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IMAGES IN CATEGORIES AS REFLECTIONS

by Hans EHRBAR and Oswald WYLER

RÉSUMÉ. Cet article, qui repose sur d'anciens travaux non publiés, définit et étudie la notion d'image "globale" d'un morphisme f dans une catégorie C, *relativement à une classe* $\mathcal{M}$ de morphismes comme étant une réflection de f vers $\mathcal{M}$ dans la catégorie des carrés commutatifs de C. Ces images sont comparées à diverses notions d'images "locales" proposées dans la littérature, en particulier dans le cas d'images obtenues à partir de factorisations quotient-image avec une propriété diagonale. La dernière section donne un théorème général d'existence d'images, et quelques exemples.

The present paper is based on joint work by the two authors, carried out in 1968 and 1969. Due to circumstances beyond the authors' control, this work was never published, except as a preliminary technical report [3] and a preprint [4]. Urged to do so by friends and colleagues, the second author has revised the 1969 paper for publication, adding some later results of his own and an important result of the first author's thesis [2]. Since contact between the first author and the mathematical community has been lost for some time, the present paper was written by the second author, who is entirely responsible for any errors and omissions which it may contain.

INTRODUCTION.

Images in a category were first defined by Grothendieck, in a footnote of his Tôhoku paper [6]. This defines the image of a morphism $f\colon A \to B$ in a strictly local and purely categorical manner, as the smallest subobject of B through which f factors, with a subobject of B defined as an equivalence class of monomorphisms with codomain

B. Since monomorphisms and epimorphisms often do not produce the "correct" subobjects and quotient objects, Mac Lane [14] considered subobjects and quotient objects axiomatically, as classes of monomorphisms and epimorphisms with certain properties. This theory was generalized by Isbell [9], Sonner [15], Kennison [12] and others. Isbell and Kennison showed that subobjects and quotient objects determine each other in a category with coimage-image factorizations; this led to subobject embeddings which need not be monomorphic, and quotient maps which need not be epimorphic.

The step from local to global subobjects was taken by Jurchescu and Lascu [10] who defined "strict" monomorphisms and epimorphisms, and by Kelly [11] who called these monomorphisms and epimorphisms "strong". Kelly's terminology has prevailed. Freyd and Kelly [5] considered "global" coimage-image factorizations in full generality. Since then, almost all category theorists whose work requires images or coimages have followed their lead, obtaining global images or coimages from coimage-image factorizations. One exception is [13] which rediscovered images and coimages as defined in [4], with some new properties.

The example of normal subgroups in the category of groups shows that subobjects and images cannot always be obtained from coimage-image or quotient map-image factorizations. Other applications suggested strongly that images need not be always monomorphic, and coimages not always epimorphic. It also became clear to the authors of this paper that locally defined images and coimages need not be sufficiently well behaved to be useful. This led both of us, independently, to define $\mathcal{M}$-images in a category C globally, as certain reflections for $\mathcal{M}$ in the category C^2 of commutative squares over C, for an arbitrary class $\mathcal{M}$ of morphisms of C. Coimages for a class E of morphisms of C are then defined dually.

The global definition of images as reflections is strictly stronger than the original local definition. On the other hand, it is more general than the global definition of images by factorizations with a diagonal property. Images obtained by factorizations are images as defined in this paper, but there are a number of examples of images in our sense, obtained "in nature", which cannot be obtained from factorizations. Images obtained as reflections, for an "image-closed" class $\mathcal{M}$ of morphisms, have all the nice categorical properties that one can hope for.

Section 1 of the present paper defines images and coimages in a category C, relative to arbitrary classes $\mathcal{M}$ and E of morphisms of C which serve as subobject embeddings or quotient maps. Images and coimages are defined as dual concepts; everything in this paper can be dualized. The nice categorical properties of images in a category with images are obtained more generally as properties of an "image closure" M^{im}; this concept and its properties are due to Ehrbar [2]. We also show that the "local" images used by many authors are in fact "global" images, as defined in this paper, under very mild conditions. Answering a question posed by the referee, we express image closure as a closure for the orthogonality relation defined by W. Tholen in [17].

Section 2 considers the case of images defined in terms of a class E of "quotient maps", with $M = E^{st}$, the class of E-strong morphisms. We show that E^{st} is "image-closed"; this class is also closed under composition. E-strong images are the images obtained from quotient map-image factorizations with a diagonal property. The theory of such factorizations is self-dual; unique (E,M)-factorizations with E and M closed under compositions, or equivalently with a diagonal property, provide M-images and E-coimages. We also obtain a strong converse: if (E,M)-factorizations exist and provide local M-images and local E-coimages, then E and M are closed under composition, the factorizations are unique, and the local images and coimages are in fact global.

Section 3 begins with a general existence theorem for images which is due to Ehrbar [2]. This theorem, previously unpublished, contains all existence theorems for images and coimages, obtained later by other authors, as more or less special cases. We also give some standard examples, a counter-example, and two applications which provide further examples. One of these applications, normal images and coimages, obtained from kernels and cokernels, provided the original motivation for this paper. The other application, perfect images, was developed by the second author. It provides a categorical definition and theory of perfect maps which is substantially different from that of [7] and [16], and much closer to the definition of perfect maps in topology.

Our list of references is deliberately small, and with one exception, mentioned above, we have not tried to establish connections between our theory and recent contributions to the theory of factorizations. Additional references may be found in the papers cited above, and the recent papers [17], [1] and [8] can serve as guides to the recent literature. It would certainly be useful to provide a unifying overview of the literature on factorizations, but this would be beyond the scope and aim of the present paper.

1. IMAGES, COIMAGES AND IMAGE CLOSURE.

1.1. **NOTATIONS.** Throughout this paper, C will denote a category, and E and M will be arbitrary classes of morphisms of C. For $f\colon A \to B$ in C, we put $\alpha f = \mathrm{id}_A$ and $\beta f = \mathrm{id}_B$, calling αf the *source* and βf the *target* of f.

We denote by C^2 the category of morphisms and commutative squares of C. Objects of C^2 are all morphisms of C, and a morphism $(u,v)\colon f \to g$ of C^2 is a quadruple $(u,v;\, f,g)$ of morphisms of C such that $vf = gu$ in C.

By a *reflection* for a class A of objects of C, we mean a reflection for the full subcategory of C with objects in A, i.e., a morphism $r\colon C \to R$ of C with codomain in A, such that every $f\colon C \to A$ with codomain in A factors uniquely in C as $f = f'r$. We call A *reflective in* C if every object of C admits a reflection for A. Coreflections, and coreflective classes of objects, are defined dually.

For a class K of morphisms of C, we denote by K^r the class of all compositions kx in C with $k \in K$ and x an isomorphism of C, and K^λ is defined dually. For convenience, we shall put $K(f) = K \cup \{f\}$ for a morphism f of C.

1.2. **DEFINITIONS.** We define an *M-image* of a morphism f of C as a pair (p,j) of morphisms of C such that:

(i) $f = jp$ in C and $j \in M$,

and (ii) whenever $vf = mu$ in C with $m \in M$, then $u = tp$ and $vj = mt$ in C for a unique morphism t of C.

We say that C has *M-images* if every morphism of C has an *M-image*.

Dually, we say that (p,j) is an *E-coimage of* f in C if (j,p) is an E-image of f in C^{op}, i.e., if

(i) $f = jp$ in C and $p \in E$,

and (ii) whenever $fu = ve$ in C with $e \in E$, then $v = jt$ and $pu = te$ in C for a unique morphism t of C.

We say that C has *E-coimages* if every morphism of C has an E-coimage.

A *local M-image* (p,j) of f in C must satisfy (i) in the definition of an M-image, and (ii) for the special case that $v = \beta f$, and $f = mu$. *Local E-coimages* are defined dually.

Examples will be given in Section 3.

1.3. PROPOSITION. *M-images have the following properties.*

(1) A pair (p,j) is an M-image of a morphism f of C iff $(p, \beta f)$: $f \to j$ is a reflection for M in C^2.

(2) If $m \in M$, then $(\alpha m, m)$ is an M-image of m.

(3) If f has an M-image (p,j), then a pair (q,h) with $h \in M$ is an M-image of f iff $h = hx$ and $p = xq$, for a (unique) isomorphism x of C.

(4) Every M-image is an M^r-image. Conversely, if f has an M^r-image, then f has an M-image.

(5) An isomorphism u of C has an M-image (p,j) iff $u \in M^r$, and then p and j are isomorphisms.

PROOF. We omit the easy proofs of (1)-(4). If $u = jp$, with p and j isomorphic and $j \in M$, then (p,j) clearly is an M-image of the isomorphism u. Conversely, if an isomorphism u has an M-image (p,j), put $t = pu^{-1}j$. Then $tp = p$ and $jt = j$; thus $t = \alpha j$, and p and j are isomorphisms.

E-coimages have the dual properties; we note only:

(1*) *A pair (p,j) is an E-coimage of a morphism f of C iff $(\alpha f, j)$: $p \to f$ is a coreflection for E in C^2.*

1.4. Many authors have defined images as local images. The following result shows that, in most applications, local images are in fact images.

PROPOSITION. *If every pair (f,m) of morphisms of C with $\beta f = \beta m$ and $m \in M$ allows a pullback $mf' = fm'$ with $m' \in M$, then every local M-image is an M-image.*

PROOF. Let (p,j) be a local M-image of f, and consider $vf = mu$ with $m \in M$. If $vm' = mv'$ is a pullback with $m' \in M$, then $f = m'r$ and $u = v'r$ for a unique r, and then $r = sp$, $j = m's$ for a unique s. Now it is easily seen that $u = tp$ and $vj = mt$ iff $t = v's$; thus (p,j) is an M-image of f.

1.5. DEFINITIONS. We define the *image closure* M^{im} of M in C as the class of all morphisms m in C such that every K-image is also a $K(m)$-image, for every class K of morphisms of C which contains M.

The *coimage closure* E^{coim} of E is defined dually.

We note that it suffices to test the defining property of M^{im} for classes $K = M(k)$; but our more general formulation is usually easier to use.

Following V. Tholen [17], we put $(p,j)^{\perp}m$, for morphisms m, p, j of C such that jp is defined in C, if for every commutative square $mu = vj$ in C, there is a unique morphism t in C such that $u = tp$ and $vj = mt$. For a class M of C, we denote by $M^{\perp}$ the class of all composable pairs (p,j) such that $(p,j)^{\perp}m$ for every m in M, and for a class F of composable pairs of morphisms, we denote by $F_{\perp}$ the class of all m in M such that $(p,j)^{\perp}m$ for every (p,j) in F. Then (p,j) is an M-image if $(p,j) \in M$ and $m \in M$, and (p,j) is an $M(j)$-image if $(p,j) \in M$ and $(p,j)^{\perp}j$. If we restrict the domain of $^{\perp}$ to pairs (p,j) which are at least $\{j\}$-images, i.e., for which $(p,j)^{\perp}j$, then it is easily observed that $M^{im} = (M^{\perp})_{\perp}$.

1.6. PROPOSITION. *Image closures in C have the following properties.*

(1) $M \subset M^{im}$, and every M-image is an M^{im}-image.

(2) If $M \subset M_1$, then $M^{im} \subset M_1{}^{im}$.

(3) $(M^{im})^{im} = M^{im}$.

(4) M^r is the class of all f in M^{im} with an M-image.

(5) If C has M-images, then $M^{im} = M^r$.

(6) M^{im} contains all isomorphisms of C.

(7) If M consists of isomorphisms of C, then M^{im} is the class of all isomorphisms of C.

PROOF. (1) and (2) follow immediately from the definitions, and (3) follows from the last observation preceding the proposition.

If $m \in M^r$ and $M \subset K$, then $m \in K^r$. Thus a K-image is a $K(m)$-image, and $m \in M^{im}$, with an M-image by 1.3. Conversely, if f in M^{im} has an M-image (p,j), then (p,j) and $(\alpha f, f)$ are M^{im}-images of f. But then p is an isomorphism of C by 1.3, and $f = jp$ is in M^r.

(5) follows immediately from (4), and (6) from the definitions.

If K is the class of all isomorphisms of C, then clearly $K \subset M^{im}$ for every M. On the other hand, every morphism f of C has a K-image $(f, \beta f)$; thus $K = K^r$ by (5). As $K^r = K$, (7) follows.

1.7. PROPOSITION. *If hg is defined in C, with hg and h in M^{im}, then $g \in M^{im}$. In particular, $M^{\lambda} \subset M^{im}$.*

PROOF. If (p,k) is a K-image of f, with $M \subset K$, and if $vf = gu$, then $hvk = hgt$ and $tp = u$ for a unique t since $hg \in M^{im}$. Then also $vkp =$

gtp, hence $vk = gt$ since (p,k) is a $K(h)$-image. Thus (p,k) is a $K(g)$-image, and $g \in M^{im}$.

For the second part, put $g = xm$ and $h = x^{-1}$, with $m \in M$ and x an isomorphism of C.

1.8. PROPOSITION. *If $mf' = fm'$ is a pullback square in C with $m \in M^{im}$, then $m' \in M^{im}$.*

PROOF. Let (p,j) be a K-image of g, with $M \subset K$, and let $m'u = vg$. Then $f'u = sp$ and $ms = fvj$ for a unique s, and $f't = s$ and $m't = vj$ for a unique t. It follows easily that $tp = u$, and that $t'p = u$ and $m't' = vj$ iff $t' = t$. Thus (p,j) is a $K(m')$-image, and $m' \in M^{im}$.

1.9. We consider diagrams D and D_1 in C with the same scheme, and with limit cones $u: L \to D$ and $v: L_1 \to D_1$ in C. For a natural transformation $\mu: D \to D_1$, there is then a unique morphism m of C, the *limit of* μ, with $vm = \mu u$.

The most important example of this is a product $\prod f_i: \prod A_i \to \prod B_i$ of morphisms, for diagrams with a discrete scheme. Another important example is an *intersection* of subobjects, obtained in the case that D_1 is a constant diagram, and the components of μ are in M.

PROPOSITION. *If diagrams* D *and* D_1 *in C with the same scheme have limits* L *and* L_1 *in C, and if $\mu: D \to D_1$ is a natural transformation with all components in M^{im}, then the limit $m: L \to L_1$ of μ is in M^{im}.*

PROOF. Let m_i be the components of μ, and u_i and v_i the projections of the two limits, so that $m_i u_i = v_i m$ for all i. If a morphism f has a K-image (p,j) with $M \subset K$, and if $yf = mx$, then we need a unique t such that $tp = x$ and $mt = yf$. This t is uniquely determined by morphisms $t_i = u_i t$, which must satisfy

$$t_i p = u_i x \quad \text{and} \quad m_i t_i = v_i yj.$$

These equations determine t_i uniquely if $m_i \in M^{im}$; thus $m \in M^{im}$ if every component of μ is in M^{im}.

1.10. If C has M-images, then M is reflective in C^2. On the other hand, $\{id_T\}$ is reflective in C^2 if C has a terminal object T, but C does not have $\{id_T\}$-images if C is not trivial. The following result answers the question which this example raises.

PROPOSITION. *C has M-images iff M is reflective in C^2, and M^r contains M^λ and all identity morphisms of C.*

PROOF. The conditions are necessary by 1.3 (1), 1.6 and 1.7.

Assume now that $(p,q): f \to j$ is a reflection for M, hence also for M^r, in C^2. If $\beta f \in M^r$, then there is $(u,v): j \to \beta f$ such that

$$(u,v)(p,q) = (f, \beta f): f \to \beta f$$

in C^2, with $u = vj$ and $vq = \beta f$ in C. Then

$$(qvj,qv)(p,q) = (j, \beta j)(p,q): f \to \beta j$$

in C^2, and $qv = \beta j$ follows if βj follows if $\beta j \in M^r$. Now v and q are inverse isomorphisms, with vj in M^λ. If $vj = kx$ with x isomorphic and $k \in M$, then

$$(x,v)(p,q) = (xp, \beta f): f \to k$$

is a reflection for M in C^2, and (xp,k) is an M-image of f by 1.3 (1).

2. COIMAGE-IMAGE FACTORIZATIONS.

2.1. **DEFINITIONS.** We recall that a morphism u of C is called *E-strong* if for every commutative square $uf = ge$ with $e \in E$, there is a unique morphism t in C such that $f = te$ and $g = ut$. Dually, we say that v is *M-costrong* in C if v is M-strong in C^{op}. We denote E^{st} the class of all E-strong morphisms of C, and by M^{cost} the class of all M-costrong morphisms.

2.2. **PROPOSITION.** *$M \subset E^{st}$ iff $E \subset M^{cost}$.*

This follows immediately from the definitions.

Thus "strong" and "costrong" define a contravariant Galois correspondence for classes of morphisms of C, with the usual properties of such a correspondence.

2.3. Our next result connects E-strong morphisms with images.

PROPOSITION. *E^{st} has the following properties.*

(1) A morphism u of C is E-strong iff $(\alpha u,u)$ is an $E(\alpha u)$-coimage of u.

(2) All isomorphisms of C are E-strong, and $E \cap E^{st}$ is the class of all isomorphisms in E.

(3) If $f = jp$ with $p \in E$ and $j \in E^{st}$, then (p,j) is an E-coimage and an E^{st}-image of f.

(4) E^{st} is closed under composition.

(5) $(E^{st})^{im} = E^{st} = (E^{coim})^{st}$.

PROOF. (1) and (3) follow immediately from the definitions, and (2) and (4) are well known.

If $u \in (E^{st})^{im}$ and $e \in E$, then the E^{st}-image $(e, \beta e)$ is also an $E^{st}(u)$-image; it follows that u is E-strong. Using the duals of this, and of 1.6, we have

$$M \subset (E^{coim})^{st} \Leftrightarrow E^{coim} \subset M^{cost} \Leftrightarrow E \subset M^{cost} \Leftrightarrow M \subset E^{st}$$

for a class M of morphisms; thus $(E^{coim})^{st} = E^{st}$.

2.4. DEFINITIONS. We say that C *has (E,M)-factorizations* if every morphism f of C factors $f = jp$ with $p \in E^{\lambda}$ and $j \in M^{r}$. We can assume $p \in E$ or $j \in M$, but not both unless $E = E^{\lambda}$ or $M = M^{r}$. We say that two factorizations $f = jp$ and $f = j'p'$ are *equivalent* if $p' = xp$ and $j = j'x$ for an isomorphism x of C, and we say that *(E,M)-factorizations are unique* if two (E,M)-factorizations of a morphism f of C always are equivalent.

We mean by $MM \subset M^{r}$ that a composition uv with u and v in M is always in M^{r}, and $EE \subset E^{\lambda}$ is defined dually.

2.5. THEOREM. *The following are logically equivalent.*

(1) C has (E,M)-factorizations, (E,M)-factorizations are unique, $EE \subset E^{\lambda}$, and $MM \subset M^{r}$.

(2) C has (E,M)-factorizations, and $M \subset E^{st}$.

(3) C has (E,M)-factorizations, and $E \subset M^{cost}$.

(4) C has E-images, and every M-image is an E^{λ}-coimage.

(5) C has E-coimages, and every E-coimage is an M^{r}-image.

(6) C has M-images, $MM \subset M^{r}$, and *$E^{\lambda} = M^{cost}$.*

(7) C has E-coimages, $EE \subset E^{\lambda}$, and *$M = E^{st}$.*

(8) C has (E,M)-factorizations; if $f = jp$ is a (E,M)-factorization, then (p,j) is always a local M^{r}-image and a local E^{λ}-coimage of f.

We say that *(E,M)*-factorizations are *coimage-image factorizations* if these conditions for E and M are satisfied.

PROOF. See e.g. [5] for the implication (1) ⇒ (2).

If (2) holds, then every f in C factors $f = jp$ with $j \in M$ and $p \in E^{\lambda}$, with (p,j) an M-image and an E^{λ}-coimage since $M \subset E^{st} = (E^{\lambda})^{st}$. By 1.3 (3) and its dual, it follows that every M-image of f is an E^{λ}-coimage; thus (4) holds.

Assume now (4), with all isomorphisms in M^r by 1.6. If $e \in M^{cost}$, then $(e, \beta e)$ is an M^r-image, hence an E^{λ}-coimage, and $e \in E^{\lambda}$. Conversely, if $e \in E^{\lambda}$, with M-image (p,j), then j is isomorphic since $(e, \beta e)$ and (p,j) are E^{λ}-coimages. Thus $(e, \beta e)$ is an M^r-image, and $e \in M^{cost}$.

Now if (p,j) is an M-image of vu, with u and v in M, then $p \in E^{\lambda}$ and $j = vs$, $sp = u$, for a unique morphism s. Since $(\alpha u, u)$ is an E^{λ}-coimage, $tp = \alpha u$ and $ut = s$ for a morphism t. But then

$$ptp = p \quad \text{and} \quad jpt = vut = vs = j,$$

and $pt = \beta p$ follows. Thus p is isomorphic, and vu in M^r.

If (6) holds and $f = pj$ is an (E,M)-factorization, then (p,j) is an M^r-image and an E^{λ}-coimage since $E^{\lambda} = M^{cost}$. If (p,j) is an M-image of f and (q,h) an M-image of p, then $j = jht$, $tp = q$ for a morphism t as jh is in M^r. As $htp = p$, it follows that $ht = \beta h$. Now $hth = h$ and $thq = q$, so that $th = \alpha h$, and h is an isomorphism. But then p is M-costrong, and $f = jp$ is an (E,M)-factorization, so that (8) holds.

If (8) is valid, then (E,M)-factorizations are unique. If an isomorphism u has an (E,M)-factorization $u = jp$, then $jt = j$ and $tp = p$, hence $t = \beta p$, for $t = pu^{-1}j$. Thus p and j are isomorphic, and $M \cap E^{\lambda}$ contains all isomorphisms. Now $MM \subset M^r$ is obtained as in the proof of (4) ⇒ (6), and $EE \subset E^{\lambda}$ by the dual argument. Thus (1) holds.

(1) and (8) are self-dual, and (2) through (7) dual in pairs, with (2) ⇔ (3) by 2.2. We obtain (3) ⇒ (5) ⇒ (7) ⇒ (8) by arguments dual to those above; this completes the proof.

3. EXAMPLES AND COMPLEMENTS.

3.1. In order to formulate a solution set condition for the existence of M-images in C, we construct for every morphism f of C a category $\mathrm{Fact}_M(f)$ of M-factorizations of f as follows.

Objects of $\mathrm{Fact}_M(f)$ are all pairs (g,m) of morphisms of C such that $f = mg$ in C and $m \in M^r$. A morphism $x: (g,m) \to (g',m')$ of $\mathrm{Fact}_M(f)$ is a morphism x of C such that

$$m = m'x \text{ and } gx = g' \text{ in } C, \text{ with } \mathrm{id}_{(g,m)} = \alpha m = \beta g.$$

Composition in $\mathrm{Fact}_M(f)$ is composition in C.

A local M-image of f is an initial object of the category $\mathrm{Fact}_M(f)$ just constructed, and $(f, \beta f)$ is a terminal object of $\mathrm{Fact}_M(f)$ if βf is in M^r.

We define a functor $H: \mathrm{Fact}_M(f) \to C$ by putting $H(g,m) = C$ if $\alpha m = C$, and $Hx = x$ for a morphism x of $\mathrm{Fact}_M(f)$. Putting $h_{(g,m)} = m$ defines a cone $h: H \to \Delta B$ if $\beta f = \mathrm{id}_B$.

3.2. THEOREM. *If C is complete and has small hom sets, then C has M-images iff M satisfies the following conditions.*

(1) M^r is closed under intersections of small cones (see 1.9).

(2) M^r is closed under pullbacks by morphisms of C.

(3) For every morphism f of C, there is a set S *of objects of* $\mathrm{Fact}_M(f)$ *such that every object of* $\mathrm{Fact}_M(f)$ *is the codomain of at least one morphism of* $\mathrm{Fact}_M(f)$ *with domain in* S.

This theorem is due, in a somewhat different form, to Ehrbar [2]. The theorem and its dual include all known existence theorems for images or coimages, and for coimage-image factorization if the condition $MM \subset M^r$, or $EE \subset E^\lambda$ for the dual, is added.

The solution set condition (3) is always satisfied if C is M-wellpowered. We also note that M^r contains all isomorphisms if (1) is satisfied; id_A is an intersection of the empty cone of morphisms in M with codomain A, for every object A of C.

PROOF. Conditions (1) and (2) are necessary by 1.8 and 1.9, and (3) is necessary since an M-image of f is an initial object of $\mathrm{Fact}_M(f)$.

Now let $f: A \to B$, and let S be the full subcategory of $\mathrm{Fact}_M(f)$ with S as its set of objects, adding (f,id_B) to S if necessary to make S connected. We denote the objects of S by $\sigma = (g_\sigma, m_\sigma)$. The category S is small; we denote by H_S and h_S the restrictions of H and h to S.

The functor H_S and the cone h_S have limits J and j in C, with $j: J \to B$ in M^r by (1). We can and shall take j to be in M. The projections p_σ of the limit of H_S satisfy $j = m_\sigma p_\sigma$, and $p_\sigma = xp_\rho$ for each $x: \rho \to \sigma$ in S. The morphisms g_σ of C define a natural transformation $\Delta A \to H_S$; thus $g_\sigma = p_\sigma q$, for all σ in S, for a unique q in C, with $p_\sigma: (q,j) \to (g_\sigma, m_\sigma)$ in $\mathrm{Fact}_M(f)$ for every σ. We claim that (q,j) is the desired M-image of f.

By our construction, there is a morphism $(q,j) \to (g,m)$ in $\mathrm{Fact}_M(f)$ for every object (g,m) of $\mathrm{Fact}_M(f)$. If x and y are two such morphisms, let e be an equalizer of x and y and C. The morphisms j and m are the components of a cone with the pair of morphisms x and

y as domain. The intersection of this cone is je ; thus je is in M^r by (1). Since $xq = g = yq$, there is q' in C with $eq' = q$, hence with $e: (q',je) \to (q,j)$ in $\mathrm{Fact}_M(f)$. There is $r: (g_\rho, m_\rho) \to (q',je)$ in $\mathrm{Fact}_M(f)$ for some $\rho \in S$, and then $p_\sigma er: \rho \to \sigma$ in S, hence $p_\sigma erp_\rho = p_\sigma$, for all $\sigma \in S$. But then $erp_\rho = \beta j$, and $x = y$ follows. Thus (q,j) is a local M-image of f ; this is an M-image by condition (2) and 1.4.

3.3. EXAMPLES. In every category C, we obtain coimage-image factorizations by taking either $M^r = C$, with E^λ all isomorphisms of C, or $E^\lambda = C$, with M^r all isomorphisms of C.

In set-based algebraic categories, we obtain the traditional coimage-image factorization by taking E^λ to be the class of all surjective homomorphisms, and M the class of all subalgebra inclusions.

In the category of topological spaces, the category of Hausdorff spaces, and in other topological categories such as uniform spaces or convergence spaces, there are several useful (and well known) factorizations, as follows.

(1) E all bijective maps, M all coarse maps.

(2) Dually, E all fine maps, M all bijective maps.

(3) E all quotient maps, M all injective maps.

(4) Dually,, E all surjective maps, M all embeddings.

(5) E all dense maps, M all closed embeddings.

3.4. EXAMPLE. We give an example of unique factorizations which are not coimage-image factorizations. This example also shows the necessity of some of the conditions in 2.5; all conditions in 2.5 can be "nailed down" by such examples.

Let $C = \{0,1,2\}$, ordered by $0 \leq 1 \leq 2$, considered as a category with morphisms $x \to y$ for $x \leq y$. Let E consist of $0 \to 2$, and all identity morphisms $x \to x$, and let M consist of all morphisms except $0 \to 2$.

Every morphism f has exactly one (E,M)-factorization $f = jp$, and then (p,j) is a local E-coimage of f. However, the local E-coimage $(1 \to 1,\ 1 \to 2)$ is not an E-coimage of $1 \to 2$.

C has M-images, but the M-image $(0 \to 1,\ 1 \to 2)$ of $0 \to 2$ is not the E-coimage $(0 \to 2,\ 2 \to 2)$ of $0 \to 2$.

We note that E is closed under composition; M is not. M^{cost} consists of all identity morphisms, and $(M^{cost})^{st} = C$. On the other side, E^{st} consists of the identity morphisms and of $0 \to 1$, and $(E^{st})^{cost}$ is E with $1 \to 2$ added.

3.5. PERFECT IMAGES. We consider a reflective class A of objects of C, with reflections r_X: X → RX. Reflections define a functor R on C, given by $Rf.r_X = r_Y f$ for f: X → Y in C. We recall that f: X → Y in C is called (strongly) *A-perfect* if $r_Y f = Rf.r_X$ is a pullback square in C, and *uniquely A-extendible* if for every u: X → A in C with A $\in$ A there is a unique v: Y → A in C such that $u = vf$. Clearly, f is uniquely A-extendible iff Rf is an isomorphism of C.

In our discussion of perfect images, we let E be the class of all uniquely A-extendible morphisms, H the class of all morphisms with domain and codomain in A, and M the class of all A-perfect morphisms. Then E and M contain all isomorphisms of C and are closed under composition in C. Moreover, E contains all reflections for A, and $H \subset M$. Thus every morphism of C with codomain in A has an *(E,M)-factorization*.

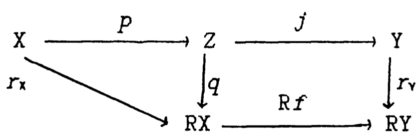
LEMMA. *If (p,j) is a K-image with $H \subset K$, then $p \in E$.*

PROPOSITION. *With E, H and M as above, the following statements are logically equivalent for a morphism f of C which factors f = jp in C.*

(1) $p \in E$ and $j \in M$.

(2) (p,j) is an M-image of f.

(3) There is in C a commutative diagram



with a pullback square on the right and q a reflection for A.

(4) (p,j) is an E-coimage of f, and $j \in M$.

Moreover, $E = H^{cost} = M^{cost}$, and $E^{st} = H^{im} = M^{im}$.

PROOF. Assume that f is $X \xrightarrow{p} Z \xrightarrow{j} Y$.

For the Lemma, we have Rf in H ; thus there is a morphism q for which the diagram of (3) is commutative. Then $q = zr_Z$ for a morphism z in H, with $z.\, Rp.r_X = qp = r_X$, hence $z.\, Rp = id_{RX}$. We have

$$Rp.q.p = Rp.r_X = r_Z p \quad \text{and} \quad Rj.\, Rp.q = r_Y j = Rj.r_Z,$$

hence $Rp.q = r_Z$ as $Rj \in H$. Now

$$Rp.z.r_Z = Rp.q = r_Z, \quad \text{hence} \quad Rp.z = id_{RZ}.$$

Thus Rp and z are inverse isomorphisms, and $p \in E$ follows.

If (1) holds, and $vf = mu$ with $m\colon A \to B$, then $Rm.s = r_B vj$ for $s = Ru.(Rp)^{-1}.r_Z$; thus there is t such that $r_A t = s$ and $mt = vj$. Then

$$r_A tp = sp = r_A u, \quad \text{and} \quad mtp = vf = mu,$$

hence $tp = u$. If also $mt' = vj$ and $t'p = u$, then $sp = Ru.r_X = r_A tp$, hence $r_A t' = s$ since $p \in E$. But then $t' = t$, and (2) holds.

If (2) is valid, then $p \in E$ by the Lemma, and the diagram in (3) commutes for $q = (Rp)^{-1} r_Z$; this is a reflection for A. Now $r_Y j = Rj.r_Z$ and $r_Z.\mathrm{id}_Z = Rp.q$ are pullback squares; thus the composite square $r_Y j = Rf.q$ is a pullback square.

If (3) holds, then $q = zr_Z$ for an isomorphism z, with $z.\,Rp.r_X = qp = r_X$. Thus $z.\,Rp = \mathrm{id}_{RX}$, and $p \in E$. Composing the pullback square $q.\mathrm{id}_Z = zr_Z$ with the pullback square in (3), we get $r_Y j = Rj.r_Z$ as a pullback square. Thus $j \in M$, and (1) holds.

Now (1) $\Leftrightarrow$ (2). We get $E = M^{cost}$ by applying this and the dual of 2.3 (1) to pairs $(p, \beta p)$, and (1) $\Leftrightarrow$ (4) follows immediately. Since $M \subset H^{im}$ by 1.8, and $H \subset M$, we have $H^{cost} = M^{cost}$ by the dual of 2.3.

Finally, $H^{im} \subset M^{im} \subset E^{st}$, by 2.1 and 2.3. Conversely, if (p,j) is a K-image with $H \subset K$, then $p \in E$ by the lemma; it follows that (p,j) is a $K(m)$-image if $m \in E^{st}$. Thus $E^{st} \subset H^{im}$, completing the proof.

3.6. DISCUSSION AND EXAMPLES. Herrlich and Strecker intensively studied (Q,Q^{st})-factorizations for Q the class of A-extendible epimorphisms of C ; see [7] and [16] for accounts of their work and further references. Their class Q^{st} of A-perfect morphisms can be much larger than our class M. The results of 3.5 above, in particular the construction of A-perfect images in 3.5 (3), are new.

By 3.5, C has A-perfect images iff every pullback of a reflection r_Y for A by a morphism Rf is a reflection for A. This is the case if C is the category of completely regular Hausdorff spaces or the category of completely regular topological spaces (without T_2), and A the class of compact Hausdorff spaces or of realcompact Hausdorff spaces. 3.5 also works if C = TOP, and A the class of sober spaces. The first example motivates the name "perfect", and the last example provides morphisms f with a coimage-image factorization $f = jp$ such that p is not epimorphic in C, and j not monomorphic.

The dual of 3.5 applies to coreflective classes A in a topological category C over sets. In this situation, coreflections for A are bijective maps $r_X\colon RX \to X$, with RX having the final structure for all morphisms $f\colon A \to X$ in C with A in A. It follows that a pushout of a corefletion r_X by a map Rf, with $\alpha f = \mathrm{id}_X$, is always a coreflection. Thus every morphism of C has an A-coperfect coimage.

3.7. NORMAL IMAGES. We consider a pointed category C, with a zero morphism 0_{AB}: A → B for every pair of objects A and B. An equalizer of f: A → B and of 0_{AB} is then called a *kernel of f*, and a *cokernel of f* is defined dually. An image for the class of all kernels is called a *normal image*, and a coimage for the class of cokernels is a *normal coimage*. We denote by ker f a "canonical" kernel of f in C if C has kernels, and dually by coker f a cokernel of f.

THEOREM. *If C is a pointed category with kernels and cokernels, then every morphism f of C has a normal image (p,k) and a normal coimage (q,j), with*

$$k = \text{ker coker } f \text{ and } q = \text{coker ker } f.$$

In this situation, f = kuq with (u,k) a normal image of j and (q,u) a normal coimage of p.

PROOF. Let 0 denote an arbitrary zero morphism in C. If g = coker f, then $gf = 0$; thus f factors $f = kp$ for k = ker g. If $vf = mw$ for m = ker s, then $svk = 0$; thus $sv = v'g$ for a morphism v'. Now $svk = 0$, hence $vk = mt$ for a morphism t. As m is monomorphic, t is unique, and $tp = w$. Thus (p,k) is a normal image of f. Dually, f has a normal coimage (q,j), with q = coker ker f.

Since k is mono, $fx = 0$ iff $px = 0$, and so f and p have the same kernels. Thus p has a normal coimage (q,u), with $ku = j$ since q is epi. Dually, (u,k) is a normal image of j.

3.8. EXAMPLES. We note that our present definition of normal images and normal coimages is somewhat more restrictive than that of [18]. The two definitions are equivalent if every morphism of C has a kernel and a cokernel, and normal images and coimages as defined in [18] are always images and coimages as defined in this paper.

Our examples refer to concrete pointed categories with finite limits and colimits, and with an underlying set functor which preserves finite limits. In this situation, there is a zero object Z with a singleton as underlying set. This object can be regarded as a subobject and a quotient object of every object, and ker f is an embedding $f^{\leftarrow}(Z)$ → A for f: A → B in C. If cokernels in C are surjective, it follows easily that a composition of cokernels is always a cokernel. Cokernels are surjective in all our examples.

For pointed sets, every injective morphism is a kernel. A surjective morphism f: A → B is a cokernel if it sends a subset of A to the base point of B and is bijective outside of that subset.

For groups, an injective homomorphism $f\colon A \to B$ iff $f^{\leftarrow}(A)$ is a normal subgroup of B; thus a composition of kernels is not always a kernel. Every surjective group homomorphism is a cokernel.

For meet semilattices, an injective homomorphism $f\colon A \to B$ is a kernel iff $f^{\leftarrow}(A)$ is a filter in B. Not every surjective homomorphism is a cokernel. For example if ϕ is a filter in a meet semilattice A, but not all of A, then the characteristic function of ϕ is a surjective homomorphism $A \to \{0,1\}$, but in general not a cokernel. The composition of two kernels for meet semilattices is again a kernel.

Groups and meet semilattices define full subcategories of the category of monoids, and the embeddings preserve kernels and cokernels. Thus for monoids, the composition of kernels is not always a kernel, and not every surjective homomorphism is a cokernel.

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