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DAVID N. YETTER

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CATEGORIES OF MODULES IN SYNTHETIC DIFFERENTIAL GEOMETRY

by David N. YETTER

RÉSUMÉ. On montre que dans tout modèle d'une version faible des axiomes de la Géométrie Différentielle synthétique, la sous-catégorie pleine de R-mod formée de tous les modules vérifiant la forme vectorielle de l'axiome 1^w ou de l'axiome 1 (ou d'autres axiomes de force intermédiaire) est une classe de Serre. Si le topos sous-jacent est de Grothendieck; et dans le cas de l'Axiome 1 dans un topos avec NNO, cette sous-catégorie pleine est réflexive, et comme conséquence a une structure monoïdale symétrique. On examine le comportement de ces modules par rapport aux foncteurs produit fibré et on caractérise les applications linéaires comme les applications pointées sur un voisinage infinitésimal de 0.

INTRODUCTION.

Classically, in both differential and algebraic geometry, modules over "the line", whether R or C, arise with tremendous frequency: as fibres of various bundles, as elementary examples of manifolds or affine varieties, as underlying modules for algebras of functions, and so forth.

One property of such modules, usually overlooked as obvious, is that when endowed with their usual differentiable structures they are isomorphic to their own tangent fibre at any of their points, in particular at 0. It is, however, this very property which is the fullest justification for the classical name of "vector spaces".

In the context of synthetic differential geometry (SDG), Kock's [2] "vector form of Axiom 1" may be restated as:

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"The map $X \to X^{.D}$ transpose to the multiplication map $X \times D \to X$ is an isomorphism, where D is the generic tangent vector and $(-)^{.D}$ denotes the object of pointed maps (0 serving as the point of X)".

Thus modules satisfying this property may be regarded as "vector spaces" over the line. In [4] the author studied the properties of the category of such modules, showing in particular that they form a reflective, exactly embedded, full abelian subcategory of the category of R-modules closed under extensions, whenever R satisfies Axiom 1 and D is tiny (i.e., $(-)^{.D}$ has a *right* adjoint).

This paper extends the result of the author's dissertation to Weil prolongation functors other than the tangent-bundle functor. In particular it is shown that modules satisfying a "vector form of Axiom 1^{W}" form a Serre class, and in Grothendieck topos models of SDG, that this full subcategory is reflective.

1. DEFINITIONS AND MODELS.

Throughout we fix a topos of discourse, $\mathbf{E}$, defined over a boolean base topos $\mathbf{B}$, and a field object K in $\mathbf{B}$. When it is necessary to assume that $\mathbf{E}$ is a Grothendieck topos, we will tacitly assume $\mathbf{B}$ = Sets.

DEFINITION 1. A *Weil algebra* is a finite dimensional commutative K-algebra W which decomposes as $K \oplus I$, where K is the subalgebra generated by 1, and I is a nilpotent ideal (i.e., for some $n \in \mathbf{N}$ any n-fold product of elements from I to 0).

DEFINITION 2. For $\mathcal{W}$ a family of Weil algebras in $\mathbf{B}$, a commutative ring R in $\mathbf{E}$ is a *$\mathcal{W}$-line* if it satisfies

L1. For all $W \in \mathcal{W}$, the map $R \otimes_K W \to R^{Spec(W)}$ is an isomorphism, where Spec(W) is the object $Hom_{R\text{-alg}}(R \otimes_K W, R)$ in $\mathbf{E}$ and this map is the transpose of the evaluation map.

and

L2. For all $W \in \mathcal{W}$, Spec(W) is tiny (i.e., $(-)^{Spec(W)}$ has a right adjoint).

DEFINITION 3. If R is a $\mathcal{W}$-line, an R-module X is a *$\mathcal{W}$-vector space* it it satisfies:

V1. For all $W \in \mathcal{W}$, the map $\alpha_{X,W}: X \otimes_K W \to X^{Spec(W)}$, transpose to

$$id_X.ev_{R\otimes W}: (X\otimes_K W) \times Spec(W) = (X\otimes_R(R\otimes_K W)) \times Spec(W) \longrightarrow X,$$

is an isomorphism.

We denote the full subcategory of R-mod whose objects are all $\mathcal{W}$-vector spaces by $\mathcal{W}$-v.s.

Note that Spec(K) = 1, and thus all R-modules are {K}-vector spaces, while by tradition $Spec(K[\epsilon]/\epsilon^2)$ is denoted D, and thus {D}-vector spaces are the R-vector spaces of Yetter [3]. The example of $\mathcal{W}$ = {1} may make the use of the name "vector space" seem abuse, but as this example is also the trivial case of all theorems contained herein, it may safely be ignored.

Before proceeding to examine the structure of $\mathcal{W}$-v.s. for various $\mathcal{W}$, it is appropriate to mention some models for $\mathcal{W}$-lines. If K is any field in a boolean topos, the generic K-algebra in the classifying topos for K-algebras is a $\mathcal{W}$-line for $\mathcal{W}$ the class of all Weil algebras over K. Likewise, the synthetic line in any of the standard models of SDG (the Dubuc topos, cf. [1]; etc.) is a $\mathcal{W}$-line for $\mathcal{W}$ the class of all Weil algebras over R. Further models may be constructed using the techniques of Yetter [4].

It is also appropriate to provide some separating examples. In any of the models above, the following are not $\mathcal{W}$-vector spaces: any R-module which is Spec(W)-discrete (in the sense of Yetter [4]) (e.g., the Spec(W)-discrete reflection of any R-module) for any $W \in \mathcal{W}$; the submodule of X generated by the image of all *pointed* maps $W \to X$ for $W \in \mathcal{W}$, when X is any R-module; and the free R-module on any object A, which is not Spec(W)-discrete for all $W \in \mathcal{W}$. If, however, A is Spec(W)-discrete for all $W \in \mathcal{W}$, then the free R-module on A is a $\mathcal{W}$-vector space. (Verifications are easy and left to the reader.) On the other hand we will see that there are other examples of $\mathcal{W}$-vector spaces.

2. THE STRUCTURE OF $\mathcal{W}$-v.s. $\subset$ R-mod.

Throughout we now fix the family $\mathcal{W}$ of Weil algebras, and a $\mathcal{W}$-line R.

Rather than examine the inclusion functor directly, we first consider the functors $(-)\otimes_K W$ and $(-)^{Spec(W)}$ considered as endofunctors on R-mod.

LEMMA 4. The functors $(-)\otimes_K W$ *and* $(-)^{Spec(W)}$ *are both exact.*

PROOF. For the first, W is a flat K-module. For the second, let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be an exact sequence in R-mod. Consider the result of applying $(-)^{Spec(W)}$. Exactness at $A^{Spec(W)}$ and at $C^{Spec(W)}$ are evident since any exponential functor preserves monics, while by tininess of Spec(W), its exponential functor preserves epis. Of course, the resulting sequence is exact at $B^{Spec(W)}$ since

$$\beta^{Spec(W)}(\phi) = \phi\beta : Spec(W) \to C$$

is 0 precisely when ϕ factors through $Ker(\beta) = A$.

We are now ready to prove

THEOREM 5. $\mathcal{W}$-v.s. *is an exactly embedded abelian subcategory of* R-mod *; moreover it is closed under extensions.*

PROOF. After observing that the maps $\alpha_{X,W}$: $X\otimes_K W \to X^{Spec(W)}$ are natural in X, the theorem follows almost immediately from Lemmas 4 and 6.

LEMMA 6. If F *and* G *are two exact endofunctors on an abelian category* **A**, *and* δ: F $\to$ G *is a natural transformation, then the full subcategory of objects* A *such that* δ_A *is an isomorphism is an exactly embedded abelian subcategory closed under extensions.*

PROOF. An easy exercise in the use of the 5-Lemma.

In order to construct the reflection functor in the case where **E** is a Grothendieck topos, it is necessary to introduce an alternative characterization of $\mathcal{W}$- vector spaces. Throughout the following if $A \subset B$ is a pair of objects, and X a pointed objects, then $X^{(A,B)}$ denotes the subobject of X^B consisting of all maps which map A to the point of X. Modules are always pointed by 0.

LEMMA 7. If $\mathcal{W}$ *is a class of* K-*Weil algebras closed under quotients,* R *a* $\mathcal{W}$-*line, and* A *an* R-*module such that for all* $W \in \mathcal{W}$,

$$A \to A^{(\mathrm{Spec}(W/\langle\mu\rangle),\mathrm{Spec}(W))}$$

is an isomorphism whenever $\mu \in I_W^{\,n-1} - \{0\}$, *where* I_W *is the nilpotent maximal ideal of* W *and* $I_W^{\,n} = 0$, *and the map is the map induced on kernels in the diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^{(\mathrm{Spec}(W/\langle\mu\rangle),\mathrm{Spec}(W))} & \longrightarrow & A^{\mathrm{Spec}(W)} & \longrightarrow & A^{\mathrm{Spec}(W/\langle\mu\rangle)} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & A\otimes W & \longrightarrow & A\otimes(W/\langle\mu\rangle) & \longrightarrow & 0
\end{array}$$

then A *is a* $\mathcal{W}$*-vector space.*

PROOF. We proceed by induction on dim(W). Let

$$\mathcal{W}_k = \{W \in \mathcal{W} \mid \dim(W) \leq k\}.$$

Thus for A to be a $\mathcal{W}$-vector space is equivalent to being a $\mathcal{W}_k$-vector space for all $k \in \mathbf{N}$. We proceed by induction on k.

For $k = 1$, any $W \in \mathcal{W}_1$ is isomorphic to K, and thus Spec(W) = 1, so any R-module is a $\mathcal{W}_1$-vector space.

Now suppose we have shown that A is a $\mathcal{W}_k$-vector space. Let $W \in \mathcal{W}_{k+1} - \mathcal{W}_k$ and $\mu \in I_W^{\,n-1} - \{0\}$, where $I_W^{\,n} = 0$. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^{(\mathrm{Spec}(W/\langle\mu\rangle),\mathrm{Spec}(W))} & \longrightarrow & A^{\mathrm{Spec}(W)} & \longrightarrow & A^{\mathrm{Spec}(W/\langle\mu\rangle)} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & A\otimes W & \longrightarrow & A\otimes(W/\langle\mu\rangle) & \longrightarrow & 0
\end{array}$$

The right hand vertical is an isomorphism by our induction hypothesis, the left hand is an isomorphism by hypothesis, and thus by the 5-Lemma the middle vertical is too.

Thus, A is a $\mathcal{W}_{k+1}$-vector space, and by induction a $\mathcal{W}$-vector space.

We are now ready to prove

THEOREM 8. *If* $\mathbf{E}$ *is a Grothendieck topos,* $\mathcal{W}$ *a family of Weil algebras, closed under quotients, then* $\mathcal{W}$-v.s. *is a reflective subcategory of* R-mod.

PROOF. Replace $\mathcal{W}$ with a set of representatives for the isomorphism classes of Weil algebras represented in $\mathcal{W}$. Let $\mathcal{G}$ be a set of generators for $\mathbf{E}$, and γ be a regular cardinal strictly greater than $\sup_{G\in\mathcal{G}, W\in\mathcal{W}}(|\mathrm{Sub}(W\times G)|)$. Given any R-module A, we define a γ-indexed family of R-modules as follows: let $A_0 = A$; for successor ordinals, let $A_{\alpha+1}$ be the colimit of the diagram consisting of a single copy of A_α and a copy of each $A_\alpha^{(\mathrm{Spec}(W/\langle\mu\rangle),\mathrm{Spec}(W))}$'s with the maps induced on kernels as in the proof of Lemma 7; for limit ordinals, let A_λ be the colimit of the A_α's with $\alpha < \lambda$ and the evident inclusions.

We claim that A_γ is the $\mathcal{W}$-vector space reflection of A. It is obvious how to extend the above construction to maps between R-modules.

To verify the claim, note first that if A were a $\mathcal{W}$-vector space, then all the objects in the construction above are isomorphic to A by the maps used in defining the various colimits, and thus $A = A_\gamma$.

Now by Lemma 7 to show that A_γ is a $\mathcal{W}$-vector space, it suffices to show, for all (W,μ) as in Lemma 7, that the induced map $A_\gamma \to A_\gamma^{(\mathrm{Spec}(W/\langle\mu\rangle),\mathrm{Spec}(W))}$ is an isomorphism. Now note that A_γ is the colimit of a large diagram including all the A_α's and all the $A_\alpha^{(\mathrm{Spec}(W/\langle\mu\rangle),\mathrm{Spec}(W))}$'s. Noting that the A_α's and $A_\alpha^{(\mathrm{Spec}(W/\langle\mu\rangle),\mathrm{Spec}(W))}$'s (for our particular choice of (W,μ)) both form cofinal sub-diagrams, it suffices to show that the given map is the colimit of the maps

$$A_\alpha \to A_\alpha^{(\mathrm{Spec}(W/\langle\mu\rangle),\mathrm{Spec}(W))} \quad \text{for} \quad \alpha < \gamma.$$

The only difficulty with this is in showing that $A_\gamma^{(\mathrm{Spec}(W/\langle\mu\rangle),\mathrm{Spec}(W))}$ is the colimit of the $A_\alpha^{(\mathrm{Spec}(W/\langle\mu\rangle),\mathrm{Spec}(W))}$'s. Let

$$\phi\colon G \to A_\gamma^{(\mathrm{Spec}(W/\langle\mu\rangle),\mathrm{Spec}(W))}$$

be a map from one of the generators. Let $\underline{\phi}\colon G\times\mathrm{Spec}(W) \to A_\gamma$ be its exponential transpose (note that $\underline{\phi}|_{G\times\mathrm{Spec}(W/\langle\mu\rangle)}$ is identically 0). Now by the tininess of Spec(W), there are maps $\psi\colon \coprod G_\iota \twoheadrightarrow G$ for some coproduct of generators, and a map

$$\underline{\phi}'\colon (\coprod G_\iota)\times\mathrm{Spec}(W) \to A_\alpha \quad \text{for some} \quad \alpha < \gamma$$

such that

$$(\psi \times \mathrm{id}_{\mathrm{Spec}(W)})\underline{\phi} = \underline{\phi}' i_\alpha ,$$

where i_α is the canonical map from A_α to A_γ. If we knew that $\underline{\phi}'|_{(\coprod G_\iota)\times\mathrm{Spec}(W/\langle\mu\rangle)}$ were identically 0, we would be done by taking its exponential transpose. However this is not necessarily the case.

Now for any ordinal β with $\alpha < \beta < \gamma$, let ϕ_β denote the map $\underline{\phi}'\chi_{\alpha,\beta}$, where $\chi_{\alpha,\beta}$ is the map from A_α to A_β. Then

$$K_{\iota,\beta} = \phi_\beta^{-1} \cap (G_\iota\times\mathrm{Spec}(W/\langle\mu\rangle))$$

form a non-decreasing family of subobjects of $G_i\times\mathrm{Spec}(W/\langle\mu\rangle)$ indexed by the ordinal interval $[\alpha,\gamma)$. Let $K_i = \bigcup K_{i,\beta}$. First note that K_i must be all of $(G_i\times\mathrm{Spec}(W/\langle\mu\rangle))$ since otherwise the exponential transpose of $(\psi\times id_{\mathrm{Spec}(W)})\phi$ would not factor through $A_\gamma^{(\mathrm{Spec}(W)/\langle\mu\rangle),\mathrm{Spec}(W))}$. Now let

$$F_i = \{\beta \mid \alpha < \beta < \gamma,\ (\beta' < \beta) \Rightarrow K_{i,\beta'} \neq K_{i,\beta}\}.$$

Note that $|F_i| < \gamma$, since F_i is in 1-1 correspondence with a subset of $\mathrm{Sub}(G_i\times\mathrm{Spec}(W/\langle\mu\rangle))$. Moreover the cardinality of any ordinal interval $[\beta,\beta']$ for β and β' in F_i is less than γ, and thus $\lambda_i = \sup_{F_i}(\beta) < \gamma$, so $K_i = K_{i,\lambda_i}$. Now since $|\mathcal{G}| < \gamma$, $\lambda = \sup \lambda_i < \gamma$. Thus if Φ: $\coprod G_i \to A_\lambda^{\mathrm{Spec}(W)}$ is the exponential transpose of ϕ_λ: $\coprod G_i\times\mathrm{Spec}(W/\langle\mu\rangle) \to A_\lambda$, Φ factors through $A_\lambda^{(\mathrm{Spec}(W)/\langle\mu\rangle),\mathrm{Spec}(W))}$, and thus is a local lift for ϕ. Thus $A_\gamma^{(\mathrm{Spec}(W)/\langle\mu\rangle),\mathrm{Spec}(W))}$ is the colimit of the diagram of the $A_\alpha^{(\mathrm{Spec}(W)/\langle\mu\rangle),\mathrm{Spec}(W))}$'s.

It appears impossible to avoid this use of the axiom of choice in the metalanguage unless one can impose some sort of smallness restriction on the $\Omega^{\mathrm{Spec}(W)}$. In one case, however, a completely internal construction for the reflection functor can be given. This is the case of $\mathcal{W} = \{K, K[\epsilon]/\epsilon^2\}$, in a topos with natural numbers object, which was considered in Yetter [4]:

THEOREM 9. *If* $\mathbb{E}$ *has a natural numbers object and* $\mathcal{W} = \{K, K[\epsilon]/\epsilon^2\}$, *then* $\mathcal{W}$-v.s. *is a reflective subcategory of* R-mod.

PROOF. Observe that by Lemma 7, an R-module is a $\mathcal{W}$-vector space iff the map $A \to A^{(1,D)}$ transpose to multiplication is an isomorphism. Using the usual means of internalizing arguments involving the natural numbers, one can carry out the obvious inductive construction:

$$V(A) = \mathrm{colim}\ (A \longrightarrow A^{(1,D)} \longrightarrow A^{(1,D\vee D)} \longrightarrow \ldots),$$

where the maps are transposes to the restrictions of multiplication to D.

The critical stages in the internalized version are the construction of smash products indexed by internal finite cardinals, the use of the generic cardinal to construct an internal version of the colimit above, and demonstration that the resulting object and the object of pointed maps from D to it are isomorphic because the latter arises as the colimit of a cofinal subdiagram of the diagram defining the former.

We now examine $\mathcal{W}$-v.s. in the context of the closed symmetric monoidal structure on R-mod.

THEOREM 10. *The internal hom-functor,* $Hom_{R\text{-}mod}(-,-)$, *restricted to* $\mathcal{W}$-v.s., *is a* $\mathcal{W}$-v.s. *valued bifunctor whenever* $\mathcal{W}$ *is a class of* K-*Weil algebras closed under quotients. If, moreover,* $\mathcal{W}$-v.s. *is a reflective subcategory of* R-mod *with reflection functor* V, *then the bifunctors* $Hom_{R\text{-}mod}(-,-)$ *and* $V(-\otimes_R-)$ *provide a closed symmetric monoidal structure on* $\mathcal{W}$-v.s.

PROOF. The second statement follows easily from the first and the universal properties of the functors involved. For the first statement, note that there is an isomorphism between

$$Hom_{R\text{-}mod}(A,B)^{(Spec(W)/\langle\mu\rangle),Spec(W))}$$

and

$$Hom_{R\text{-}mod}(A,B^{(Spec(W)/\langle\mu\rangle),Spec(W))}),$$

whenever (W,μ) is as in the statement of Lemma 7. (Both may be canonically identified with the subobject of $B^{A\times Spec(W)}$ consisting of maps with are fibre-wise linear in A and identically 0 on $A\times Spec(W/\langle\mu\rangle)$.) This identification moreover identifies the two evident maps to Hom_R-mod(A,B), thus since B is a $\mathcal{W}$-vector space, this map is an isomorphism and thus by Lemma 7 $Hom_{R\text{-}mod}(A,B)$ is a $\mathcal{W}$-vector space.

3. MISCELLANEOUS RESULTS.

We conclude with several results concerning $\mathcal{W}$-vector spaces, one of which describes their behaviour under pullback functors (the preservation of $\mathcal{W}$-lines by slicing is a consequence of Freyd's Theorem (see Yetter [3] or [4]) on the preservation of tininess and of the fact that Δ_B is logical for any B.) The others deal with the "local character" of maps between them. Taken together these results suggest that $\mathcal{W}$-vector spaces could serve as model objects for the description of "finite and infinite dimensional manifolds" in the topos (these terms are used lososely since no obvious notion of dimension for $\mathcal{W}$-vector spaces presents itself).

Throughout the remainder of the paper we will assume that $\mathcal{W}$ is closed under quotients, and thus the characterization of $\mathcal{W}$-vector spaces given in Lemma 7 holds.

THEOREM 11. *If* $X \to B$ *is a* $\mathcal{W}$*-vector spaces in* $\mathbf{E}/B$ *for* B *a well-supported object in* $\mathbf{E}$*, then* $\Pi_B(X \to B)$*, the object of global sections, is an* R*-vector space in* $\mathbf{E}$*.*

PROOF. The R-module structure is obvious, being the restriction of the R^B-module to "constant scalars". To see that the geometric condition is satisfied, consider the following sequence of natural bijections:

maps $\quad Z$ to $[\Pi_B(X \to B)]^{(\mathrm{Spec}(W)/\langle\mu\rangle),\mathrm{Spec}(W))}$ $\quad$ in $\mathbf{E}$,

commutative squares

$$\begin{array}{ccc} Z\times\mathrm{Spec}(W/\langle\mu\rangle) & \longrightarrow & 1 \simeq \Pi_B(B \to B) \\ {\scriptstyle id\times i}\downarrow & & \downarrow{\scriptstyle \Pi_B(0)} \\ Z\times\mathrm{Spec}(W) & \longrightarrow & \Pi_B(X \to B) \end{array} \qquad \text{in } \mathbf{E},$$

commutative squares

$$\begin{array}{ccc} \Delta_B(Z\times\mathrm{Spec}(W)/\langle\mu\rangle) & \longrightarrow & 1 \\ {\scriptstyle \Delta_B(id,i)}\downarrow & & \downarrow{\scriptstyle (0)} \\ \Delta_B(Z\times\mathrm{Spec}(W)) & \longrightarrow & (X \to B) \end{array} \qquad \text{in } \mathbf{E}/B,$$

maps $\quad \Delta_B(Z) \to (X \to B)$ $\quad$ in $\mathbf{E}/B$ by Lemma 7,

maps $\quad Z$ to $\Pi_B(X \to B)$ $\quad$ in $\mathbf{E}$. •

We also have

THEOREM 12. If A *and* B *are* $\mathcal{W}$*-vector spaces,* $K[\epsilon]/\epsilon^2 \in \mathcal{W}$, then

$$A \simeq D\langle A\rangle^{(1,D)} \quad \textit{(and similarly for B)}$$

and

$$\mathrm{Hom}_{R\text{-mod}}(A,B) \simeq D\langle B\rangle^{(1,D\langle A\rangle)},$$

where $D\langle X\rangle$ *denotes the union of the images of all pointed maps from* D *to* X *(for any pointed object* X*) and the isomorphism is induced by restriction to* $D\langle B\rangle$. (Note any pointed map from $D\langle B\rangle$ to A must factor through $D\langle A\rangle$.)

PROOF. By considering the isomorphisms between A and $A^{(1,D)}$ and between B and $B^{(1,D)}$ it is clear that any linear map $A \to B$ can be recovered from its restriction to D⟨A⟩, and thus that the restriction map is monic.

It thus suffices to show that any (internal) map ϕ: D⟨A⟩ → D⟨B⟩ has a *linear* extension via $(-)^{(1,D)}$. For multiplicativity, note that the identification of A with $D\langle A\rangle^{(1,D)}$ and of R with $R^{(1,D)} \simeq D^{(1,D)}$ identify scalar multiplication with composition, and thus multiplicativity of $\phi^{(1,D)}$ follows from the associativity of composition.

For additivity, we use generalized elements: Let a, $\alpha \in A$, then the isomorphism $A \simeq D\langle A\rangle^{(1,D)}$ identifies these with $a\hat{}$, $\alpha\hat{}$: $D \to A$ given by

$$a\hat{} = \lambda d.da \quad \text{and} \quad \alpha\hat{} = \lambda d.d\,\alpha.$$

Let F: $D^2 \to A$ be $\lambda(d,\delta).da + \delta\alpha$. Thus $\Delta F = a\hat{} + \alpha\hat{}$. Now $F\phi$: $D^2 \to B$ is of the form

$$\lambda(d,\delta)(db + \delta\beta + d\delta\gamma)$$

for some b, β, $\gamma \in B$. And thus

$$\phi^{(1,D)}(a\hat{}) = a\hat{}\,\phi = (\mathrm{id}_D,0)F\phi = \lambda d.db,$$
$$\phi^{(1,D)}(\alpha\hat{}) = \alpha\hat{}\phi = (0,\mathrm{id}_D) = \lambda d.d\beta,$$

and

$$\phi^{(1,D)}(a\hat{} + \alpha\hat{}) = (a\hat{} + \alpha\hat{})\phi = \Delta F\phi = \lambda d.db + d\beta = \lambda d.db + \lambda d.d\beta$$
$$= \phi^{(1,D)}(a\hat{}) + \phi^{(1,D)}(\alpha\hat{}).$$

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School of Mathematics
Institute for Advanced Study
PRINCETON, N.J. 08540. U.S.A.