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HOMOTOPY OVER B AND UNDER A
by K.A. HARDIE and K.H. KAMPS

RÉSUMÉ. Dans cet article, on décrit une certaine catégorie d'homotopie cohérente $\underline{H}_B{}^A$ d'espaces au-dessus d'un espace B et sous un espace A. Le problème d'isomorphisme et le problème de classification sont résolus. On indique aussi les liens avec les classes de composition secondaires.

0. INTRODUCTION.

An object X of $\underline{H}_B{}^A$ is a diagram

(0.1) $$A \xrightarrow{\ \sigma\ } X \xrightarrow{\ \rho\ } B$$

where X is a space and A and B are fixed spaces. An arrow from X to X' in $\underline{H}_B{}^A$ will be a certain equivalence class of homotopy commutative diagrams of the form

where u is a continuous map and σ_t and ρ_t are homotopies. Our first result (Theorem 1.2) is to the effect that such an arrow is an isomorphism in $\underline{H}_B{}^A$ whenever the map u is (an ordinary) homotopy equivalence. As an immediate corollary we obtain (Corollary 1.8) that allowing σ and ρ to vary up to homotopy does not change the isomorphism class of X.

We shall denote the set of morphisms in $\underline{H}_B{}^A$ from X to X' by $\pi(A/X,X'/B)$. If $u: X \to X'$ is a continuous map, let $\pi_1{}^x(X';u)$ denote the u-based track group. (See [3, 2].) Then $\pi_1{}^x(X';u)$ depends up to isomorphism only on the homotopy class $\{u\}$ of u. In Section 2 we classify the elements of $\pi(A/X,X'/B)$ in the sense that we exhibit a bijection between $\pi(A/X,X'/B)$ and the union over certain classes $\{u\}$ of sets of double cosets in $\pi_1{}^x(X';u)$. The result generalizes classifications obtained in [2] for $\pi(X,X'/B)$ (the case A = ∅) and for $\pi(A/X,X')$ (the case B is a singleton).

In view of Corollary 1.6 the set $\pi(A/X,X'/B)$ is determined by the spaces A,X,X',B and by the homotopy classes $\{\sigma\},\{\sigma'\},\{\rho\},\{\rho'\}$. Thus it is an invariant that is defined whenever we are given two factorizations of a homotopy class. In Section 3 we show, in a special case, that it is related to, and in a sense measures, the set of possible nontrival secondary homotopy compositions of a certain type.

1. THE EQUIVALENCE RELATION.

If $h_t: h_0 \simeq h_1$ and $k_t: k_0 \simeq k_1$ are homotopies from Y to Z with the property that $h_0 = k_0$ and $h_1 = k_1$ then h_t and k_t are *relatively homotopic*, denoted $h_t \equiv k_t$, if there exists a homotopy of homotopies $H_{t,s}: Y \to Z$ such that

$$h_t = H_{t,0}, \quad k_t = H_{t,1}, \quad H_{0,s} = h_0 = k_0 \ (s \in I), \text{ and } H_{1,s} = h_1 = k_1 \ (s \in I).$$

The *track* $\{h_t\}$ of a homotopy h_t is its relative homotopy class. The track of the constant homotopy $h_0 \simeq h_0$ is denoted by $\{h_0\}$.
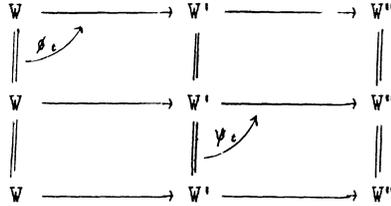
For each homotopy commutative diagram 0.2 there is a triple $(\{\rho_t\},u,\{\sigma_t\})$, where $\{\sigma_t\}$ and $\{\rho_t\}$ are tracks. We obtain an equivalence relation in the set of such triples if, whenever $u_t: X \to X'$ is a homotopy with $u_0 = u$, we define

(1.1) $(\{\rho_t + \rho'u_t\}, u_1, \{u_{1-t}\sigma + \sigma_t\}) \sim (\{\rho_t\},u,\{\sigma_t\})$.

In 1.1 the + refers to the usual track addition of homotopies. Denoting the set of equivalence classes by $\pi(A/X,X'/B)$ we obtain a category with composition induced by juxtaposition of diagrams. Formally we set

$(\{\rho'_t\},u',\{\sigma'_t\}) \circ (\{\rho_t\},u,\{\sigma_t\}) = (\{\rho_t + \rho'_t u\}, u'u, \{u'\sigma_t + \sigma'_t\})$.

To check that composition respects the equivalence relation may at
first glance present a problem, but we may bear in mind that a
diagram

represents equally well the tracks $\{\psi_0\phi_t + \psi_t\phi_1\}$ and $\{\psi_t\phi_0 + \psi_1\phi_t\}$, for
we have

$$\psi_0\phi_t + \psi_t\phi_1 \equiv \psi_t\phi_0 + \psi_1\phi_t.$$

The required verifications are now easily done. The identity morphism
$X \to X$ in $\underline{H}_B{}^A$ is the equivalence class of the triple $(\{\rho\},1_x,\{\sigma\})$. The
equivalence class of a triple $(\{\rho_t\},u,\{\sigma_t\})$ will be denoted simply by
$\{\rho_t,u,\sigma_t\}$.

THEOREM 1.2. *The arrow $\{\rho_t,u,\sigma_t\}$ is an isomorphism in $\underline{H}_B{}^A$ if and only
if the map $u: X \to X'$ is a homotopy equivalence.*

PROOF. If $\{\rho_t,u,\sigma_t\}$ is an isomorphism then it is obvious that u is a
homotopy equivalence. Suppose conversely that u is a homotopy equi-
valence. Then by [6] there exists a homotopy inverse u' of u and
homotopies

$$\phi_t: uu' \simeq 1_{x'}, \quad \psi_t: u'u \simeq 1_x \text{ such that}$$

(1.3) $\phi_t u \equiv u \psi_t.$

It is claimed that

$$\{\rho'\phi_{1-t} + \rho_{1-t}u',u',\ u\sigma_{1-t} + \psi_t\sigma'\}$$

is inverse to $\{\rho_t,u,\sigma_t\}$ in $\underline{H}_B{}^A$, for firstly

$$\{\rho_t,u,\sigma_t\} \circ \{\rho'\phi_{1-t} + \rho_{1-t}u',u',\ u\sigma_{1-t} + \psi_t\sigma'\}$$

is represented by the following composite diagram

$$
\begin{array}{ccccccccc}
A & = & A & = & A & = & A & = & A \\
\sigma'_{1-t}\downarrow & & \sigma'\downarrow & & \sigma\downarrow & & \sigma'_{t}\downarrow & & \sigma'\downarrow \\
\sigma'\downarrow & & X & = & X & & \sigma\downarrow & & \sigma'\downarrow \\
& & u\downarrow \psi_t & & \| & & & & \\
X' & = & X' & \to & X & = & X & \to & X' \\
\| & & \phi_{1-t}\| & & u\downarrow \rho_{1-t} & & \rho_t & & \rho'\downarrow \\
X' & = & X' & = & X' & & \rho\downarrow & & \\
\rho'\downarrow & & & & \rho'\downarrow & & & & \rho'\downarrow \\
B & = & B & = & B & = & B & = & B
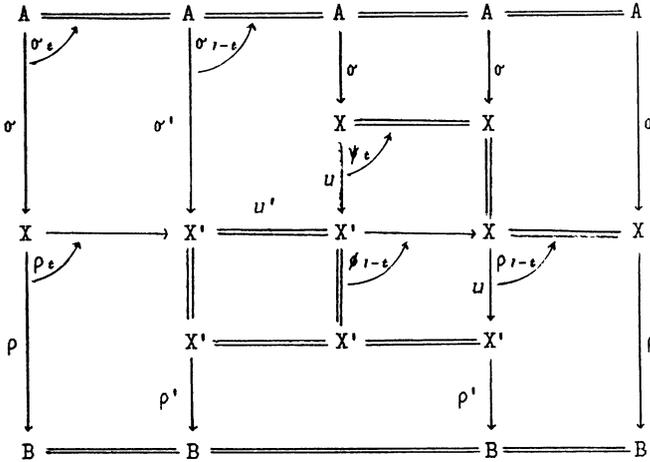\end{array}
$$

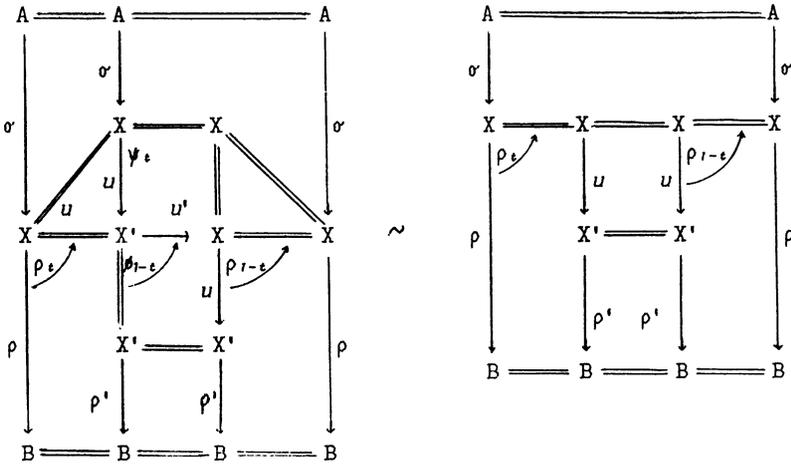whose associated triple is equal to that of

$$
\sim
$$

(using 1.3) which is equal to $(\{\rho'\}, 1_{X'}, \{\sigma'\})$. Secondly the composite

$$
\{\rho'\phi_{1-t} + \rho_{1-t}u', \; u', \; u\psi_{1-t} + \psi_t\sigma'\} \circ \{\rho_t, u, \sigma'_t\}
$$

is represented by the composite diagram

whose associated triple is equal to

(using 1.3) which is equal to $(\{\rho\}, 1_x, \{\sigma\})$, completing the proof.

COROLLARY 1.4. *Each object X in* $\mathbf{H}_B{}^A$ *is isomorphic to an object X' such that* σ' *is a closed cofibration and* ρ' *a Hurewicz fibration.*

PROOF. Let

$$A \xrightarrow{\ \sigma\ } X \xrightarrow{\ \rho\ } B$$

be the given object. By the mapping cylinder construction we can factor σ as qj where j is a closed cofibration and q is a homotopy equivalence. By [5], Proposition 2 (see also [4], Remark (d)) we can factor pq as pi with p a fibration and i a closed cofibration and a homotopy equivalence. Now apply Theorem 1.2 twice to the diagram

(1.5)

Now suppose that $\sigma_t\colon A \to X$ and $\rho_t\colon X \to B$ are homotopies. Applying Theorem 1.2 to the diagram

we obtain the following corollary.

COROLLARY 1.6. *The isomorphism type of an object* X *of* $\underline{H}_B{}^A$ *depends only on the homotopy classes of* σ *and* ρ.

Let $f\colon A \to B$ be a fixed map and consider the class of all objects X of $\underline{H}_B{}^A$ such that $\rho\sigma = f$. Let $\underline{HR}_B{}^A$ be the category with these objects whose arrows are equivalence classes under the relation 1.1 of triples $(\{\rho_t\},u,\{\sigma_t\})$ for which the tracks $\{\rho_t\}$ and $\{\sigma_t\}$ satisfy the additional condition

(1.7) $\{\rho_t\sigma\} + \{\rho'\sigma_t\} = \{f\}$,

where $\{f\}$ denotes the track of the constant homotopy f. Note that the relation 1.1 still makes sense for such triples and that the inclusion $\underline{HR}_B{}^A \rightsquigarrow \underline{H}_B{}^A$ is an embedding.

COROLLARY 1.8. *An arrow (ρ_t, u, σ_t) of $\underline{HR}_B{}^A$ is an isomorphism if and only if u is a homotopy equivalence.*

PROOF. If $(\{\rho_t\}, u, \{\sigma_t\})$ satisfies (1.7) then it can be checked that the triple

$$(\{\rho'\phi_{1-t} + \rho_{1-t}u\}, u', \{u'\sigma_{1-t} + \psi_t\sigma\})$$

as constructed in the proof of Theorem 1.2 also satisfies condition 1.7.   •

REMARK 1.9. Since the diagram 1.5 is strictly commutative, a result corresponding to Corollary 1.4 also holds in $\underline{HR}_B{}^A$.

2. THE KERVAIRE DIAGRAM.

Let

(2.1)

be a fixed homotopy commutative diagram over B and under A. Let

$$U = (\mu_t, u) =$$

and

$$V = (u, \chi_t) =$$

be the induced homotopy commutative diagrams over B and under A whose equivalence classes are

$$U^\sim \in \pi(X, X'/B) \quad \text{and} \quad V^\sim \in \pi(A/X, X')$$

respectively. Let us also denote by $U^\sim V^\sim$ the element of $\pi(A/X,X'/B)$ represented by 2.1.

Consider the following interlocking (Kervaire) diagram of groups and pointed sets with base points as indicated.

$$(2.2)$$

$$\lambda(A/X,X';V) \xrightarrow{(\rho'.)i} \pi_1{}^x(B;\rho'u) \xrightarrow{m_u = r_B n_{uv}} \pi(X,X'/B)_{u^-} \xrightarrow{(.\sigma)d} \pi(A,X')_{\cdot}$$

with diagonal maps i, ρ', n_{uv}, r_B, d, $.\sigma$ and middle row
$$\pi_1{}^x(X';u) \qquad \pi(A/X,X'/B)_{u^- v^-} \qquad \pi(X,X')_u$$
with j, $.\sigma$, n^{vu}, r^A, c, $\rho'.$

$$\lambda(X,X'/B;U) \xrightarrow{(.\sigma)j} \pi_1{}^A(X';uv) \xrightarrow{m^v = r^A n^{vu}} \pi(A/X,X')_{v^-} \xrightarrow{(\rho'.)c} \pi(X,B)_{\cdot}$$

Here, $\lambda(A/X,X';V)$ and $\lambda(X,X'/B;U)$ are subgroups of $\pi_1{}^x(X';u)$ defined as the kernel of the induced group homomorphism

$$.\sigma\colon \pi_1{}^x(X';u) \to \pi_1{}^A(X';uv), \text{ respectively } \rho'.\colon \pi_1{}^x(X';u) \to \pi_1{}^x(B;\rho'u);$$

the maps i and j are inclusions, r_B, r^A, d and c are the obvious restriction operators and n_{uv}, n^{vu} are defined by the rules
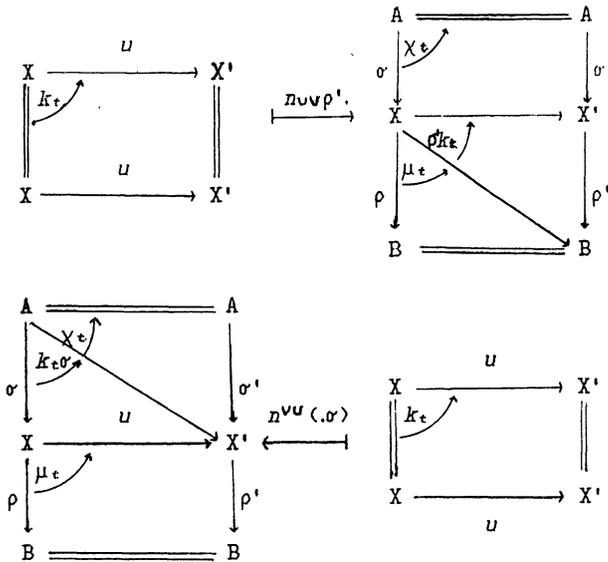
$$(2.3)$$

$$(2.4)$$

THEOREM 2.5. *The Kervaire diagram 2.2 is commutative and its four interlocking sequences of homomorphisms and pointed maps are exact. Moreover*

$$\bullet \xrightarrow{\;(.\sigma)j\;} \bullet \xrightarrow{\;n^{\nu\upsilon}\;} \bullet$$

is exact of type E3 (see [2], 1.A) at $\pi_1^A(X';u\sigma)$.

PROOF. We prove commutativity from $\pi_1^X(X';u\sigma)$. We have that

The two sequences passing through $\pi_1^X(X';u)$ are exact by definition of $\lambda(A/X,X';V)$, $\lambda(X,X'/B;U)$ and by [2], Theorem 4.3 and its dual. Thus the following proposition remains to be proved.
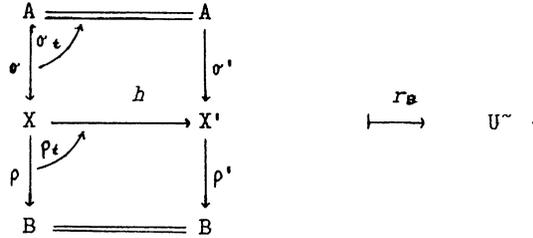
PROPOSITION 2.6. *The sequence*

$$\lambda(X,X'/B;U) \xrightarrow{\;(.\sigma)j\;} \pi_1^A(X';u\sigma) \xrightarrow{\;n^{\nu\upsilon}\;} \pi(A/X,X'/B)_{u\sigma\nu\sigma^-} \xrightarrow{\;r_B\;}$$
$$\xrightarrow{\hspace{2cm}} \pi(X,X'/B)_{\upsilon^-} \xrightarrow{\;(.\sigma)d\;} \pi(A,X')_\sigma.$$
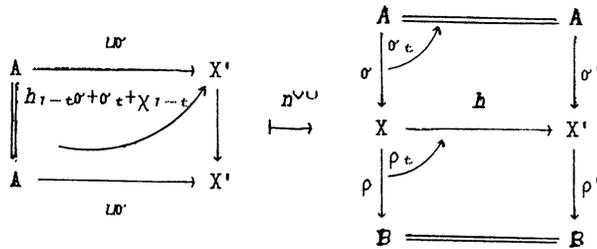
is exact. At $\pi_1^A(X';u\sigma)$ *it is exact of type* E3.

PROOF. The exactness at $\pi(X,X'/B)_{\upsilon^-}$ is obvious.

Exactness at $\pi(A/X,X'/B)_{u-v-}$: n^{vu} is given by 2.4. Clearly r_B applied to the composite square on the right of 2.4 yields $U^\sim$. Now suppose that

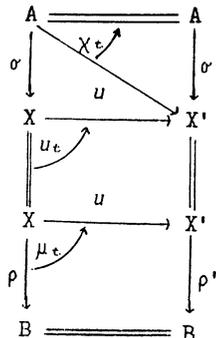Then there exists h_t: $X \to X'$ with $h_0 = h$ such that $h_1 = u$ and $\rho_t + \rho'h_t \equiv \mu_t$ and it can be checked that
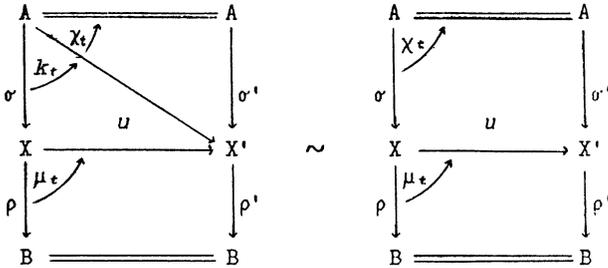

as required.

Exactness at $\pi_1{}^A(X';u\sigma)$: Let $\{u_t\} \in \lambda(X,X'/B;U)$, i.e., $\{u_t\}$ is in $\pi_1{}^x(X';u)$ such that $\rho'u_t \equiv \rho'u$, whence

(2.7) $\mu_t + \rho'u_t \equiv \mu_t$.

Now $n^{vu}(.\sigma)\{u_t\}$ is represented by the diagram

and in view of the relation 2.7, we have that $n^{vv}(.\sigma)\{u_t\} = U^\sim V^\sim$. Conversely, if $\{k_t\} \in \pi_1^A(X';u\sigma)$ is such that

then there exists a homotopy $u_t: X \to X'$ with $u_0 = u_1 = u$ such that

(2.8) $\mu_t + \rho' u_t \equiv \mu_t$

                                                                            and

(2.9) $u_{1-t}\sigma + k_t + \chi_t \equiv \chi_t$.

By 2.8 $\{u_t\}$ belongs to $\lambda(X,X'/B;U)$ and by 2.9 we have that

$$(.\sigma)\{u_t\} = \{k_t\}.$$

   Exactness of type E3 is proved by checking that

$$n^{vv}\{k_t\} = n^{vv}\{k'_t\} \qquad \text{iff} \qquad \{k'_t\}\{k_t\}^{-1} \in \text{Im}(.\sigma) j$$

and applying ordinary exactness at $\pi_1^A(X';u\sigma)$.   •

   Applying [2], Theorem 2.13 and Lemma 2.11, we obtain

COROLLARY 2.10. *The sequence*

$$\lambda(A/X,X';V) \times \lambda(X,X'/B;U) \xrightarrow{\theta} \pi_1^x(X';u) \xrightarrow{\Delta} \pi(A/X,X'/B)_{U-V-}$$

$$\longrightarrow \pi(A/X,X')_V - \times \pi(X,X'/B)_U -$$

is exact, where $\Delta = n^{vv}(.\sigma)$ *and* $\theta(\alpha,\beta) = \beta^{-1}\alpha$.
   Moreover the images of two elements under Δ *coincide iff they belong to the same double coset of the subgroups* $\lambda(X,X'/B;U)$ *and* $\lambda(A/X,X';V)$.

   Let $K(u,U,V)$ denote the set of double cosets in $\pi_1^x(X';u)$ of the subgroups $\lambda(X,X'/B;U)$ and $\lambda(A/X,X';V)$.

COROLLARY 2.11. *There is a bijection*

$$\pi(A/X,X'/B) \longleftrightarrow \bigcup_S K(u,U,V),$$

where

$$S = \{(u,U,V) \mid U^\sim \in d^{-1}\{u\}, V^\sim \in c^{-1}\{u\}, \{u\} \in (.\sigma)^{-1}\{\sigma'\} \cap (\rho'.)^{-1}\{\rho\}\}.$$

REMARK 2.12. The homomorphisms

$$\rho'.: \pi_1{}^x(X';u) \longrightarrow \pi_1{}^x(B;\rho'u) \quad \text{and} \quad .\sigma: \pi_1{}^x(X';u) \longrightarrow \pi_1{}^A(X';u\sigma)$$

can, in certain special cases, be computed as discussed by Rutter [3].

3. SECONDARY COMPOSITION CLASSES.

In this final section we examine in a special case some inter-
actions between elements of the set $\pi(A/X,X'/B)$ and secondary comp-
osition classes.

We consider pointed topological spaces A,B,X,X' and the case in
which $\rho: X \to B$ and $\sigma': A \to X'$ are the trivial maps (denoted by *). The
following operators may be defined.

(3.1) $R: \pi(A/X,X'/B) \longrightarrow \pi(X,X')$.

Set $R\{\rho_t,u,\sigma_t\} = \{u\}$.

(3.2) $M: \pi(A/X,X'/B) \longrightarrow \pi(\sigma,\rho')$.

Here $\pi(\sigma,\rho')$ refers to the homotopy pair set, for details see [1]. Set

$$M\{\rho_t,u,\sigma_t\} = \{*, *, \rho_t\sigma + \rho'\sigma_t\}.$$

Let $\alpha = \{\rho_t,u,\sigma_t\}$ denote an arbitrary element of $\pi(A/X,X'/B)$. Let

$$\{\{\rho'\},\{R\alpha\},\{\sigma\}\} \subset \pi(\Sigma A,B)$$

denote the Toda bracket coset of $\{\sigma\}$, $\{\rho'\}$ and $\{R\alpha\}$.
We have the following result.

THEOREM 3.3. *The following are equivalent.*
(i) $0 \in \{\{\rho'\},\{R\alpha\},\{\sigma\}\}$.

(ii) $M\alpha = 0.$

PROOF. Allowing diagrams to represent elements we have

By [2], Proposition 3.14 it follows that (i) and (ii) are equivalent. •

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