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MAREK GOLASIŃSKI

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HOMOTOPY GROUPS OF SMALL CATEGORIES AND DERIVED FUNCTORS

by Marek GOLASIŃSKI

RÉSUMÉ. Dans cette Note on montre que les groupes d'homotopie $\pi_n(C)$ d'une catégorie pointée C peuvent être réalisés comme foncteurs dérivés du groupe fondamental. On examine aussi le cas relatif.

INTRODUCTION.

In [15], Thomason showed that all homotopy types could be represented by categories, more precisely that the homotopy category of small categories is equivalent to that of CW-spaces. Other algebraic models for homotopy types are known, for instance simplicial groups model all connected pointed homotopy types. The importance of the homotopy types of small categories is, however, due more to the fact that in many situations, and especially in algebraic K-theory, a homotopy type comes most naturally in the form of a small category. Many authors have passed from a small category to its nerve and thus to its classifying space, but this is known not to be necessary and as the category encodes the information more directly, it may even be a hindrance to interpreting the homotopy groups of the category in terms of the original algebraic and geometric problem. It should perhaps be mentioned that Grothendieck in [8] essentially expresses tha above opinion and puts **Cat**, the category of small categories, into pride of place amongst the possible settings for algebraic models of homotopy types.

One is therefore led to consider the question of calculating, in hopefully as direct a way as possible, homotopy invariants of categories. In particular, if C is a pointed category, it would be useful to be able to obtain new descriptions of its homotopy groups. In this Note we give a description of $\pi_n(C)$ as the n-1^{st} derived functor of the fundamental group functor π_1 with respect to the class of all free categories. Here we are using the language of simplicial derived functors as developed by Barr-Beck (cf. [2]), Tierney-Vogel [16] and Keune [10].

These homotopy groups and in fact a useable internally defined homotopy theory for small categories based on cubical methods has been developed by Evrard [5] and the author [6]. This theory naturally leads to a notion of a fibration of categories [4] and this is used here to obtain a relative form of the description mentioned above. This should have applications to relative K-theory and possibly to the more recent developments in the multirelative theory if applied to Quillen's Q-construction in [13].

BACKGROUND.

Let $\mathbb{C}$ be a small category, **Ab** the category of abelian groups and **$\mathbb{C}$-Mod** the category of all $\mathbb{C}$-modules (i.e., all functors from C to **Ab**). Put $H^n(\mathbb{C},M)$ for the n-th cohomology group of $\mathbb{C}$ with coefficients in M. Then there exists an isomorphism

$$H^n(\mathbb{C},M) \cong \mathrm{Ext}^n_{Z\mathbb{C}}(\Delta Z,\ M) \qquad \text{for} \quad n \geqslant 0$$

(cf. [11], [14] and [17]), where $Z\mathbb{C}$ denotes the ringoid of $\mathbb{C}$ over the integers Z [11] and ΔZ the constant functor determined by Z. In particular,

$$\mathrm{Hom}^0(\mathbb{C},M) \simeq \mathrm{Hom}_{Z\mathbb{C}}(\Delta Z,M).$$

$H^n(\mathbb{C},M)$ can also be described by n-fold extensions (cf. [7]).

A *derivation* (cf. [9] and [11]) from $\mathbb{C}$ to M is a mapping d such that

a) $d(f) \in M(C')$, for $f\colon C \to C'$,

b) $d(f.g) = d(f) + fd(g)$, where f and g are composable maps of $\mathbb{C}$ and we have written $fd(g)$ for $M(f)(d(g))$.

We note that if τ assigns to each $C \in |\mathbb{C}|$ a $\tau(C) \in M(C)$, then d_τ given by $d_\tau(f) = \tau(C') - f\,\tau(C)$, for $f\colon C \to C'$ is a derivation from $\mathbb{C}$ to M. A derivation of this form is called *an inner derivation* [11]. Denote by $\mathrm{Der}(\mathbb{C},M)$ and $\mathrm{Int}(\mathbb{C},M)$ the abelian groups of all derivations and all inner derivations, respectively. There are actually functors

$$\mathrm{Der}(\mathbb{C},-),\ \mathrm{Int}(\mathbb{C},-)\colon \mathbb{C}\text{-}\mathbf{Mod} \to \mathbf{Ab}.$$

REMARK [9]. There is a natural isomorphism

$$H^1(\mathbb{C},M) \simeq \mathrm{Der}(\mathbb{C},M)/\mathrm{Int}(\mathbb{C},M).$$

This is clear, since derivations are just cocycles and inner derivations are coboundaries in a suitable cochain complex [17].

Now let ZX denote the free abelian group generated by X. Then we can define a $\mathbb{C}$-module $\Sigma_{C\in|C|}Z\mathbb{C}(C,-)$. The augmentation ideal $I(\mathbb{C})$ is given by the exact sequence

$$0 \longrightarrow I(\mathbb{C}) \longrightarrow \Sigma_{C\in|C|}Z\mathbb{C}(C,-) \xrightarrow{\ \epsilon\ } \Delta Z \longrightarrow 0,$$

where ϵ is the augmentation $\mathbb{C}$-module map. One can easily see that $I(\mathbb{C})(C)$ is the free abelian group generated by the set of elements $id_C - f$, for all maps $f\colon C' \to C$. $I(\mathbb{C})$ represents the functor of derivations from $\mathbb{C}$ to M by the following lemma.

LEMMA 1.2. *There is a natural isomorphism*

$$\gamma\colon \mathbb{C}\text{-}\mathbf{Mod}(I(\mathbb{C}),M) \simeq \mathrm{Der}(\mathbb{C},M)$$

given by $\gamma(\phi)(f) = \phi(C')(id_{C'}-f)$, *for* $\phi \in \mathbb{C}\text{-}\mathbf{Mod}(I(\mathbb{C}),M)$ *and a map* $f\colon C \to C'$.

PROOF. We define

$$\delta\colon \mathrm{Der}(\mathbb{C},M) \to \mathbb{C}\text{-}\mathbf{Mod}(I(\mathbb{C}),M)$$

the inverse of γ by

$$\delta(d)(C')(id_{C'}-f) = d(f), \quad \text{for} \quad d \in \mathrm{Der}(\mathbb{C},M) \text{ and } f\colon C \to C'.$$

Then

$$\gamma\delta = id_{\mathrm{Der}(\mathbb{C},M)} \quad \text{and} \quad \delta\gamma = id_{\mathbb{C}\text{-}\mathrm{Mod}(I(\mathbb{C}),M)}.$$

It suffices to prove that $\gamma(\phi)$ is a derivation if ϕ is a $\mathbb{C}$-module map and $\delta(d)$ is a $\mathbb{C}$-module map, if d is a derivation. If $\phi\colon I(\mathbb{C}) \to M$ is a $\mathbb{C}$-module map then the following diagram is commutative:

$$\begin{array}{ccc} I(\mathbb{C})(C') & \xrightarrow{\ \phi(C')\ } & M(C') \\ {\scriptstyle I(\mathbb{C})(f)}\downarrow & & \downarrow{\scriptstyle M(f)} \\ I(\mathbb{C})(C'') & \xrightarrow{\ \phi(C'')\ } & M(C'') \end{array}$$

for each map $f: C' \to C''$. Hence

$$\phi(C'')I(\mathbb{C})(f)(id_{C'}-g) = M(f)\phi(C')(id_{C'}-g),$$

for a map $g: C \to C'$, but

$$\phi(C'')I(\mathbb{C})(f)(id_{C'}-g) = -\phi(C'')[(id_{C''}-f) - (id_{C''}-fg)]$$
$$= -\gamma(\phi)(f) + \gamma(\phi)(fg)$$

and

$$M(f)\phi(C')(id_{C'}-g) = f\,\gamma(\phi)(g).$$

Therefore

$$\gamma(\phi)(fg) = \gamma(\phi)(f) + f\,\gamma(\phi)(g).$$

The second implication is also straightforward.

LEMMA 1.3. *If* C *is a free category on free generators* $f_i: C_i \to C'_i$ *for* $i \in I$, *then each derivation* d *from* $\mathbb{C}$ *to* M *is uniqely determined by the family* $(d(f_i))_{i\in I}$ *of its values on the generators.*

PROOF. By definition, the free category $\mathbb{C}$ consists of the paths

$$f = (C_0 \xrightarrow{f_1} C_1 \longrightarrow \dots \xrightarrow{f_n} C_n)$$

in the generators. The composition of two paths is obtained by juxtaposition. Now a derivation d satisfies the equation

$$d(f.g) = d(f) + f\,d(g),$$

for composable maps f and g. Therefore, d is completely determined by its values $d(f_i) \in M(C'_i)$ on the free generators f_i.

Conversely, given $m_i \in M(C'_i)$ we may set $d(f_i) = m_i$ and define $d(f)$ by induction on the length of the path f by the formula

$$d(f_i f) = f_i d(f) + m_i,$$

if f_i and f are composable.

By Lemma 1.2 the derivations d from $\mathbb{C}$ to M correspond one-one to the C-module maps $\gamma: I(\mathbb{C}) \to M$. In particular

$$d(f_i) = \gamma(C'_i)(id_{C'}-f_i).$$

Thus the lemma above states that the $\mathbb{C}$-module maps γ are determined in one-one fashion by their values on $id_{C'_i} - f_i \in I(\mathbb{C})(C'_i)$. Hence for each exact diagram of $\mathbb{C}$-modules

$$\begin{array}{ccccc} & & I(\mathbb{C}) & & \\ & & \downarrow \psi & & \\ M & \xrightarrow{\phi} & N & \longrightarrow & 0 \end{array}$$

there exists a $\mathbb{C}$-module map $\chi\colon I(\mathbb{C}) \to M$ such that $\phi.\chi = \psi$. This means that $I(\mathbb{C})$ is a projective $\mathbb{C}$-module. One concludes

PROPOSITION 1.4. *For a free category* $\mathbb{C}$, $H^n(\mathbb{C},M) = 0$ *for* $n > 1$.

This sort of result has also been obtained by Mitchell [11], but here we give an explicit proof of the above proposition. Using these methods one can also prove that for a free groupoid C, $H^n(\mathbb{C},M) = 0$, for $n > 1$.

Now let $(Ar\mathbb{C})^{-1}\mathbb{C}$ be the fundamental groupoid of a pointed small category $\mathbb{C}$ and M an $(Ar\mathbb{C})^{-1}\mathbb{C}$-module. Then the canonical functor $p\colon \mathbb{C} \to (Ar\mathbb{C})^{-1}\mathbb{C}$ determines an isomorphism

$$\mathrm{Hom}(\Delta Z,M) \xrightarrow{\simeq} \mathrm{Hom}(\Delta \mathbf{Ab}, M.p)$$

(i.e., $H^0((Ar\mathbb{C})^{-1}\mathbb{C}, M) \simeq H^0(\mathbb{C}, M.p)$). Similarly

$$\mathrm{Der}((Ar\mathbb{C})^{-1}\mathbb{C}, M) \xrightarrow{\simeq} \mathrm{Der}(\mathbb{C}, M.p)$$

and

$$\mathrm{Int}((Ar\mathbb{C})^{-1}\mathbb{C}, M) \xrightarrow{\simeq} \mathrm{Int}(\mathbb{C}, M.p).$$

Therefore

$$H^1((Ar\mathbb{C})^{-1}\mathbb{C}, M) \simeq H^1(\mathbb{C}, M.p).$$

If C is free, then by Proposition 1.4

$$H^n((Ar\mathbb{C})^{-1}\mathbb{C}, M) \simeq H^n(\mathbb{C}, M.p), \quad \text{for} \quad n > 1.$$

Of course, the functor $p\colon \mathbb{C} \to (Ar\mathbb{C})^{-1}\mathbb{C}$ induces also isomorphisms of the following homotopy groups

$$\pi_0(\mathbb{C}) \xrightarrow{\simeq} \pi_0((Ar\mathbb{C})^{-1}\mathbb{C}) \quad \text{and} \quad \pi_1(\mathbb{C}) \xrightarrow{\simeq} \pi_1((Ar\mathbb{C})^{-1}\mathbb{C}).$$

Therefore, by Quillen's result [12] the functor p is a weak homotopy equivalence (i.e., the induced map of the classifying spaces is a homotopy equivalence). But functors π_n vanish on groupoids, for $n > 1$. Finally, one concludes:

PROPOSITION 1.5. For a free category $\mathbb{C}$, $\pi_n(\mathbb{C}) = 0$ *for* $n > 1$.

THE MAIN RESULTS.

Put **Grph*** for the category of pointed graphs. The underlying graph functor U: **Cat*** → **Grph*** has a left adjoint F: **Grph*** → **Cat***, the free category functor. Following Tierney-Vogel [16] and Keune [10] we can construct for each pointed small category $\mathbb{C}$ a free simplicial resolution (unique up to homotopy) ϵ: $\mathbb{C}_* \to \mathbb{C}$ (i.e., $\mathbb{C}_*$ is free and $U(\mathbb{C}_*) \to U(\mathbb{C})$ an aspherical object in **Grph***). We refer the reader to [3] for the basic results on the homotopy theory of simplicial groups.

LEMMA 2.1. For any pointed connected small category $\mathbb{C}$ *there exists an isomorphism*

$$\pi_{n-1}(\pi_1(\mathbb{C}_*)) \simeq \pi_n(\mathbb{C}) \quad \textit{for} \quad n \geq 1,$$

where $\pi_{n-1}(\pi_1(\mathbb{C}_*))$ *is the (n-1)-st homotopy group of the simplicial group* $\pi_1(\mathbb{C}_*)$.

PROOF. Let ϵ: $\mathbb{C}_* \to \mathbb{C}$ be a free simplicial resolution of $\mathbb{C}$. Applying the nerve functor **N** one obtains a simplicial resolution of the simplicial set $\mathbf{N}\mathbb{C}$, $\mathbf{N}\epsilon$: $\mathbf{N}\mathbb{C}_* \to \mathbf{N}\mathbb{C}$. By Artin-Mazur [1] there exists a spectral sequence

$$E^2_{n,m} = \pi_n(\pi_m \, N(\mathbb{C}_*)) \Rightarrow \pi_{n+m} \, N(\mathbb{C}).$$

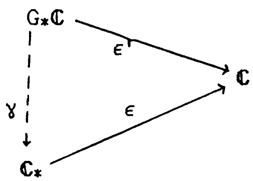
But, by Proposition 1.5, $\pi_k(\mathbb{C}_n) = 0$, for $k > 1$ and $n \geq 0$. So this spectral sequence collapses and $\pi_{n-1}(\pi_1(\mathbb{C}_*)) \simeq \pi_n(\mathbb{C})$.

Now let $L^F_n\pi_1$ be the n-th left-derived functor of π_1 with respect to the class F of all free categories, for $n \geq 0$ (cf. [10]). Summarizing the above, we obtain:

THEOREM 2.2. *For any pointed connected small category* $\mathbb{C}$ *there exists an isomorphism*

$$(L^F_n\pi_1)(\mathbb{C}) \simeq \pi_{n+1}(\mathbb{C}), \quad \textit{for} \quad n \geqslant 0.$$

Let ϵ': $G_*\mathbb{C} \to \mathbb{C}$ be the free cotriple resolution of $\mathbb{C}$ [2], then by the Comparison Theorem [10] there exists a map of resolutions



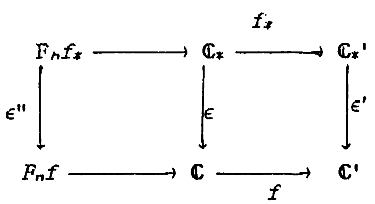
Using the methods of Lemma 2.1 we deduce that

$$\pi_{n-1}(\pi_1(G_*\mathbb{C})) \simeq \pi_n(\mathbb{C}) \quad \text{for} \quad n \geqslant 1$$

and finally $\pi_1(\gamma)$: $\pi_1(G_*\mathbb{C}) \to \pi_1(\mathbb{C}_*)$ is a weak homotopy equivalence of simplicial groups. Therefore ϵ': $G_*\mathbb{C} \to \mathbb{C}$ can also be used to calculate $(L^F_n\pi_1)(\mathbb{C})$.

Moreover, if $f\colon \mathbb{C} \to \mathbb{C}'$ in $\mathbf{Cat}^*_c$ is surjective (i.e., $U(f)$ is surjective in $\mathbf{Grph}^*$) then so is G_*f: $G_*\mathbb{C} \to G_*\mathbb{C}'$ and we also put f_*: $\mathbb{C}_* \to \mathbb{C}_*'$ for this simplicial map.

Let F_hf be the homotopy fibre of f (cf. [4]) and F_hf_* a simplicial category such that $(F_hf_*)_n = F_hf_n$, for $n \geqslant 0$ and its simplicial structure is determined by that of $\mathbb{C}_*$ and $\mathbb{C}_*'$. Then we obtain the following commutative diagram



Each functor f_n: $\mathbb{C}_n \to \mathbb{C}_n'$ is surjective and $\mathbb{C}_n'$ are free, hence there exists a functor s_n: $\mathbb{C}_n' \to \mathbb{C}_n$ such that $f_n.s_n = id_{\mathbb{C}_n'}$.

Therefore, $\pi_1(f_n)$: $\pi_1(\mathbb{C}_n) \to \pi_1(\mathbb{C}_n')$ is surjective, for $n \geqslant 0$ and $\pi_1(f_*)$: $\pi_1(\mathbb{C}_*) \to \pi_1(\mathbb{C}_*')$ is a surjection of simplicial groups. Applying the homotopy long exact sequence for each functor f_n: $\mathbb{C}_n \to \mathbb{C}_n'$ we obtain by Proposition 1.5 short exact sequences

$$1 \longrightarrow \pi_1(F_n f_n) \longrightarrow \pi_1(\mathbb{C}_n) \longrightarrow \pi_1(\mathbb{C}_n') \longrightarrow 1$$

and $\pi_k(F_n f_n) = 0$ for $k > 1$ and $n \geqslant 0$. Finally, we have that Ker $\pi_1(f_*) = \pi_1 F_n(f_*)$. But by [10] the n-th relative left derived functor of π_1 is given by

$$(L^F_n \pi_1)(f\colon \mathbb{C} \to \mathbb{C}') = \pi_n \operatorname{Ker}\pi_1(f_*).$$

Hence

$$(L^F_n \pi_1)(f\colon \mathbb{C} \to \mathbb{C}') = \pi_n(\pi_1 F_n f_*)).$$

We note that $\epsilon''\colon F_n f_* \to F_n f$ is surjective and from the above that

$$\pi_k(F_n f_n) = 0 \quad \text{for} \quad k > 1 \quad \text{and} \quad n = 0,1,\ldots .$$

Therefore, using the methods of Lemma 2.1 we obtain an isomorphism

$$\pi_{n-1}\pi_1(F_n f_*) \simeq \pi_n(F_n f) \quad \text{for} \quad n \geqslant 1.$$

Finally, we have

THEOREM 2.3. *If $f\colon \mathbb{C} \to \mathbb{C}'$ is surjective and $\mathbb{C}$ is connected, then there exists an isomorphism*

$$(L^F_n \pi_1)(f\colon \mathbb{C} \to \mathbb{C}') \simeq \pi_{n+1}(F_n f), \quad \textit{for} \quad n \geqslant 0.$$

In particular, if $f\colon \mathbb{C} \to \mathbb{C}'$ is a fibration (in Evrard's sense [4]) and $\mathbb{C}'$ is connected, then f is surjective and $\pi_n(F_n f) \simeq \pi_n(Ff)$ for $n \geqslant 0$ (cf. [4]), where Ff is the fibre of f. Summarising, we have

COROLLARY 2.4. *If $f\colon \mathbb{C} \to \mathbb{C}'$ is a fibration and $\mathbb{C}'$ is connected, then there exists an isomorphism*

$$(L^F_n \pi_1)(f\colon \mathbb{C} \to \mathbb{C}') \simeq \pi_{n+1}(Ff), \quad \textit{for} \quad n \geqslant 0$$

and the Keune sequence for f (cf. [10]) is isomorphic to the homotopy long exact sequence.

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Institute of Mathematics
Nicholas Copernicus University
ul. Chopina 12/18
87-100 TORUN, POLAND