ is not monotone; and if $f\colon X \to Y$ is not epi, $f\colon X_I \to Y_I$ is not Darboux. Similarly, in either case, the singleton convexity fails to induce a functor.

PROPOSITION 3.2. *The category CPrealn of convex prealigned spaces is a coreflective subcategory of Prealn.*

(Recall that a subcategory is reflective (resp. coreflective) if the inclusion has a left (resp. right) adjoint.)

PROOF. The functor

$$C: (X,K) \mapsto (X, K \cup \{X\}), \quad f \mapsto f$$

is right adjoint to the inclusion *CPrealn* ↪ *Prealn*. •

COROLLARY 3.2.1. *The category CAln of convex aligned spaces is a coreflective subcategory of Aln.*

PROPOSITION 3.3. *The categories Cxy_1 and $Precxy_1$ of* S_1 *(pre)convexity spaces are reflective subcategories of Cxy and Precxy respectively.*

PROOF. The functor

$$S: (X,K) \mapsto (X, K \cup \{\{x\} \mid x \in X\}): \quad f \mapsto f$$

is left adjoint to the inclusions. •

PROPOSITION 3.4. *The functors* T: *Set* → *Cxy*, T: *Set* → *Precxy*, D: *Set* → Cxy_1, D: *Set* → $Precxy_1$, P: *Set* → *Aln, and* P: *Set* → *Prealn are left adjoint to the appropriate forgetful functors; and the functors* T: *Set* → Aln, T: *Set* → *Prealn*, I: *Set* → *CPrealn*, I: *Set* → *CAln*, P: *Set* → *Precxy and* P: *Set* → *Cxy are right adjoint to the appropriate forgetful functors.*

PROOF. We will prove one case; the rest follow the same pattern. Let U: Cxy_1 → *Set* be the forgetful functor, X a set, and (Y,L) an S_1 convexity space. Then any Darboux function f: $X_D \to (Y,L)$ is also a set map $f: X \to Y$; and, conversely, for any set map $g: X \to Y$, then $g: X_D \to (Y,L)$ is Darboux, as all the singleton sets in (Y,L) are convex. Thus, we have a natural isomorphism between the hom-sets $[X, U(Y,L)]_{Set}$ and $[X_D, (Y,L)]_{Cxy_1}$. •

COROLLARY 3.4.1. *The forgetful functors from Precxy, Cxy, Prealn, and Aln to Set preserve all limits and colimits that exist in their domains.* •

Thus, we know what the underlying sets of any (co)limits that exist in *Precxy, Cxy, Prealn*, or *Aln* are; in the next section we shall determine when they exist and what the appropriate convex structures are.

PROPOSITION 3.5. *Aln is a coreflective subcategory of Prealn; Cxy is a reflective subcategory of Precxy.*

PROOF. For any preconvexity $\mathcal{K}$ on $\mathbf{X}$, let

$$\mathcal{K}^- = \{\cup_\Gamma \mathbf{A}_\gamma \mid \mathbf{A}_\gamma \in \mathcal{K},\ \{\mathbf{A}_\gamma\} \text{ directed by } \Gamma\}.$$

This is a preconvexity; for let $\{\mathbf{A}_{\beta\gamma} \mid \gamma \in \Gamma_\beta\}$ be directed sets, with unions $\mathbf{A}_\beta$, and let Γ be the poset product of the posets Γ_β. Any product of directed posets is itself directed; and

$$\cap_{\beta\in B}\mathbf{A}_\beta = \cap_B \cup_{\Gamma_\beta} \mathbf{A}_{\beta\gamma} = \cup_{(\gamma_\beta)\in\Gamma} \cap_B \mathbf{A}_{\beta\gamma_\beta}\ .$$

Now, $\cap_B \mathbf{A}_{\beta\gamma_\beta} \in \mathcal{K}$; and

$$\{\cap_{\beta\in B}\mathbf{A}_{\beta\gamma_\beta} \mid (\gamma_\beta) \in \Gamma\}$$

is directed by Γ, so $\cap_{\beta\in B}\mathbf{A}_\beta \in \mathcal{K}^-$. Furthermore, any nested union of directed unions is itself directed; so $\mathcal{K}^-$ is a convexity. Any element of $\mathcal{K}$ is also in $\mathcal{K}^-$, so the identity map $i\colon \mathbf{X} \to \mathbf{X}$ induces a Darboux map $(\mathbf{X},\mathcal{K}) \to (\mathbf{X},\mathcal{K}^-)$ and a monotone map $(\mathbf{X},\mathcal{K}^-) \to (\mathbf{X},\mathcal{K})$; these are the desired reflection and coreflection. •

We will refer to the reflector as *Cx: Precx → Cxy*, and to the coreflector as *Cx: Prealn → Aln*. As they act in the same way on spaces and on maps, this should not lead to confusion.

DEFINITION 3.6. A preconvexity space will be said to be *downclosed* if every subset of a convex set is also convex.

We have already come across some examples of downclosed preconvexity spaces; see, for instance, Examples 2.10 and 2.11. Any downclosed convex preconvexity space has, of course, the powerset convexity! The two examples just mentioned, however, show that there are also non-trivial downclosed preconvexities.

PROPOSITION 3.7. *The category DcPrealn of downclosed prealigned spaces is a coreflective subcategory of Prealn; and DcPrecxy is a reflective subcategory of Precxy. Also, DcAln is a coreflective sub-*

category of Aln, DcCxy is a reflective subcategory of Cxy, and the (co)reflector Dc commutes with Cx.

PROOF. Let $(X,\mathcal{K})$ be a preconvexity space, and define

$$\mathcal{K}^{dc} = \{A \mid (\exists K \in \mathcal{K})(A \subset K)\}.$$

$\mathcal{K} \subset \mathcal{K}^{dc}$; so the identity map $i: X \to X$ induces the reflection $(X,\mathcal{K}) \mapsto (X,\mathcal{K}^{dc})$ and the coreflection $(X,\mathcal{K}^{dc}) \mapsto (X,\mathcal{K})$. To see that *Cx* and *Dc* commute, suppose that $A \subset K$, and that K is the union of the directed family $\{K_\gamma\}$; then A is the union of the directed family $\{A \cap K_\gamma\}$. •

Jamison-Waldner observed in [6] that many of the properties of aligned spaces depend only on the hulls of their finite subsets. The next two results show a new way to consider this "finitary property", via an equivalence between downclosed convexity spaces and downclosed spaces with only finite convex sets. For any set X, let $\mathcal{F}(X)$ be the family of finite subsets of X.

DEFINITION 3.8. A preconvexity space will be said to be *finitary* if every convex set is finite.

PROPOSITION 3.9. *The category FinDcPrecxy of finitary downclosed preconvexity spaces is a reflective subcategory of DcPrecxy.*

PROOF. *Fin:* $(X,\mathcal{K}) \mapsto (X, \mathcal{K}\cap\mathcal{F}(X))$ is a functor, for the image of any finite convex set under a Darboux map is again a finite convex set. If $(X,\mathcal{K})$ is finitary, then $f: (X,\mathcal{K}) \to (Y,\mathcal{L})$ is Darboux iff $f: (X,\mathcal{K}) \to Fin(Y,\mathcal{L})$ is Darboux; thus *Fin* is right adjoint to the inclusion

$$FinDcPrecxy \hookrightarrow DcPrecxy.$$ •

Note that there is not an analogously-defined functor for prealigned spaces, as the preimage of a finite set is not in general finite.

THEOREM 3.10. *The functors Fin and Cx give an equivalence between the categories FinDcPrecxy and DcCxy.*

PROOF. If a finite set F is the directed union of sets $\{F_\gamma\}$, $F \in \{F_\gamma\}$; thus $Fin \circ Dc = Fin$. Similarly, $Cx \circ Fin \circ Dc = Cx \circ Dc$; for if a set A is the directed union of convex sets $\{A_\gamma\}$, the finite subsets of the sets $\{A_\gamma\}$ form a directed family whose union is A; and in a downclosed space, those finite subsets are themselves convex. Combining the two identities just proved,

$$Fin \circ Cx \circ Fin \circ Dc = Fin \circ Cx \circ Dc = Fin \circ Dc$$

and

$$Cx \circ Fin \circ Cx \circ Dc = Cx \circ Fin \circ Dc = Cx \circ Dc.$$

Thus (Fin, Cx) form an equivalence of categories between *FinDcPrecxy* and *DcCxy*. •

4. LIMITS AND COLIMITS.

Jamison-Waldner has shown [6] that *CAln* is complete and cocomplete. In this section, we will consider the more general case of arbitrary prealigned spaces, and the corresponding (but in some ways quite different) behaviour of preconvexity spaces. From Corollary 3.4.1, we know that any such limits or colimits that exist must have the same underlying sets as the corresponding limits or colimits in *Set* ; it remains to determine when there exists an appropriate convexity structure, and what it is.

THEOREM 4.1. *PreAln is complete and cocomplete.*

PROOF. Given a prealigned space $(X,\mathcal{K})$ and a set epimorphism $f\colon X \twoheadrightarrow Z$, f induces a quotient prealignment

$$f^*\mathcal{K} = \{A \mid f^{-1}A \in \mathcal{K}\}$$

on Z such that $f\colon (X,\mathcal{K}) \to (Z,f^*\mathcal{K})$ is monotone, and such that $h\colon (Z,f^*\mathcal{K}) \to (Y,\mathcal{L})$ is monotone whenever hf is monotone. The coequalizer of a pair

$$\{f_i\colon (X,\mathcal{K}) \rightrightarrows (Y,\mathcal{L})\}$$

is the set of equivalence classes of Y under the relation $f_1(y) \simeq f_2(y)$ with the quotient prealignment. Similarly, given a set monomorphism $f\colon Z \rightarrowtail X$, f induces a subobject prealignment

$$ {}^*f(\mathsf{K}) = \{f^{-1}A \mid A \in \mathsf{K}\} \ ; $$

and the equalizer of the pair given above is the set

$$ \{x \mid x \in \mathbf{X},\ f_1(x) = f_2(x)\}, $$

with the subobject alignment.

The coproduct of prealigned spaces $(\mathbf{X}_i,\mathsf{K}_i)$ must, by Corollary 3.4.1, be the disjoint union of the sets $\mathbf{X}_i$ with a suitable prealignment. As the injection from each space $(\mathbf{X}_i,\mathsf{K}_i)$ must be monotone, no set in $\coprod_I \mathbf{X}_i$ can be convex whose restriction to any $\mathbf{X}_i$ is not convex. The finest such prealignment on $\dot{\cup}_I \mathbf{X}_i$ is of the form

$$ \mathsf{K} = \{\dot{\cup}_I \mathbf{K}_i \mid \mathbf{K}_i \in \mathsf{K}_i\}. $$

If the coproduct prealignment were any coarser, the injections

$$ \{q_i\colon (\mathbf{X}_i,\mathsf{K}_i) \to \textstyle\coprod_I (\mathbf{X}_i,\mathsf{K}_i)\} $$

would not factor through $(\dot{\cup}_I \mathbf{X}_i,\mathsf{K})$; so K is the coproduct prealignment.

Similarly, the product of prealigned spaces $(\mathbf{X}_i,\mathsf{K}_i)$ must be the cartesian product of the sets $\mathbf{X}_i$ with a suitable prealignment. The projection to each factor space must be monotone; so every set of the form $p_i^{-1}(\mathbf{K})$, $\mathbf{K} \in \mathsf{K}_i$, must be convex, as must their intersections. But this requires that every set of the form $\mathsf{X}_I \mathbf{K}_i$, $\mathbf{K}_i \in \mathsf{K}_i$, must be convex; and this must be the product prealignment, as we cannot factor the projections through any finer prealignment on $\mathsf{X}_I \mathbf{K}_i$.

This concludes the proof, as any category with all (co)products and binary (co)equalizers is (co)complete. •

COROLLARY 4.1.1. *Aln is complete and cocomplete.*

PROOF. *Aln* is a coreflective subcategory of *Prealn*, and thus the inclusion reflects all colimits. Furthermore (see, e.g., [5], p. 280), a coreflective subcategory of a complete category is complete, and the limit of any diagram in the subcategory is the coreflection of the limit in the larger category. Thus equalisers in *Aln* are the same as *Prealn* equalisers, while

$$ \textstyle\prod_{Aln}(\mathbf{X}_i,\mathsf{K}_i) = Cx\prod_{Prealn}(\mathbf{X}_i,\mathsf{K}_i). $$ •

COROLLARY 4.1.2. *DcPrealn and DcAln are complete and cocomplete.*

PROOF. This follows the same pattern as the proof of the previous corollary; colimits and equalisers are reflected, while the product in *Dc(Pre)Aln* is the downclosure of the product in *(Pre)Aln*. •

Note that the product in *Prealn* is the *box product*, whose convex sets are cartesian products of convex sets in the factor spaces, while the other products are coreflections of the box product. The *Aln* product only differs from the *Prealn* product when there are infinitely many nontrivial factors. We may write the various products as follows:

Category	infix product	prefix product
Prealn	$(\mathbf{X},\mathcal{K}) \square (\mathbf{Y},\mathcal{L})$	$\square_{I} (\mathbf{X}_i,\mathcal{K}_i)$
Aln	$(\mathbf{X},\mathcal{K}) \hat{\square} (\mathbf{Y},\mathcal{L})$	$\square_{I} (\mathbf{X}_i,\mathcal{K}_i)$
DcPrealn	$(\mathbf{X},\mathcal{K}) \boxdot (\mathbf{Y},\mathcal{L})$	$\boxdot_{I} (\mathbf{X}_i,\mathcal{K}_i)$
DcAln	$(\mathbf{X},\mathcal{K}) \hat{\boxdot} (\mathbf{Y},\mathcal{L})$	$\hat{\boxdot}_{I} (\mathbf{X}_i,\mathcal{K}_i)$

As an immediate consequence of these constructions, we have (in any of the four categories considered above):

OBSERVATION 4.1.3. *(Co)limits of convex spaces are convex; (co)limits of S_1 spaces are* S_1; and products and colimits of connected spaces are connected. •

In the case of (pre)convexity spaces, the situation is not quite so straightforward. It will be seen that *Cxy* and *Precxy* are neither complete nor cocomplete; the following theorem summarises their general completeness properties.

THEOREM 4.2. *Cxy and Precxy have all coproducts and all equalisers, and these are reflected by the inclusion.*

PROOF. Given a preconvexity space $(\mathbf{X},\mathcal{K})$ and a set monomorphism f: $\mathbf{Z} \rightarrowtail \mathbf{X}$, f induces a subobject preconvexity

$$_{*}f\mathcal{K} = \{\mathbf{A} \mid f\mathbf{A} \in \mathcal{K}\}$$

on $\mathbf{Z}$. The equaliser of a set of maps $\{f_i\colon (\mathbf{X},\mathcal{K}) \rightrightarrows (\mathbf{Y},\mathcal{L})\}$ is the set

$$\{x \mid x \in \mathbf{X},\ f_i(x) = f_j(x)\}$$

with the subobject preconvexity. If $\mathbf{K}$ is a convexity on $\mathbf{X}$, and $\{\mathbf{A}_\gamma\}$ is a directed family of elements of $_f\!\mathbf{K}$, then $\bigcup f\mathbf{A}_\gamma \in \mathbf{K}$, and $\bigcup\mathbf{A}_\gamma$ is in $_f\!\mathbf{K}$; thus if the domain of a set of Darboux maps is a convexity space, so is their equaliser.

By Corollary 3.4.1, if a set of spaces $(\mathbf{X}_i,\mathbf{K}_i)$ has a coproduct in *Precxy*, it will be the disjoint union of the sets with the suitable preconvexity. In order that the injections q_i: $(\mathbf{X}_i,\mathbf{K}_i) \to (X,\mathbf{K})$ may be Darboux, it is necessary that every set $\mathbf{A}$ which is convex in some $(\mathbf{X}_i,\mathbf{K}_i)$ must be convex in the coproduct preconvexity. As before, it is not possible to factor these projections through any finer preconvexity; so $(\bigcup\mathbf{X}_i,\bigcup\mathbf{K}_i)$ is the coproduct of spaces in *Precxy*. As the ranges of different injections are disjoint, and every convex set lies in one such range, any directed family of convex sets in $\coprod(\mathbf{X}_i,\mathbf{K}_i)$ is the image of a directed family of convex sets in some summand space. If that summand space is a convexity space, then the union of the directed family will also be convex, as will its image in the coproduct space. •

A consequence of the coproduct construction is the following:

COROLLARY 4.2.1. *Any preconvexity space has a unique representation as a coproduct of connected preconvexity spaces; and if it is downclosed, S_1, or a convexity space, the connected summands will share these properties.*

PROOF. A *component* is a maximal connected subset of a preconvexity space (with the subspace preconvexity). For any point x, if x is contained in no convex set, $(\{x\},\emptyset)$ is a component; otherwise, the union of all connected sets containing x is a component. Thus, every point is in at least one component. But it cannot be in more than one component, as given y in C_1 connected to x by a chain of intersecting convex sets $\{\mathbf{A}_i\}$, and z in C_2 connected to x by a chain of intersecting convex sets $\{\mathbf{B}_j\}$, the chain $\mathbf{A}_1,\mathbf{A}_2,\ldots,\mathbf{A}_n,\mathbf{B}_m,\ldots,\mathbf{B}_2,\mathbf{B}_1$ is a chain of intersecting convex sets connecting y to z. Thus $C_1 \cup C_2$ is connected, contradicting our assumption of maximality.

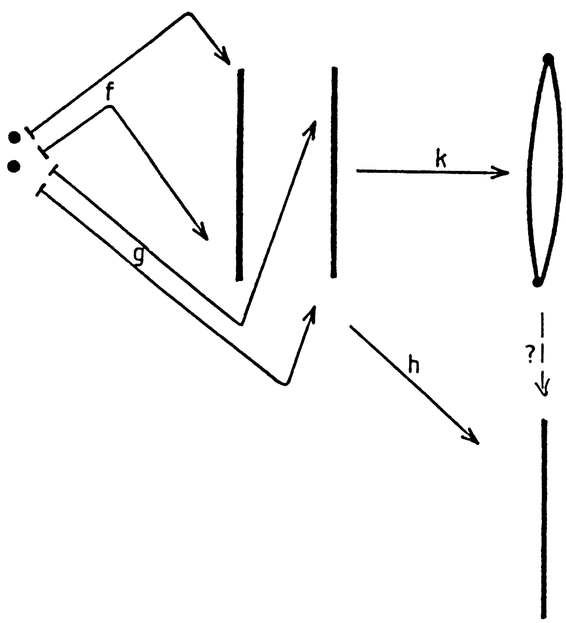
Thus, the underlying set $\mathbf{X}$ of a preconvexity space $(\mathbf{X},\mathbf{K})$ is the disjoint union of the underlying sets of the components of $(\mathbf{X},\mathbf{K})$. But any convex set is connected, hence contained in a component; so $(\mathbf{X},\mathbf{K})$ is the coproduct of its components. Finally, if a space is downclosed, S_1, or a convexity space, so are its subspaces. •

We may contrast this with the situation in *Prealn*, in which coproducts of connected spaces are connected (Observation 4.1.3); or in *Top*, where some, but not all, spaces (consider the rationals with the usual topology) can be decomposed into a coproduct of connected spaces.

EXAMPLE 4.3. *Precxy* does not have coequalisers. For let **X** be {0,1} with the discrete convexity, and let **Y** be {0,1}×[0,1] with the disjoint union convexity on the two copies of [0,1]. Then

$$f\colon 0 \mapsto (0,0),\ 1 \mapsto (0,1) \quad \text{and} \quad g\colon 0 \mapsto (1,0),\ 1 \mapsto (1,1)$$

are both Darboux from **X** to **Y**. If there were a coequaliser for these two maps, it would have to consist of the two unit intervals with their corresponding endpoints identified (Figure 6); but then the two endpoints would, together, be a convex set in the coequaliser preconvexity. However, the map $h\colon (i,j) \mapsto j$, which maps **Y** onto [0,1], also satisfies $h\circ f = h\circ g$; but it cannot be factored through the set quotient map k in *Precxy*, as {0,1} is not a convex set in [0,1]. •



It is not difficult to see where the problem arises. In constructing coequalisers, or other quotients, two previously disjoint convex sets may have intersecting images, and their intersection may have to be included as a new convex set. This addition, however, may make the induced preconvexity too strong to factor other Darboux functions through. In the case of downclosed spaces, however, the subsets of a convex set are all already convex, so this problem does not arise.

THEOREM 4.4. *DcPrecxy has coequalisers and is cocomplete.*

PROOF. Given a downclosed preconvexity space $(\mathbf{X},\mathcal{K})$ and a set epimorphism $f: \mathbf{X} \twoheadrightarrow Z$, f induces a downclosed quotient preconvexity $f_*\mathcal{K} = \{f\mathbf{A} \mid \mathbf{A} \in \mathcal{K}\}$ on Z. The coequaliser construction follows as before. •

The situation with regard to products is similar. In general, attempts to construct a product fail due to the necessity for convex sets that do not arise from the projection maps, but from closing up under intersection. The next theorem shows that when we can prevent this, there is a product.

THEOREM 4.5. *Two preconvexity spaces $(\mathbf{X},\mathcal{K})$, $(\mathbf{Y},\mathcal{L})$ have a product in Precxy if one of the following conditions is satisfied:*

a) $\mathbf{X}$ or $\mathbf{Y}$ has only singleton convex sets.

b) $\mathbf{X}$ and $\mathbf{Y}$ are downclosed.

PROOF. If (a) or (b) is satisfied, the *DcPrealn* product $(\mathbf{X},\mathcal{K}) \boxminus (\mathbf{Y},\mathcal{L})$ is also a product in *DcPrecxy*. Suppose, on the other hand, that (b) is not satisfied. Then there exists $\mathbf{A}$ convex in (WLOG) $\mathbf{X}$, with $\mathbf{B} \subset \mathbf{A}$ non-convex. If (a) is not satisfied either, there exists a non-singleton convex set $\mathbf{C} \subset \mathbf{Y}$. Select $b \in \mathbf{B}$, c_1, $c_2 \in \mathbf{C}$, and let Z be $\mathbf{A} \cup \mathbf{C}$ with the singleton convexity. Define f, g_1 and g_2 as follows:

$$f: Z \to \mathbf{X},\ z \mapsto z \text{ for } z \in \mathbf{A}, \qquad \text{otherwise } z \mapsto b\ ;$$
$$g_1: Z \to \mathbf{Y},\ z \mapsto c_1 \text{ for } z \in \mathbf{A}, \qquad \text{otherwise } z \mapsto z\ ;$$
$$g_2: Z \to \mathbf{Y},\ z \mapsto c_1 \text{ for } z \in \mathbf{B},\ z \mapsto c_2 \text{ for } z \in \mathbf{A}\backslash\mathbf{B}, \text{ else } z \mapsto z\ ;$$

All of these are Darboux; so if there is a product preconvexity on $\mathbf{X}\times\mathbf{Y}$, $(f,g_1): Z \to \mathbf{X}\times\mathbf{Y}$ must be Darboux, and $\mathbf{A}\times c_1 \cup b\times\mathbf{C}$ must be convex (see Figure 7). Similarly, (f,g_2) must be Darboux, and so the image of Z under that map, $\mathbf{B}\times c_1 \cup (\mathbf{A}\backslash\mathbf{B})\times c_2 \cup b\times\mathbf{C}$, must be convex (Figure 8).

Figure 7:

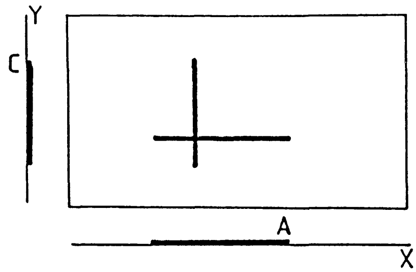
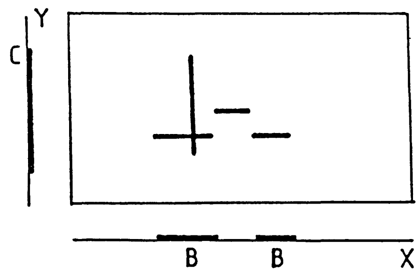


Figure 8:



PROPOSITION 4.6. *DcPrecxy, DcCxy, DcPrecxy*$_1$ *and DcCxy*$_1$ *are complete and cocomplete.*

PROOF. It is easily verified that the *DcPrealn* product $\boxdot$ is also a product in *DcPrecxy*; and the *DcAln* product $\hat{\boxdot}$ is also a product in *DcCxy*. As these products preserve the S_1 property, they are also the products in the categories *DcPrecxy*$_1$ and *DcCxy*$_1$. •

PROPOSITION 4.7. *If J is a diagram in Dc(Pre)cxy whose morphisms are monotone as well as Darboux, its limit in Dc(Pre)cxy is the same as its limit in Dc(Pre)aln.*

PROOF. It only remains to show that, for a downclosed preconvexity space $(\mathbf{X},\mathcal{K})$ and a set monomorphism $f\colon Z \rightarrowtail \mathbf{X}$, the subobject preconvexity and the subobject prealignment agree, as this implies that equalisers agree: and if equalisers and products agree, all limits do. For any space $(\mathbf{X},\mathcal{K})$, we have

$$_*f(\mathcal{K}) = \{\mathbf{A} \mid f\mathbf{A} \in \mathcal{K}\} \subset \{f'\mathbf{B} \mid \mathbf{B} \in \mathcal{K}\} = {}^*f(\mathcal{K}).$$

However, if $(\mathbf{X},\mathcal{K})$ is downclosed,

$$(\mathbf{A} \in {}^*f(\mathbb{K})) \Rightarrow (\mathbf{A} = f^{-1}\mathrm{B},\ \mathrm{B} \in \mathbb{K}) \Rightarrow (f\mathbf{A} = f(f^{-1}\mathrm{B}) \subset \mathrm{B} \in \mathbb{K})$$
$$\Rightarrow (f\mathbf{A} \in \mathbb{K}) \Rightarrow (\mathbf{A} \in {}_*f(\mathbb{K})). \qquad \bullet$$

5. IND-FINITENESS.

It is often of interest to extend a category which does not have some sort of limit by giving it formal limits of that type. It is frequently found that the extended category shares some nice properties of the original. For instance, if we adjoin cofiltered limits to the category of finite groups, we obtain the category of *profinite* groups. As cofiltered limits and filtered colimits are particularly suited to the preservation of properties related to finiteness, profinite groups retain some of the good behaviour of finite objects.

The next theorem describes the ind-completion (closure under filtered colimits) of the category of finite downclosed preconvexity spaces. (For a clear introduction to ind- and pro-completions, the reader is referred to [7], Ch. VI, §1, whence Definition 5.2 has been adapted.)

DEFINITION 5.1. For a concrete category *C*, let C_f be the full subcategory of finite objects in *C* (that is, objects with finite underlying sets).

DEFINITION 5.2. For a small category *C*, *Ind-C* is a category whose objects are all small filtered diagrams in *C*. If $D: J \to C$ and $E: K \to C$ are two such diagrams, a morphism $f: D \to E$ is a family

$$(f_j : j \in \mathrm{ob}(J)),$$

where each f_j is an equivalence class of morphisms from $D(j)$ to objects in the image of *E*, satisfying the compatibility condition that if $g: j \to j'$ is a morphism of *J*, and if $h: D(j') \to E(k) \in f_{j'}$, then $h \circ D(g) \in f_j$ (Figure 9).

The equivalence relation defining the classes f_j makes two morphisms

$$g: D(j) \to E(k) \quad \text{and} \quad g': D(j) \to E(k')$$

equivalent iff there exist $h: k \to k''$ and $h': k' \to k''$ in *K* such that $E(h)\circ g = E(h')\circ g'$ (Figure 10).

Figure 9:

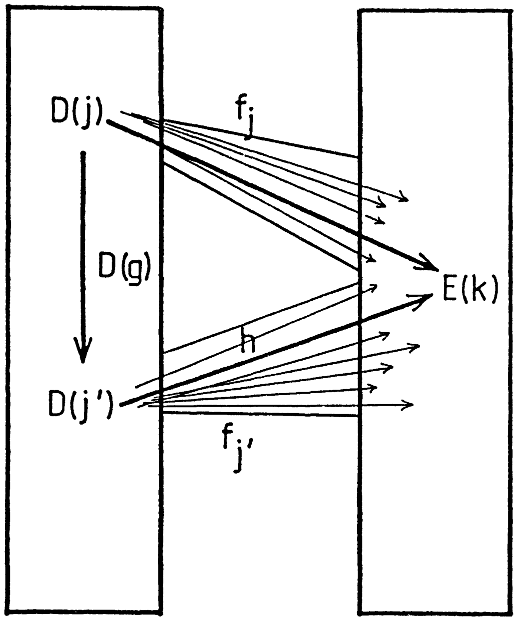
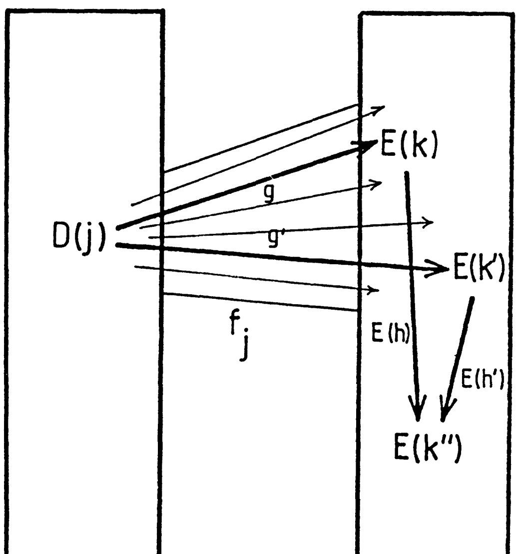


Figure 10



THEOREM 5.3. *Ind-DcPrecxy$_f$* ≃ *FinDcPrecxy*.

PROOF. *DcPrecxy* is cocomplete, so *K: D* ↦ colim *(D)* is an isomorphism from *Ind-DcPrecxy$_f$* to a subcategory of *DcPrecxy*. It remains to identify the objects of this subcategory and to show that it is full. Colimits in *DcPrecxy* are preserved by the forgetful functor to *Set*, and have preconvexities of the form

$$\{q_j A_{jk} \mid A_{jk} \in \mathbf{K}_j,\ q_j : X_j \to X \text{ is the cocone map}\}.$$

Thus any colimit of finite spaces has only finite convex sets, and

$$K(\textit{Ind-DcPrecxy}_f) \subset \textit{FinDcPrecxy}.$$

Conversely, any object of *FinDcPrecxy* is the colimit of its finite subspaces, filtered by inclusion. Thus,

$$\mathrm{ob}(K(\textit{Ind-DcPrecxy}_f)) = \mathrm{ob}(\textit{FinDcPrecxy}).$$

Furthermore, the restrictions of a Darboux map between two downclosed preconvexity spaces to their finite subspaces obey the compatibility condition of Definition 5.2; so any morphism of *FinDcPrecxy* corresponds to a morphism of *Ind-DcPrecxy$_f$*. •

COROLLARY 5.3.1. *Ind-DcPrecxy$_f$* ≃ *DcCxy*.

PROOF. Apply Theorem 3.10 to the preceding result. •

We can define a convexity on a finite simplicial complex in which the simplexes are convex sets; in this convexity, any simplicial map is Darboux. Furthermore, any S_1 finite downclosed preconvexity space can be so generated. (If it is not S_1, there are points which are not contained in any convex set.) If we let *SC* be the category of simplicial complexes and simplicial maps, it follows that

$$SC_f \simeq (DcPrecxy_1)_f.$$

As colimits of S_1 spaces are themselves S_1, we can easily deduce:

PROPOSITION 5.4. *Ind-SC$_f$* ≃ DcCxy$_1$. •

This suggests that downclosed convexity spaces may be regarded as generalised simplicial complexes. This may be used as a basis for a homology theory of preconvexity spaces (see [3], now being prepared for more general publication).

In the case of aligned spaces, the corresponding result is even more straightforward; the interested reader should have no difficulty proving:

PROPOSITION 5.5. *Ind-Prealn*$_f$ $\simeq$ *Aln.*

6. CONCLUSION.

We have seen that several categorical properties of convex aligned spaces (as introduced by Jamison-Waldner) can be extended to the larger category of prealigned spaces. We have also developed, in parallel, the theory of categories of preconvexity spaces. This latter category is not complete or cocomplete; it has, however, a largest full subcategory which is complete (and cocomplete), namely the category of downclosed preconvexity spaces.

Finally, we have shown that the category of downclosed convexity spaces is equivalent to the inductive completion of the familiar category of finite simplicial complexes. This provides yet another way to view the relationship between convexity spaces and their finite subsets, which has been used in [3] to develop a very simple axiomatisation of the homology theory of preconvexity spaces.

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