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ON THE NERVE OF AN n-CATEGORY

by Michael JOHNSON and R.F.C. WALTERS

RÉSUMÉ. Street a défini [5] le nerf d'une n-catégorie en obtenant une description de la n-catégorie libre sur le n-simplex. Le présent article introduit les notions de domaine et codomaine d'un complexe simplicial orienté et les utilise pour caractériser les complexes "composables". Le principal résultat est que la collection des sous-complexes composables du n-simplex forme une n-catégorie - la n-catégorie libre sur le n-simplex. Les calculs dans cette n-catégorie sont particulièrement simples car les compositions sont données par la réunion des complexes simpliciaux.

The nerve of a category C is the simplicial set [2] whose n-dimensional elements are composable n-tuples of arrows of C [4]. Equivalently, the n-dimensional component of the nerve is the set of functors from the ordered set $[n] = \{0,1,2,\ldots,n\}$ (considered as a category with a morphism from i to j if and only if $i \leq j$) into the category C. Geometrically, an element of the nerve of dimension n is an n-simplex whose (1-dimensional) edges are arrows of C, and whose faces are all commutative.

John E. Roberts is credited [5] with the generalization of this notion to r-categories. The nerve of an r-category C should be the simplicial set whose n-dimensional elements are n-simplices with m-cells of C in each face of dimension m. Making this notion precise has proved to be difficult. Roberts [3] was able to describe explicitly the nerve of a 3-category but was unable to do so for $r > 3$.

Ross Street realized as early as 1980 that an explicit description of the nerve construction depends upon an understanding of "the free n-category on the n-simplex". Recently [5] Street has succeeded in defining this n-category which he calls the n-th oriental and writes as O_n.

This paper reexamines the construction of the free n-category on the n-simplex, and obtains a new description of Street's orientals.

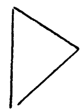
Our approach is to consider the elements of the combinatorial n-simplex of dimension m as m-cells with a chosen orientation. In Section 2 we show how to calculate the domain and codomain of an arbitrary subcomplex of the n-simplex, and in Section 3 we use this to characterize the *well-formed subcomplexes* of the n-simplex - those which can be obtained from individual elements by legitimate n-categorical compositions. Section 4 is devoted to demonstrating that the compositions involved are free in the sense that composing two well-formed subcomplexes can never create a loop of any dimension. Finally, Section 5 shows that the well-formed subcomplexes of the n-simplex do indeed form an n-category.

This paper was presented at the Bangor Conference on Homotopical Algebra at Bangor, Wales, in July 1985. We would like to thank Ross Street for providing a preprint of his paper [5].

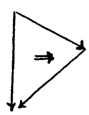
1. PRELIMINARIES.

PROBLEM: To construct an n-category on the n-simplex.

EXAMPLE 1. Consider a 2-simplex



We wish to construct a 2-category which has as 0-cells the 0-dimensional edges (vertices) of the simplex, as non-identity 1-cells the 1-dimensional edges, and a single non-identity 2-cell corresponding to the 2-dimensional interior of the simplex. The 0-cells are thus fully determined. For each 1-dimensional edge we choose, for the moment, an arbitrary orientation for the corresponding 1-cell. Similarly, we choose an orientation of the 2-cell to obtain



(Here the 2-cell has as domain the vertical 1-cell and as a codomain the composite of the other two 1-cells). The 2-category generated by this diagram by inserting identity 1- and 2-cells and all possible compositions (the only non-trivial composition is the codomain of the 2-cell) will be called O_2.

We begin with some definitions.

DEFINITIONS. For the set of integers $\{0,1,\ldots,k\}$ write $[k]$; for $\{0,1,\ldots\}$ write ω. If A is any set, and n a natural number, write $\left[{A \atop n}\right]$ for the set of all n-element subsets of A. Let Y be the standard ω-simplex with vertices the natural numbers (thus the set of n-dimensional elements of Y, Y_n, is precisely $\left[{\omega \atop n+1}\right]$). Recall that for each natural number n, greater than 0, Y is equipped with n+1 face maps $\delta_0,\ldots,\delta_n$: $Y_n \to Y_{n-1}$ defined by

$$\delta_i(\{y_0,y_1,\ldots,y_n\}) = \{y_0,y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_n\}$$

and that Y is a graded set (i.e., a sequence of sets indexed by the natural numbers).

DEFINITION. If $X = (X_i)_{i\in\omega}$ is a sub-graded set of Y (i.e. $X_i \subset Y_i$ for all $i \in \omega$) which is finite dimensional (there exists a least integer n, called the *dimension* of X, such that $X_m = \emptyset$ for all $m > n$) and finite (each X_i is finite) we call X a *simplicial structure*.

DEFINITION. A simplicial structure which is closed under the application of the δ_i (i.e., for $n > 0$, $x \in X_n$, $i \in [n]$ implies $\delta_i x \in X_{n-1}$) will be called a *simplicial complex*.

In what follows, a simplicial structure will always be assumed empty at unspecified dimensions.

Notice that in Example 1 we had to choose orientations for all the elements of dimension greater than 0. We specify now, once and for all, our choice of orientation.

DEFINITIONS. If X is a simplicial structure, and $x \in X$ is of dimension n greater than 0, x will be *oriented* away from its

$$\text{odd faces} = \{\ \delta_i x \mid i \in [n],\ i \text{ odd}\},$$

and towards its

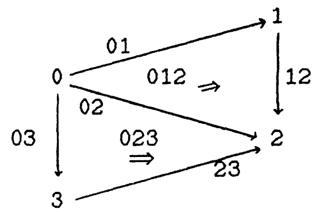
$$\text{even faces} = \{\delta_i x \mid i \in [n],\ i \text{ even}\}.$$

The element x will be said to *begin* at its odd faces and *end* at its even faces. If x, x' are n-dimensional elements and an n-dimensional element y is an even face of x and an odd face of x' then y will be said to be *between* x *and* x'.

EXAMPLE 2. Consider the simplicial complex

$$\{\{0,2,3\},\{0,1,2\}\}, \quad \{\{0,1\},\{1,2\},\{2,3\},\{0,2\},\{0,3\}\}, \quad \{\{0\},\{1\},\{2\},\{3\}\}$$

which may be represented (omitting braces and indicating orientation) as



Notice that {0,3} begins at its odd image {0}, and ends at its even image {3}; {0,2,3} begins at {0,3} and ends at {0,2} and {2,3}; and that {0,2} is between {0,2,3} and {0,1,2}.

To construct an n-category explicitly we must determine all of its cells. One aspect of this is easy: the k-cells must include all the k-dimensional elements of the simplex. Thus in Example 1, the three 1-dimensional faces of the simplex all occur as 1-cells. However, in general many other k-cells will occur because all the permissible n-category compositions must be considered. Furthermore, because the n-category compositions allow the 'pasting' of j- and k-cells with $j \neq k$, a general k-cell will have structure at several dimensions. This suggests that a k-cell might best be considered as a simplicial complex built from simplices of various dimensions, each of which is generated by an edge of the n-simplex. Our problem then becomes: To determine which of the sub-simplicial complexes of an n-simplex can be obtained by legitimate n-categorical compositions of subsimplices of the n-simplex.

A partial answer is again easy: a k-dimensional complex which has k-dimensional elements with contradictory orientations, e.g.

$$\{\{0,2\},\{1,2\}\} \qquad \qquad \text{i.e.,} \quad 0 \xrightarrow{02} 2 \xleftarrow{12} 1$$
$$\{\{0\},\{1\},\{2\}\}$$

cannot be the result of an n-category composition.

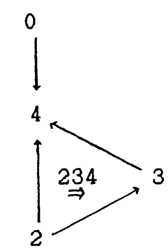
DEFINITION. A simplicial complex of dimension k is called *compatible* if

$$x,y \in X_k, \quad \delta_i x = \delta_j y, \ i \equiv j \ (\text{mod } 2) \quad \text{implies} \quad x = y$$

(i.e., there are no two distinct k-dimensional elements which begin or end in the same k-1-dimensional element). By convention a compatible 0-dimensional complex is a singleton (Street [5]).

Notice that compatibility of a complex depends only upon the nature of its highest dimensional non-empty component. That this is appropriate can be seen by considering Example 1 as a well-formed 2-cell and noticing the arrangement of 1-cells. However, this means that an incompatibility may be hidden in a lower dimension of a compatible complex.

EXAMPLE 3.



Notice that Example 3 is a compatible simplicial complex but cannot be obtained by legitimate 2-category compositions.

The difference between Examples 1 and 3 is that in Example 1 the incompatibility is between elements of the domain and codomain of the 2-cell, while in Example 3 the incompatibility occurs within the domain of the 2-cell. To formalize this we need to be able to calculate the domain of an arbitrary simplicial complex.

2. DOMAIN AND CODOMAIN.

NOTATION. Let x be a k-dimensional element of a simplicial complex Y and let A be any subset of $[k] = \{0,1,\dots,k\}$, say $A = \{a_1,a_2,\dots,a_i\}$. Write $R_A(x)$ (remove A) for the k-i-dimensional element of Y obtained by deleting the a_1-st, a_2-nd, a_i-th vertices of x.

If the elements of A are all even (odd) integers write $E_A(x)$ ($B_A(x)$) for $R_A(x)$.

If X is a set of k-dimensional elements of a simplicial complex Y write $R_A(X)$ for the set $\cup_{x\in X} R_A(x)$. Similarly $E_A(X)$, $B_A(X)$.

If i is an integer less than or equal to k write

$$R_i(X) = \cup_{A\subset[k], |A|=k-i} R_A(X)$$

and similarly $E_i(X)$, $B_i(X)$. Thus $R_i(X)$ is the set of all i-dimensional elements of Y which can be obtained by deleting vertices from elements of X, while $E_i(X)$ is the set of such elements obtainable by deleting only even positioned vertices from elements of X.

Write $R(X)$ ($E(X)$, $B(X)$) for the simplicial structure $(R_i(X))_{i\in[k]}$ ($(E_i(X))_{i\in[k]}$, $(B_i(X))_{i\in[k]}$), and if Y is a simplicial structure of dimension k write $R(Y)$ ($E(Y)$, $B(Y)$) for $R(Y_k)$ ($E(Y_k)$, $B(Y_k)$).

Notice that $R(X)$ is the sub-simplicial complex of Y obtained by taking X and all the elements underlying it (the image of X by all the δ_i, the image of that by all the δ_i, etc.). However, $E(X)$ and $B(X)$ are not in general simplicial complexes. In fact, if X is k-dimensional, $E(X)$ is empty at all dimensions less than $[(k-1)/2]$ and $B(X)$ is empty at all dimensions less than $[k/2]$ (one cannot remove more than about half the vertices if one only removes evenly positioned ones!). The utility of $E(X)$ and $B(X)$ is that they are precisely the ends and beginnings respectively of elements of X in the following sense.

Consider an element x of dimension n. By definition of orientation x ends at its even images $E_{n-1}(x)$. What other elements might we say are at the end of x ? All those elements of lower dimension which are between elements of the end are themselves at the end and there seems to be no reason to include any others.

PROPOSITION 1. *Let w be an n-dimensional element of some simplicial complex and suppose $k < n-1$, then $x \in E_k(w)$ iff x is between two elements y,z of $E_{k+1}(w)$.*

PROOF. Suppose $x \in E_k(w)$, say $x = E_{\{a_1,a_2,\dots,a_{n-k}\}}(w)$, then let

$$y = E_{\{a_2,a_3,\dots,a_{n-k}\}}(w) \quad \text{and} \quad z = E_{\{a_1,a_3,a_4,\dots,a_{n-k}\}}(w).$$

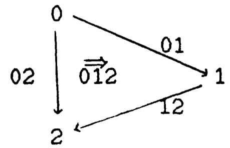
Conversely, suppose $y,z \in E_{+1}(w)$ are given and suppose $x \in E_k(y)$, $x \in B_k(z)$. Then $y \neq z$ so

$$y = E_{\{a_1,a_2,\dots,\hat{a}_i,\dots,a_{n-k}\}}(w), \quad z = E_{\{a_1,a_2,\dots,\hat{a}_j,\dots,a_{n-k}\}}(w)$$

for some $a_1,a_2,\ldots,a_{n-k}$, and $x = E_{(a_1,a_2,\ldots,a_{n-k})}(w)$.

Thus elements at the end of x are precisely those in $E_k(x)$, $k = 1,2,\ldots$. For this reason we refer to E(X) as the *end* of X. Similarly B(X) will be called the *beginning* of X.

EXAMPLE 4. Let $x = \{0,1,2\}$. R(X) may be represented by



Applying the definition

$$B(x) = \{\{0,1,2\}\},\quad \{\{0,2\}\},$$
$$E(x) = \{\{012\}\},\quad \{\{0,1\},\{1,2\}\},\quad \{\{1\}\}\ .$$

Notice that for X of dimension n and cardinality greater than 1, B(X) is just the union of the beginnings of x for $x \in X_n$, and not necessarily the beginning in any global sense of the 'composite' orientation of X.

DEFINITION. For a simplicial structure Y of dimension n define the *domain* of Y, dom(Y), to be Y-E(Y) (the global beginning of the n-dimensional orientation is that which remains when all the elements which are at the end of any n-dimensional element are removed) and the *codomain* of Y, cod(Y), to be Y-B(Y).

THEOREM 2. *Let* X *be a simplicial complex, then*

$$\mathrm{dom\ dom}(X) = \mathrm{dom\ cod}(X)\ .$$

The proof depends upon two lemmas.

LEMMA 3. *If a, b are k-dimensional and there exists a sequence of k-dimensional elements $a = a_0,a_1,a_2,\ldots,a_n = b$ such that for $j = 0,1,\ldots,n-1$ there is some k-1-dimensional face between a_j and a_{j+1}, then*

$$B_{k-1}(a)\cap E_{k-1}(b) = \emptyset.$$

PROOF. See Corollary 10.

COROLLARY 4. *If* X *is a simplicial complex of dimension k then* dom(X) *is a simplicial structure (though not necessarily a complex) of dimension k-1.*

PROOF. $$(\mathrm{dom}(X))_k = X_k - E_k(X_k) = X_k - X_k = \emptyset$$

so dom(X) is at most k-1-dimensional.

To see that dom(X) is k-1-dimensional choose some $a_0 \in X_k$, and some $y_0 \in B_{k-1}(a_0)$. If $y_0 \notin \mathrm{dom}(X)$ it can only be because $y_0 \in E(a_1)$ for some $a_1 \in X_k$. Now choose any $y_1 \in B_{k-1}(a_1)$, and repeat. Notice that by Lemma 3, $a_j \neq a_i$ for all $i < j$, so since X_k is finite we must eventually locate a $y_n \in (\mathrm{dom}(X))_{k-1}$.

LEMMA 5. *Let* X *be a simplicial complex of dimension k* and suppose $x \in \mathrm{dom}(X)$ *and* $x \in E(z)$ *for some* $z \in X_{k-1}$, *then there exists a* $v \in (\mathrm{dom}(X))_{k-1}$ *such that* $x \in E(v)$.

PROOF. If $z \in \mathrm{dom}(X)$ we are done, so suppose $z = E_{(j)}(w)$ for some $w \in X_k$, $j \in [k]$. Suppose $x = E_{(a_1, a_2, \ldots, a_n)}(z)$. Now $j \leqslant a_n$ (since if $j > a_n$, $x = E_{(a_1, a_2, \ldots, a_n, j)}(w) \notin \mathrm{dom}(X)$), so suppose $j \leqslant a_h$ but $j > a_{h-1}$, where for simplicity we set $a_0 = -1$. Then

$$x = E_{(a_1, a_2, \ldots, a_{h-1}, j, a_{h+1}, \ldots, a_n)}(B_{(a_{h+1})}(w)).$$

Similarly, if $z_1 = B_{(a_{h+1})}(w) \in \mathrm{dom}(X)$ we are done, otherwise $z_1 = E_{(j_1)}(w_1)$ for some $w_1 \in X_k$, $j_1 \in [k]$, etc.

This process must terminate since, by Lemma 3, it cannot cycle (w and w_1 have a k-1-dimensional face, $B_{(a_{h+1})}(w)$, between them) and simplicial complexes are finite.

DUALITY. Notice that, exactly as in ordinary category theory, we can obtain dual results by reversing the chosen orientation of all elements of any given dimension, say k. We call a result so obtained the dual_k of the original result. Thus Lemma 5 which involves cells of two different dimensions, k and k-1, yields four lemmas: Dualizing the k-th dimension changes both occurences of 'dom(X)' to 'cod(X)' while dual_{k-1} changes both references to 'E( )' to 'B( )', and these changes are independent. Similary the dual_{k-1} of Theorem 2 says

$$\mathrm{cod}\ \mathrm{dom}(X) = \mathrm{cod}\ \mathrm{cod}(X),$$

and the dual of Corollary 4 claims that cod(X) is a k-1-dimensional simplicial structure.

PROOF OF THEOREM 2. Required to show:

$$\text{dom dom}(X) = \text{dom cod}(X),$$

$$\text{dom dom}(X) = (X-E(X)) - E(X-E(X)) = X - (E(X)\cup E(X-E(X)))$$

and

$$\text{dom cod}(X) = (X-B(X)) - E(X-B(X)) = X - (B(X)\cup E(X-B(X)))$$

so it suffices to show that

$$E(X)\cap E(X-E(X)) = B(X)\cap E(X-B(X)).$$

Suppose X is k-dimensional. Notice that both sides are empty for dimensions greater than k and both equal X_k at the k-th dimension.
$\subset$: Suppose $x \in E(X)$, x of dimension k-1 or less, say

$$x = E_{(a_1,a_2,\ldots,a_j)}(w) \quad \text{for some} \quad w \in X_k.$$

If $x \in B(X)$ then $x \in$ RHS, so suppose $x \notin B(X)$. Then

$$x \in \text{cod}(X) \text{ and } x \in E_{(a_j)}(w),$$

hence by dual$_k$ of Lemma 5, $x \in E(X-B(X))$.

Suppose $x \in E(X-E(X))$, say

$$x = E_{(a_1,a_2,\ldots,a_{j-1})}(y) \quad \text{for some} \quad y \in X_{k-1}.$$

If $x \in B(X)$ then $x \in$ RHS, so suppose $x \in$ cod(X). Since $x \in E(y)$ and y is k-1-dimensional we can apply the dual$_k$ of Lemma 5 to obtain that $x \in E(X-B(X))$.
$\supset$: The converse is precisely the dual$_k$ of $\subset$.

3. THE CONSTRUCTION OF O_n.

Now that we have defined domain and codomain, and shown that they satisfy the basic relations dom dom = dom cod, we can formalize our desire for compatibility within domains and codomains at all dimensions.

DEFINITION. Call a simplicial complex *well-formed* if X, dom(X), cod(X), dom dom(X), cod dom(X), dom cod(X),... are all compatible simplicial complexes.

Notice that in view of Theorem 2 the binary tree of height $n+1$ in this definition is in fact a list of $n+1$ pairs

$$(X,X),\quad (\mathrm{dom}(X),\ \mathrm{cod}(X)),\ \ldots,\quad (\mathrm{dom}^n(X),\ \mathrm{cod}^n(X)).$$

DEFINITION. If n is a natural number, the set O_n (to be given an n-category structure below) is the set of well-formed sub-simplicial complexes of the standard n-simplex (considered as a simplicial complex Y whose k-dimensional component is $Y_k = \binom{[n]}{k+1}$).

We will show that O_n is non-trivial; in particular, for any $z \subset [n]$, $R(z) \in O_n$. First some

NOTATION. Let z be an n-dimensional element. If $A = \{a_1, a_2, \ldots, a_m\}$ is a set with $a_i < a_j$ and a_i even whenever i is even, odd whenever i is odd (i.e., arranged in increasing order elements of A are alternately odd and even, beginning with an odd), write $A_A{}^1(z)$ for $R_A(z)$. If elements of A are alternately even or odd, beginning with an even, write $A_A{}^0(z)$ for $R_A(z)$ (mnemonic: alternating removal, beginning with an element of parity 0). As before write $A_j{}'(z)$ for the set of all j-dimensional elements obtainable from z by removing elements of alternating parity (beginning with an odd) and

$$A_j{}'(X) = \cup_{x \in X} A_j{}'(x), \quad \text{etc.}$$

Our aim is to express $\mathrm{dom}^j(R(z))$ in terms of $A_{n-j}{}'(z)$. We begin by characterizing the ends of $A_{n-j}{}'(z)$.

LEMMA 6. *Suppose given* A, *alternately odd and even, beginning with an odd, say* $A = \{a_1, a_2, \ldots, a_j\}$ *and, for ease of exposition, let* $a_0 = -1$, *then* $x \in E(A_A{}^1(z))$ *iff there exists a set* B, *with the property that for any* $b \in$ B:

(a) *if* $a_{i-1} < b < a_i$ *then* $b \not\equiv a_i \bmod 2$,

(b) *if* $a_i \leqslant a_j < b$ *for all* i *then* $b \equiv a_j \bmod 2$,

such that $x = R_{A \cup B}(z)$ *(i.e., x has the form*

$$R_{(\underbrace{b,b,\ldots,b}_{\text{even}},a_1,\underbrace{b,b,\ldots,b}_{\text{odd}},a_2,\underbrace{b,\ldots,b}_{\text{even}},a_3,\ldots\ \ldots a_j,\underbrace{b,b,\ldots,b})}(z)\ ).$$

PROOF. (⇐) Such an x is an end of $A_A'(z)$ since $a_{i-1} < b < a_i$, $b \not\equiv a_i \bmod 2$ implies that the element of z in position b will be in an even position in $A_A'(z)$. Similarly for $b > a_j$.

(⇐) If $R_{A\cup B}(z) = x$ and B does not satisfy the property, then there exists some $b \in$ B with $a_{i-1} < b < a_i$, $b \neq a_i \bmod 2$, or with $b > a_j$ but $b \not\equiv a_j \bmod 2$. Now the element of z in position b must be removed from $A_A'(z)$ to get x but it will be in odd position so that $x \notin E(A_A'(z))$.

LEMMA 7. *Suppose z is an n-dimensional element, then*

$$\mathrm{dom}^j(R(z)) = R(A_{n-j}'(z)).$$

PROOF. By induction on j. True for $j = 0,1$.

Suppose true for j, then

$$\mathrm{dom}^{j+1}(R(z)) = \mathrm{dom}(R(A_{n-j}'(z))),$$

so it suffices to show

$$\mathrm{dom}(R(A_{n-j}'(z)) = R(A_{n-(j+1)}'(z)).$$

$\supset$: Suppose

$$x \in R(A_{n-j}'(z)) \text{ but } x \notin \mathrm{dom}(R(A_{n-j}'(z))),$$

then $x \in E(R(A_{n-j}'(z)))$ so $x \in E(A_A'(z))$ for some A, so there exists a B as in Lemma 6 with $x = R_{A\cup B}(z)$, but then $x \in R(A_{n-(j+1)}'(z))$ since there can be no $j+1$ element alternating (beginning odd) set of vertices to be deleted.

$\subset$: Suppose $x \in \mathrm{dom}(R(A_{n-j}'(z))$, then $x \in R(A_A'(z))$ for some A = $\{a_1,a_2,\ldots,a_j\}$ say, but $x \notin E(A_{n-j}'(z))$. As before write x as $R_{A\cup B}(z)$. Now since $x \notin E(A_A'(z))$, B does not satisfy the property given in Lemma 6, so either there exists $b \in$ B, $b > a_j$ with $b \not\equiv a_j \bmod 2$, whence

$$x \in R(A_{\{a_1,a_2,\ldots,a_j,b\}}'(z)) \subset R(A_{n-(j+1)}'(z)),$$

or there exists some $b \in$ B such that for some $i \in [j]$, $a_{i-1} < b < a_i$, (allowing again $a_0 = -1$) and $b \equiv a_i \bmod 2$. Furthermore, we can choose such a b so that there exists b' with $b < b' < a_i$ and $b' \not\equiv a_i \bmod 2$ (if not, then writing c_i for the least element of A∪B such that $a_{i-1} < c_i \leqslant a_i$ and $c_i \equiv a_i \bmod 2$ we see $x \in E(A_{\{c_1,c_2,\ldots,c_j\}}'(z))$ contrary to assumption) and then

$$x \in R(A_{\{a_1,\ldots,a_{i-1},b,b',a_i,a_{i+1},\ldots,a_{j-1}\}}'(z)) \subset R(A_{n-(j+1)}'(z)).$$

THEOREM 8. *Suppose n is a natural number and $z \subset [n]$ then $R(z) \in O_n$.*

PROOF. Trivially $R(z)$ is a compatible simplicial complex.

Furthermore, for all j, $\mathrm{dom}^j(R(z))$ is a simplicial complex (since, by Lemma 7, it is $R(A_{n-j}{}^1(z))$, and is compatible since if $x,y \in A_{n-j}{}^1(z)$, $x \neq y$ then $\delta_i x = \delta_h y$ implies

$$x = A_{(a_1, a_2, \ldots, a_k, a_{k+1}, a_{k+2}, \ldots, a_j)}{}^1(z)$$

and

$$y = A_{(a_1, a_2, \ldots, a_k, b_{k+1}, a_{k+2}, \ldots, a_j)}{}^1(z),$$

hence

$$a_{k+1} \equiv k+1 \equiv b_{k+1} \pmod 2,$$

and so $i \not\equiv h \bmod 2$ (since if, without loss of generality, $a_{k+1} > b_{k+1}$, then

$i \equiv k+1-k \equiv 1 \pmod 2$, and $h \equiv k+1 - (k+1) \equiv 0 \pmod 2$)).

Similarly, $\mathrm{cod}^j(R(z)) = R(A_{n-j}{}^0(z)))$ is a compatible simplicial complex.

4. THE NON-EXISTENCE OF CYCLES.

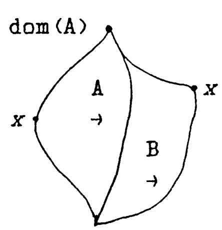
We aim to exhibit O_n with an n-category structure which uses dom and cod to build source and target maps. Such a structure must satisfy

$$\mathrm{dom}(B*A) = \mathrm{dom}(A) \qquad (\dagger).$$

It appears that composition should be set theoretic union whence

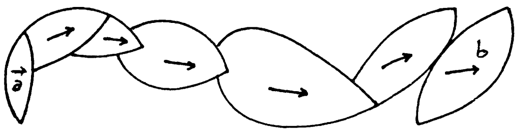
$$\mathrm{dom}(B*A) = \mathrm{dom}(B\cup A) = B\cup A - E(B\cup A) = B\cup A - (E(B)\cup E(A)).$$

Thus we will not obtain (†) if it happens that $\mathrm{dom}(A)\cap E(B) \neq \emptyset$, which may be thought of as a cyclical behaviour as shown in the figure where x is a supposed element of the intersection



In this section we will prove that such cycles do not occur in O_n and in Section 5 we will exhibit O_n as an n-category.

NOTATION. Suppose $k > 0$. If a,b are k-dimensional elements of some simplicial complex, write $a \lhd b$ when there exists a finite sequence $a = a_0, a_1, \dots, a_n = b$ of k-dimensional elements such that $0 \leqslant v < n$ implies that there exist i even, j odd in $[k]$ with $\delta_i a_v = \delta_j a_{v+1}$ (i.e., a sequence of k-dimensional elements from a to b in which successive elements have a common k-1-dimensional face which occurs as an end of the first and a beginning of the second. Pictorially



LEMMA 9. *If $a \lhd b$ then $B(a) \cap E(b) = \emptyset$.*

PROOF. By induction on the dimension of a, b, say k.

True for $k = 1$ since $a < b$ implies that there exist

$$a_0 = a = \{a_0^0, a_0^1\},\ a_1 = \{a_1^0, a_1^1\},\ a_2 = \{a_2^0, a_2^1\}, \dots a_n = b = \{a_n^0, a_n^1\}$$

with $a_i^0 < a_i^1$ and $a_i^1 = a_{i+1}^0$, hence a_n^1, the only end of b is greater than a_0^0, the only beginning of a.

Now suppose a, b k-dimensional and $a \lhd b$, then we have

$$a_0 = a = \{a_0^0, a_0^1, \dots, a_0^k\},\ a_1 = \{a_1^0, a_1^1, \dots, a_1^k\}, \dots,\ a_n = b = \{a_n^0, a_n^1, \dots, a_n^k\}$$

and a_{i+1} is obtainable from a_i by deleting an even positioned element and inserting an odd positioned one.

Now if

$$a_0^k = a_1^k = \dots = a_n^k$$

then writing $a_i - \{a_i^k\} = a'_i$ we obtain the k-1-dimensional sequence $a'_0, a'_1, a'_2, \dots, a'_n$ showing that $a'_0 \lhd a'_n$ and use induction (since $B(a_0) \cap E(a_n) \neq \emptyset$ implies $B(a'_0) \cap E(a'_n) \neq \emptyset$).

If, on the other hand, $a_0^k \neq a_j^k$ for some j, then if k is odd, $a_0^k < a_j^k \leqslant a_n^k$ and so $a_n^k \notin a_0$ but for any $x \in E(a_n)$, $a_n^k \in x$, therefore $x \notin B(a_0)$ hence $B(a) \cap E(b) = \emptyset$. Similarly, if k is even, $a_0^k > a_j^k \geqslant a_n^k$ and so $a_0^k \notin a_n$ but for any $x \in B(a_0)$, $a_0^k \in x$, therefore $x \notin E(a_n)$ hence

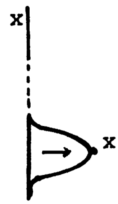
$$B(a_0) \cap E(a_n) = B(a) \cap E(b) = \emptyset.$$

COROLLARY 10. *If a, b are k-dimensional, a ◁ b, then*

$$B_{k-1}(a) \cap E_{k-1}(b) = \emptyset.$$

PROOF. Immediate from Lemma 9.

Unfortunately far more subtle cycles such as



might occur in O_n. To show that this cannot happen we will establish for O_n the following two properties.

Let A be a k-1-dimensional well-formed simplicial complex, let x be a k-dimensional element with $\mathrm{dom}(R(x)) \subset A$ and write $X = R(x)$.

P1: $E(X) \cap A = \emptyset$.

P2: If $y \in A$ and $B(X) \cap R(y) \neq \emptyset$, then $y \in B(X)$.

First we need to develop some tools to use in our inductive proof of P1 and P2.

THEOREM 11. A, X *as above, then* P1, P2 *imply that* $A \cup X$ *is a well-formed simplicial complex.*

PROOF. $A \cup X$ is trivially compatible and is a simplicial complex since A and X are. Furthermore

$$\mathrm{dom}(A \cup X) = A \cup X - E(X) = (A - E(X)) \cup (X - E(X)) = A \cup \mathrm{dom}(X) = A$$

(using P1) and so it and all $\mathrm{dom}^n(A \cup X)$, $\mathrm{cod}^n(A \cup X)$, $n \geqslant 2$ (using dual_{k-1} of Theorem 2) are compatible simplicial complexes, because A is well-formed.

It remains only to consider

$$\mathrm{cod}(A \cup X) = (A \cup X) - B(A \cup X) = A - B(X) \cup \mathrm{cod}(X)$$

which is a simplicial complex since cod(X) is (Theorem 8) and, by P2, $A - B(X)$ is. Finally $\mathrm{cod}(A \cup X)$ is compatible since suppose not, then there exists $z \neq w \in (\mathrm{cod}(A \cup X))_{k-1}$ (Corollary 4) and $i \equiv j \pmod 2$ such that $\delta_i z = \delta_j w$. Now, z, w are not both in $A - B(X)$, since if it is k-1-dimensional then it must be compatible being a subcomplex of a compatible k-1-dimensional complex (A), nor in cod(X) since it is compatible (Theorem 8). Hence without loss of generality, suppose $w \in A - B(X)$, $z \in \mathrm{cod}(X)$. Now $\delta_i z \in \mathrm{dom}(X)$ (since $\delta_i z = \delta_j w \in A$, so $\delta_i z \notin E(X)$ by P1), so by Lemma 5 there exists $v \in (\mathrm{dom}(X))_{k-1} \subset A$, $h \equiv i \pmod 2$ such that $\delta_h v = \delta_i z = \delta_j w$, contradicting the compatibility of A since $v \in B_{k-1}(X)$, $w \in A - B(X)$.

COROLLARY 12. A, X *as above,* P1, P2; *then*

$$\mathrm{cod}(A \cup X) = (A - B(X)) \cup \mathrm{cod}(X) = A \cup E(X) - B(X)$$

is well-formed.

PROOF. Immediate.

LEMMA 13 (Decomposability). *Suppose* Q *is well-formed k-dimensional and* Y *is a set of k-dimensional elements of* Q *such that, if* $y \in Y$, $w \lhd y$ *then* $w \in Y$, *then there exists an enumeration* $y_0, y_1, \ldots, y_n$ *of the elements of* Y *such that*

$$B_{k-1}(y_i) \subset \mathrm{dom}(Q) \cup E(\{y_j \mid j < i\}) - B(\{y_j \mid j < i\}), \quad i = 0,1,\ldots,n.$$

PROOF. Firstly, there exists a suitable y_0 since, choose a $y \in Y$, if $B_{k-1}(y) \not\subset \mathrm{dom}(Q)$ it can only be because there is some $y' \lhd y$ with $E_{k-1}(y') \cap B_{k-1}(y) \neq \emptyset$. Repeat to obtain $y'' \lhd y' \lhd y$, etc. This process must terminate yielding $y^{(r)} = y_0$ because Q_k is finite and, by Corollary 10, $\lhd$ is antisymmetric (so the procedure cannot cycle).

Similarly there is $y_1 \in Y - \{y_0\}$ such that

$$B_{k-1}(y_1) \subset \mathrm{dom}(Q) \cup E(y_0), \quad \text{etc.}$$

Finally notice that because of the compatibility of Y (inherited from Q), $B_{k-1}(y_i) \cap B_{k-1}(y_j) = \emptyset$, $y_i \neq y_j$ hence if

$$B_{k-1}(y_i) \subset \mathrm{dom}(Q) \cup E(\{y_j \mid j < i\})$$

then

$$B_{k-1}(y_i) \subset \mathrm{dom}(Q) \cup E(\{y_j \mid j < i\}) - B(\{y_j \mid j < i\}).$$

NOTATION. If Q is well-formed k-dimensional and $y \in Q$ write

$$\lhd\, y = \{ q \in Q \mid q \lhd y,\ q \neq y\}.$$

As usual if $Y \subset Q$ write $\lhd\, Y = \cup_{y \in Y} \lhd\, y$.

COROLLARY 14. *Suppose* Q *is well-formed k-dimensional,* $w \neq z \in Q_k$, *and, for any k-1-dimensional well-formed* $A \subset Q$, $x \in Q_k$ *with* $\mathrm{dom}(R(x)) = \mathrm{dom}(X) \subset A$, *properties P1 and P2 hold, then* $E(w) \cap E(z) = \emptyset$.

PROOF. Let $Y = \lhd\,(\{w,z\})$ and suppose there is an $a \in E(w) \cap E(z)$. By the decomposability lemma there is an enumeration $y_0, y_1, \ldots, y_n$ of elements of Y such that

$$B_{k-1}(y_i) \subset \mathrm{dom}(Q) \cup E(\{y_j \mid j < i\}) - B(\{y_j \mid j < i\}).$$

Since, by Theorem 8,

$$\mathrm{dom}(y_0) = R(B_{k-1}(y_0)) \subset R(\mathrm{dom}(Q)) = \mathrm{dom}(Q),$$

Corollary 12 applies (with $A = \mathrm{dom}(Q)$) and $\mathrm{dom}(Q) \cup E(\{y_0\}) - B(\{y_0\})$ is well-formed. Proceeding inductively, $A = \mathrm{dom}(Q) \cup E(Y) - B(Y)$ is well-formed. Furthermore either $B_{k-1}(w)$ and $B_{k-1}(z) \subset A$ and hence $\mathrm{dom}(w) \subset A$, so using Corollary 12 again $A' = A \cup E(w) - B(w)$ is well-formed, or w or $z \in Y$ (but not both). Suppose, without loss of generality, $w \in Y$ then let $A' = A$. Now in either case $a \in A'$ since $a \in E(w)$ and for any $y \in Y \cup \{w\}$, $a \notin B(y)$ (by Lemma 9), and $B_{k-1}(z) \subset A'$ hence $\mathrm{dom}(R(z)) \subset A'$ but $a \in E(z)$ contradicting P1.

COROLLARY 15. *Suppose* Q *is well-formed k-dimensional,* $y \in Q_k$, $\mathrm{dom}(R(y)) = \mathrm{dom}(Y) \subset \mathrm{dom}(Q)$ *and, for any k-1-dimensional well-formed* $A \subset Q$, $x \in Q_k$ *with* $\mathrm{dom}(R(x)) \subset A$, *properties P1 and P2 hold, then* $Q - B(Y)$ *is well-formed.*

PROOF. If $Q-B(Y)$ is k-1-dimensional then $Q-B(Y) = \mathrm{cod}(Q)$ which is well-formed, so suppose $Q-B(Y)$ is k-dimensional. Then $Q-B(Y)$ is trivially compatible since Q is, and it is a simplicial complex since suppose not, then there exists $a \in B(Y) \cap R(z)$ for some $z \in Q_k - \{y\}$. Furthermore $a \notin E(z)$ since $a \in B(Y) \subset \mathrm{dom}(Q)$, therefore $a \in R(w)$ for some $w \in B_{k-1}(z)$ and $w \in \mathrm{dom}(Q)$ or $E(z_2)$ etc. to obtain $w \in \mathrm{dom}(Q)$ with $a \in R(w)$, $w \in B_{k-1}(z_n)$ but then by P2, $w \in B_{k-1}(Y)$ and $z_n \neq y$ because $a \notin R(E_{k-1}(y))$ but $a \in R(E_{k-1}(z_n))$ contradicting Q well-formed.

Furthermore,

$$\mathrm{cod}(Q-B(Y)) = Q-B(Y)-B(Q_k-y) = Q - (B(y)\cup B(Q_k-y)) = Q-B(Q) = \mathrm{cod}(Q)$$

is well-formed and so $\mathrm{dom}^n(Q-B(Y))$, $\mathrm{cod}^n(Q-B(Y))$ are compatible simplicial complexes for $n \geqslant 2$.

It remains only to show that $\mathrm{dom}(Q-B(Y))$ is a compatible simplicial complex. Now,

$$\begin{aligned}\mathrm{dom}(Q-B(Y)) &= Q-B(Y)-E(Q_k-y) = Q-E(Q_k-y)-B(Y) = Q-E(Q_k)\cup E(Y)-B(Y)\\ &\quad (\text{since } E(X)\cap E(Y) = \emptyset,\ X, Y \in Q_k,\ X \neq Y \text{ by Corollary 14})\\ &= \mathrm{dom}(Q)\cup E(Y)-B(Y) = \mathrm{cod}(\mathrm{dom}(Q)\cup Y)\end{aligned}$$

which is a compatible simplicial complex by Theorem 11.

COROLLARY 16 *(Paring down). Suppose* Q *is well-formed k-dimensional,* $Y \subset Q_k$ *satisfies* $y \in Y$, $w \lhd y$ *implies* $w \in Y$, *and, for any k-1-dimensional well-formed* $A \subset Q$, $x \in Q_k$ *with* $\mathrm{dom}(R(x)) \subset A$, *properties P1 and P2 hold, then* $Q-B(Y)$ *is well-formed.*

PROOF. The decomposability lemma provides a sequence in which the elements of Y can be removed from Q and Corollary 15 shows that at each step what remains of Q is well-formed.

NOTATION. Suppose $k > 0$. If Q is a k-dimensional complex and $x,y \in Q_k$ write $x \lhd_Q y$ if there exist $z_0 = x$, z_1, $z_2,\ldots$, $z_n = y$ such that

$$z_i \in Q_k \quad \text{and} \quad E_{k-1}(z_i)\cap B_{k-1}(z_{i+1}) \neq \emptyset, \quad i = 0,1,\ldots n-1.$$

Let

$$\lhd_Q y = \{x \in Q_k \mid x \lhd_Q y,\ x \neq y\} \quad \text{and} \quad \lhd_Q Y = \cup_{y\in Y} \lhd_Q y.$$

EXAMPLE 5. Suppose that P1, P2 hold, that Q is well-formed k-dimensional and $y \in Q_k$, then $Q - B(\lhd_Q y)$ is well-formed.

We are now in a position to prove P1 and P2 for all A, X. The proof is by induction and uses paring down of k-1-dimensional complexes for which P1 and P2 have been established to prove P1 and P2 for k-dimensional complexes.

THEOREM 17. *Let k be any natural number greater than zero and suppose* A *is k-1-dimensional well-formed, x is a k-dimensional element,* $X = R(x)$ and $\mathrm{dom}(X) \subset A$ *then*

P1: $$E(X)\cap A = \emptyset \qquad \textit{and}$$
P2: $$\textit{If } y \in A \textit{ and } B(X)\cap R(y) \neq \emptyset \textit{ then } y \in B(x).$$

PROOF. By induction over k.

If $k = 1$ then any well-formed k-1-dimensional complex is a singleton, say $\{\{a\}\}$, and any k-dimensional element x is a pair $\{x_0,x_1\}$. To say $\mathrm{dom}(R(x)) \subset A$ is to say $a = x_0$, but $E(x) = x_1$, $B(x) = x_0$, hence $E(x)\cap A = \emptyset$ and $B(x)\cap R(y) \neq \emptyset$ implies $y = a = x_0 \in B(x)$.

Suppose true for k-1, i.e., for all well-formed k-2-dimensional complexes A, for all k-1-dimensional elements x such that $\mathrm{dom}(R(x)) \subset A$, P1 and P2 hold.

(P2) Suppose P2 is false for some k-1-dimensional well-formed A, and x a k-dimensional element with $\mathrm{dom}(R(x)) \subset A$, then there exists $y \in A$, $y \notin B(x)$, $a \in B(x)\cap R(y)$, say $a = B_{\{a_0,a_1,\dots,a_r\}}(x)$. Now there are $w,z \in B_{k-1}(x)$ with $a \in B(w)$, $a \in E(z)$ (choose $w = B_{\{a_r\}}(x)$, $z = B_{\{a\}}(x)$) and $y \notin R(x)$ (since $a \in B(x)$, $a \in R(y)$, $y \in R(x)$ imply $y \in B(x)$ contrary to assumption). In fact, we may choose y to be k-1-dimensional since $a \notin \mathrm{dom}(A)$ (because $a \in E(z)$, $z \in B(x) \subset A$) and $\mathrm{dom}(A)$ is well-formed hence a simplicial complex so $y \notin \mathrm{dom}(A)$, therefore there is some $y' \in A$, k-1-dimensional with $y \in E(y')$ and therefore with $a \in R(y')$ and $y' \notin R(x)$ (since $y \in R(x)$), hence $y' \notin B(x)$.

So suppose y, z and w are all k-1-dimensional. Now $y \lhd_A w$ since if not then $y \in A - B(\lhd_A w \cup w) = A'$ say, which is well-formed by k-1-dimensional paring down, but $a \in R(y)$, $a \notin A'$, contradiction. Similarly $z \lhd_A y$ (using the dual_{k-1} of paring down for k-1-dimensional complexes). However $z \lhd_A y \lhd_A w$, $z,w \in B(x)$, $y \notin B(x)$ contradicts the following lemma.

LEMMA 18. *Suppose A is a k-1-dimensional well-formed simplicial complex and x is a k-dimensional element such that* $\mathrm{dom}(R(x)) \subset A$. *Suppose that $z = B_{\{a_0\}}(x)$, $w = B_{\{a_r\}}(x)$, $a_0 < a_r$, then*

(i) $z \lhd_A w$.

(ii) If v is k-1-dimensional and $z \lhd_A v \lhd_A w$ then $v \in B_{k-1}(x)$.

PROOF. (i) $z \lhd_A w$ since $E_{\{a_{r-1}\}}(z) = B_{\{a_r\}}(w)$.

(ii) Suppose

$$z = z_0 \lhd_A z_1 \lhd_A \dots \lhd_A z_n = w \quad \text{with} \quad E_{k-2}(z_i)\cap B_{k-2}(z_{i+1}) \neq \emptyset.$$

We show by induction that $z_i \in B_{k-1}(x)$.

True for $z_0 = B_{\{a_0\}}(x)$.

Suppose $z_i \in B(x)$, say $z_i = B_{(a)}(x)$ then for some b, q, $z_{i+1} = E_{(b)}(B_{(a)}(x)) \cup \{q\}$ where q is odd positioned. Now $a_0 \leqslant a \leqslant a_r$ since $z \lhd z_i < w$ and $b > a$ since all elements in positions less than a occur in w, so if one were removed it would need to be replaced which could only be done by removing/inserting an even lower positioned element, which in turn must be inserted/removed, etc. But then an odd image of z_{i+1} is

$$E_{(b)}(B_{(a)}(x)) = B_{(a)}(B_{(b+1)}(x))$$

which is an odd image of $B_{(b+1)}(x)$, contradicting the well-formedness of A unless $z_{i+1} = B_{(b+1)}(x)$.

PROOF OF THEOREM continued.
(P1) Suppose A is k-1-dimensional well-formed, x a k-dimensional element such that $\text{dom}(R(x)) \subset$ A, and $a \in E(x) \cap$ A. The proof will follow from three lemmas.

LEMMA 19. A, X, *a as above, then* $a \in \text{dom}(A)$.

PROOF. Suppose $a \in \text{dom}(A)$, then the unique end of x of minimal dimension, $a' \in \text{dom}(A)$ and $a' \in E(w)$ where $w = E_{(j)}(x)$, $j = k$ for k even, k-1 for k odd. Now $B_{k-2}(w) \subset$ A (since $B_{k-2}(w) \cap E(x) = \emptyset$), hence writing

$$Y = \{y \in A_{k-1} \mid E_{k-2}(y) \cap B_{k-2}(w) \neq \emptyset\},$$

$A - B(\lhd_A Y \cup Y) = A'$ say is well-formed, and $B_{k-2}(w) \subset \text{dom}(A')$.

But dom(A') is k-2-dimensional well-formed, w is k-1-dimensional, $a' \in \text{dom}(A')$ (since

$$\text{dom}(A') \supset A - B(Y) - E(A) = \text{dom}(A) - B(Y)$$

and by Lemma 9, $a' \notin B(Y)$ since $y \in Y_{k-1}$ implies $y \lhd w$ and $a' \in E(w)$), and $a' \in E(w)$ contradicting the inductive hypothesis.

LEMMA 20. A, X, *a as above, then a is not k-2-dimensional.*

PROOF. Suppose a is k-2-dimensional and note that, by Lemma 19, $a \notin \text{dom}(A)$, and by dual_{k-1} of Lemma 19, $a \notin \text{cod}(A)$.

Now A is k-1-dimensional. If $j < k$-1 and $a, b \in A_j$, write $a \lhd_A b$ when there exist $\alpha, \beta \in A_{k-1}$, $\alpha \lhd_A \beta$ with $a \in B(\alpha)$, $b \in E(\beta)$. Let

$$Y = \lhd_A \{y \in A_{k-1} \mid a \in E(y)\} \cup \{y \in A_{k-1} \mid a \in E(y)\}$$

and let $w = E_{(k)}(x)$ if k is even, and $E_{(k-1)}(x)$ if k is odd. Notice that A' = A-B(Y) is, by k-1-dimensional paring down, well-formed k-1-dimensional and $a \in$ dom(A'). Hence if $B_{k-2}(w) \subset$ A' we may obtain a contradiction exactly as in the proof of Lemma 19. So suppose there exists $b \in B_{k-2}(w)$ with $b \triangleleft_A a$ (i.e., $b \in B$(Y)). Similarly, for $z = E_{(0)}(x)$, there exists $c \in E_{(k-2)}(z)$ with $a \triangleleft_A c$ (otherwise we could use the dual$_{k-1}$ of the proof of Lemma 19 to obtain a contradiction to the dual$_{k-1}$ of P1 which is assumed true). But $b \triangleleft_A a$ implies that there exists $\beta, \alpha \in A_{k-1}$, $\beta \triangleleft_A \alpha$ with $b \in B_{k-2}(\beta)$, $a \in E_{(k-2)}(\alpha)$, $a \triangleleft_A c$ implies that there exists $\alpha', \gamma \in A_{k-1}$, $\alpha' \triangleleft_A \gamma$ with $a \in B_{k-2}(\alpha')$, $c \in E_{(k-2)}(\gamma)$. Notice $\beta \triangleleft_A \alpha \triangleleft_A \alpha' \triangleleft_A \gamma$ and, using the compatibility of A, $\beta, \gamma \in B_{k-1}(x)$ hence by Lemma 18, $\alpha, \alpha' \in B_{k-1}(x)$ too. But then by Proposition 1, $a \in B_{k-2}(x)$ which contradicts $a \in E_{(k-2)}(x)$.

LEMMA 21. A, X *as above; then* A' = (A∪X)−B(X) *is well-formed k-1-dimensional.*

PROOF. A' = (A−B(X))∪(X−B(X)) is a union of simplicial complexes (by Theorem 8 and k-dimensional P2 which has already been proved) and hence is a simplicial complex.

A' is compatible since suppose

$$u, v \in A'_{k-1} \quad \text{with} \quad \delta_i u = \delta_j v, \quad i \equiv j \pmod 2;$$

then u, v are not both in A−B(X) nor X−B(X) = cod(X) by the compatibility of those complexes so without loss of generality suppose $u \in$ A−B(X), $v \in$ cod(X). If $\delta_j v \in$ dom(X) proceed as in the proof of Theorem 11. If $\delta_j v \notin$ dom(X) then $\delta_j v \in E_{(k-2)}$(X) but

$$\delta_j v = \delta_i u \in A-B(X) \subset A,$$

contradicting Lemma 20.

Furthermore, dom(A) = dom(A') since:

dom(A) ⊂ dom(A'): A' ⊃ A−B(X) ⊃ dom(A) (elements of B(X) are k-1-dimensional or ends of k-1-dimensional elements), E(A') ⊂ E(A)∪E(X) (using Lemma 5), and

$$E(A) \cap \mathrm{dom}(A) = E(X) \cap \mathrm{dom}(A) = \emptyset$$

(using Lemma 19).

dom(A') ⊂ dom(A): A ⊃ dom(A') (elements of E(X) are k-1-dimensional or ends of k-1-dimensional elements), E(A) ⊂ E(A')∪B(X) (using dual$_k$ of Lemma 5), and

$$E(A')\cap \mathrm{dom}(A') = \emptyset = B(X)\cap \mathrm{dom}(A')$$

since if $x \in B(X)$, $x \notin A'$. Dually$_{k-1}$ cod(A') = cod(A), so all lower dimensional domains and codomains are compatible simplicial complexes.

PROOF OF THEOREM (continued). Now finally the minimal dimensional end of X, $a' \in$ AUX $- B(X) = A'$ and $a' \notin$ dom(A), so if P1 is assumed false for X, A, then $a' \in E(w')$ for some $w' \in A$. But we have seen that $a' \in E(w)$ for $w = E_{(j)}(x)$ ($j = k$ or $k-1$). Now $w \neq w'$ since $E_{k-1}(x)\cap A = \emptyset$ (because if not, $E_{k-2}(x)\cap A \neq \emptyset$ contradicting Lemma 20). But $w' \in A'$ (since if $w' \in B(X)$, then $a' \notin E(X)$ as $a' \in R(w')$), contradicting Corollary 14.

5. O_n AS AN n-CATEGORY.

DEFINITIONS (Street [5]). A *category* $(A,s,t,*)$ consists of a set A, functions $s,t\colon A \to A$ satisfying the equations

$$ss = ts = s, \quad tt = st = t$$

and a function

$$*\colon \{(A,B) \in A\times A \mid s(A) = t(B)\} \to A$$

satisfying the equations

$$s(A*B) = s(B), \quad t(A*B) = t(A)$$

and the axioms:

(right identity) $s(A) = t(V) = V$ implies $A*V = A$,

(left identity) $U = s(U) = t(A)$ implies $U*A = A$,

(associativity) $s(A) = t(B)$, $s(B) = t(C)$ imply $A*(B*C) = (A*B)*C$.

A *2-category* $(A,s_0,t_0,*_0,s_1,t_1,*_1)$ consists of two categories $(A,s_0,t_0,*_0)$, $(A,s_1,t_1,*_1)$ satisfying the conditions

(i) $s_1s_0 = s_0 = s_0s_1 = s_0t_1, \quad t_1t_0 = t_0 = t_0t_1 = t_0s_1.$

(ii) $s_0(A) = t_0(A')$ implies
$s_1(A*_0 A') = s_1(A)*_0\, s_1(A')$ and $t_1(A*_0 A') = t_1(A)*_0\, t_1(A')$.

(iii) $s_1(A) = t_1(B)$, $s_1(A') = t_1(B')$, $s_0(A) = t_0(A')$ imply
$$(A*_1B)*_0(A'*_1B') = (A*_0 A')*_1(B*_0 B').$$

An *n-category* $(A,(s_i,t_i,*_i)_{i\in[n]})$ consists of categories $(A,s_i,t_i,*_i)$ for each $i \in [n]$ such that $(A,s_j,t_j,*_j,s_i,t_i,*_i)$ is a 2-category for all $j < i$ and s_n,t_n: $A \to A$ is the identity map.

DEFINITION. Suppose $n > 0$. For $i = 0,1,\dots,n$, define s_i,t_i: $O_n \to O_n$ by: let X be a k-dimensional element of O_n, then

$$s_i(X) = t_i(X) = X \quad \text{if} \quad i \geq k,$$
$$s_i(X) = \mathrm{dom}^{k-i}(X) \quad \text{if} \quad i < k,$$
$$t_i(X) = \mathrm{cod}^{k-i}(X) \quad \text{if} \quad i < k.$$

Notice $s_i(X)$, $t_i(X)$ are i-dimensional for $i < k$, k-dimensional for $i \geq k$.

LEMMA 22. *Let i be a natural number. Suppose* A, B *are well-formed simplicial complexes such that $s_i(B) = t_i(A)$, then*

(i) $A\cap B = s_i(B) = t_i(A)$.

(ii) $s_i(A\cup B) = s_i(A)$, $t_i(A\cup B) = t_i(B)$.

(iii) $s_j(A\cup B) = s_j(A)\cup s_j(B)$, $t_j(A)\cup t_j(B)$, $j > i$.

(iv) $A\cup B$ *is a well-formed simplicial complex.*

PROOF. By induction on the dimension of $A\cup B$.

If $A\cup B$ is of dimension less than or equal to i, then $s_i(B) = t_i(A)$ implies A = B, so (i), (ii), (iii) and (iv) follow.

Suppose $A\cup B$ is of dimension i+1.

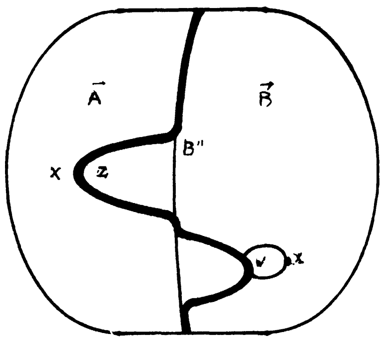
(i) Suppose $x \in A\cap B$ but $x \notin s_i(B) = t_i(A)$. Then A, B are both i+1-dimensional (since if either were not, then $s_i(B) = B$ or $t_i(A) = A$, whence $x \notin s_i(B) = t_i(A)$ implies $x \notin A\cap B$), so

$$s_i(B) = \mathrm{dom}(B), \quad t_i(A) = \mathrm{cod}(A),$$

and $x \notin s_i(B)$ implies $x \in E(w)$ for some $w \in B_{i+1}$. Now dom(B) is well-formed, so
$$B' = \mathrm{dom}(B) \cup E(\triangleleft_B w) - B(\triangleleft_B w)$$

is well-formed too (by Corollary 12 and Lemma 13). Similarly, $x \notin t_i(A)$ implies $x \in B(z)$ for some $z \in A_{i+1}$ and if

$$z' \triangleright z \text{ and } y \in E(z')\cap\mathrm{cod}(A) = E(z')\cap\mathrm{dom}(B),$$



then $y \in B'$ (by Lemma 9). Thus

$$B'' = B' \cup B(\triangleright_A z \cup z) - E(\triangleright_A z \cup z)$$

is well-formed (duals$_k$ of Corollary 12 and Lemma 13) and $x \in B''$. But dom$(w) \subset B''$ and $x \in E(w) \cap B''$, contradicting Theorem 17. Thus

$$A \cap B = s_i(B) = t_i(A).$$

(ii)
$$s_i(A \cup B) = \mathrm{dom}(A \cup B) = A \cup B - E(A \cup B)$$
$$= (A - E(A_{i+1}) - E(B_{i+1})) \cup (B - E(B_{i+1}) - E(A_{i+1}))$$
$$= (s_i(A) - E(B_{i+1})) \cup (s_i(B) - E(A_{i+1})) = s_i(A)$$

(since

$$s_i(B) - E(A_{i+1}) = t_i(A) - E(A_{i+1}) \subset s_i(A)$$

and by (i), $E(B_{i+1}) \cap A = \emptyset$).

Dually,

$$t_i(A \cup B) = \mathrm{cod}(A \cup B) = t_i(B).$$

(iii) If $j > i$, $A \cup B$ of dimension $i+1$, then

$$s_j(A) = t_j(A) = A, \quad s_j(B) = t_j(B) = B, \quad s_j(A \cup B) = t_j(A \cup B) = A \cup B.$$

(iv) $A \cup B$ is compatible because $x \in E_i(B)$ implies $x \notin$ dom(B) which implies (using (i)) $x \notin A$ and similarly $E_i(A) \cap B = \emptyset$, and is a simplicial complex because A and B are. Finally

$$\mathrm{dom}(A \cup B) = s_i(A) \quad \text{and} \quad \mathrm{cod}(A \cup B) = t_i(B)$$

are well-formed because A and B are.

Now, suppose for all well-formed A, B with AUB of dimension less that h ($h > i+1$) and $s_i(B) = t_i(A)$, (i), (ii), (iii) and (iv) hold, and suppose AUB is of dimension h, $s_i(B) = t_i(A)$, then:

(i) Suppose $x \in A \cap B$ but $x \notin s_i(B) = t_i(A)$. We may suppose x is of dimension less than h since if not, choose any $v \in B_{h-1}(x)$ then $v \in A \cap B$, v is of dimension $h-1$, and $v \notin t_i(A) = s_i(B)$, so v will do. Let

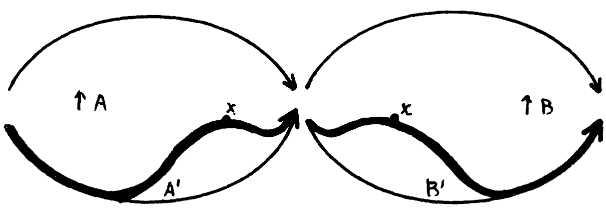
$$P = \{a \in A_h \mid x \in E(a)\}$$

(possibly empty, otherwise a singleton by Corollary 14) and

$$Q = \{b \in B_h \mid x \in E(b)\}.$$

Put

$$A' = s_{h-1}(A) \cup E(\triangleleft_A P \cup P) - B(\triangleleft_A P \cup P),$$
$$B' = s_{h-1}(B) \cup E(\triangleleft_B Q \cup Q) - B(\triangleleft_B Q \cup Q),$$



then A', B' are well-formed and

$$s_i(B') = s_i(B) = t_i(A) = t_i(A')$$

(Corollary 12, Lemma 13 and Theorem 11). But $x \in A' \cap B'$, $x \in s_i(B') = t_i(A')$ and A', B' are of dimension less than h, contradicting (i) in the inductive hypothesis.

(ii)
$$\begin{aligned} s_i(A\cup B) &= s_i(s_{h-1}(A\cup B)) = s_i(\mathrm{dom}(A\cup B)) \\ &= s_i((s_{h-1}(A)-E(B_h)) \cup (s_{h-1}(B)-E(A_h))) \\ &= s_i(s_{h-1}(A)\cup s_{h-1}(B)) && \text{by (i) and Theorem 2} \\ &= s_i(s_{h-1}(A)) && \text{inductive hypothesis (ii)} \\ &= s_i(A). \end{aligned}$$

Similarly

$$t_i(A\cup B) = t_i(B).$$

(iii)
$$s_j(A\cup B) = A\cup B = s_j(A)\cup s_j(B)$$

if $j \geq h$. If $j < h$ then

$$s_j(A\cup B) = s_j(s_{h-1}(A\cup B)) = s_j(\mathrm{dom}(A\cup B))$$

$= s_j(s_{h-1}(A) \cup s_{h-1}(B))$ as above
$= s_j(s_{h-1}(A)) \cup s_j(s_{h-1}(B))$ inductive hypothesis (iii)
$= s_j(A) \cup s_j(B)$;

and similarly

$$t_j(A \cup B) = t_j(A) \cup t_j(B).$$

(iv) $A \cup B$ is a compatible simplicial complex because A and B are and

$$E_{h-1}(A) \cap E_{h-1}(B) = \emptyset = B_{h-1}(A) \cap B_{h-1}(B).$$

Furthermore

$$\mathrm{dom}(A \cup B) = s_{h-1}(A) \cup s_{h-1}(B) \quad \text{and} \quad \mathrm{cod}(A \cup B) = t_{h-1}(A) \cup t_{h-1}(B)$$

are well-formed by inductive hypothesis (iv). Thus $A \cup B$ is well-formed.

THEOREM 23. *$(O_n, (s_i, t_i, \cup)_{i \in [n]})$ is an n-category.*

PROOF. Straightforward verification of the definition using Lemma 22 and Theorem 2.

Thus the O_n are ω-categories. In fact, the collection $(O_n)_{n \in \omega}$ is a co-simplicial object in the category ω-*cat* of ω-categories and ω-functors and so for any ω-category X, ω-*cat*$(O-,X)$ is a simplicial set - the *nerve* of the ω-category X.

Furthermore the left Kan extension [1] of O along the Yoneda embedding yields a construction of ω-categories on simplicial sets which is of importance in its own right. This construction will be taken up elsewhere.

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