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INTEGRABILITY OBSTRUCTIONS FOR EXTENSIONS OF LIE ALGEBROIDS

by Kirill MACKENZIE

RÉSUMÉ. L'auteur a démontré dans une autre publication [10] que, pour qu'un algébroïde de Lie transitif sur une base simplement connexe soit intégrable, il faut et il suffit qu'une certaine obstruction cohomologique à l'intégrabilité soit, en un certain sens, discrète. Dans le présent article, on examine le problème de l'intégrabilité pour les extensions des algébroïdes de Lie transitifs, et on démontre que ce problème équivaut au problème d'intégrabilité pour un certain algébroïde de Lie associé. La démarche cruciale est de démontrer qu'une extension de fibrés principaux (ou de groupoïdes de Lie) équivaut à un seul groupoïde 'transverse' équipé d'un groupe d'opérateurs.

This paper gives necessary and sufficient conditions for the integrability of an extension of transitive Lie algebroids of the form

(1) $$K \rightarrowtail A' \twoheadrightarrow A\Omega$$

where K is a Lie algebra bundle and Ω an α-connected Lie groupoid. The method used is to show that the integrability of (1) is equivalent to the integrability of a certain inverse image Lie algebroid, whose base is the universal cover of the total space of the principal bundle corresponding to Ω, and then to apply the integrability results of [10]. In particular we recover a result of Almeida and Molino [2] in which the integrability problem for transitive Lie algebroids on multiply connected base spaces was reconducted to that for simply-connected bases.

The key to the equivalence between the integrability of (1) and that of the inverse image Lie algebroid is a concept which we have called "PBG-groupoid", and its infinitesimal version, "PBG-Lie algebroid". A PBG-groupoid is a Lie groupoid T whose base is the total space of a principal bundle P(B, G) and on which G acts by Lie groupoid automorphisms. It is shown in §§1,2 that PBG-groupoids on

P(B, G) are naturally equivalent to principal bundle extensions of P(B, G). An important technical step in this process is Theorem 2.2 which establishes that the (almost) purely algebraic conditions by which the PBG-groupoid structure on a Lie groupoid T is defined are sufficient to make T itself into a principal bundle with respect to the action of G. It seems certain that this equivalence between PBG-groupoids and extensions will be central to a future cohomological classsification of extensions of principal bundles.

Extensions such as (1) arise as infinitesimal linearizations of extensions of principal bundles or Lie groupoids. As one always expects of a linearization, their theory is more tractable than that of the extensions which they approximate; the extension theory and cohomology theory of transitive Lie algebroids exist and one can, for example, classify extensions of the form

$$P{\times}V/G \rightarrowtail A' \twoheadrightarrow A\Omega$$

where Ω corresponds to a principal bundle P(B, G) and V is a G-vector space, by equivariant de Rham cohomology $H^2(P, V)^G$ ([10], IV §2). There is also a spectral sequence for the cohomology of $A\Omega$, with arbitrary coefficients, which can be used to analyze the geometric properties of extensions of $A\Omega$ ([10], IV §5). The integrability criteria given here provide a method by which such results on extensions of Lie algebroids can be forced to yield information on extensions of Lie groupoids, or of principal bundles. The consequences of this point of view will be developed further elsewhere.

The first two sections of the paper present the equivalence between PBG-groupoids on P(B, G), and extensions of P(B, G) or its associated groupoid $P{\times}P/G$. We have included slightly more than is strictly necessary for the results of §4, having in mind later applications elsewhere. Section 3 deals, more briefly, with the relevant corresponding results for Lie algebroids. The integrability criteria themselves are in Section 4.

1. EXTENSIONS AND THEIR TRANSVERSE GROUPOIDS.

Throughout the paper, except for one passing reference to analytic manifolds, we work with real C^∞ paracompact manifolds with at most countably many components.

Given a principal bundle $P(B, G, p)$, denote by $P{\times}G//G$ the Lie group bundle associated to $P(B, G)$ by the conjugation action of G on itself; thus elements of $P{\times}G//G$ are equivalence classes $\langle u, g\rangle$, $u \in P$, $g \in G$, with

$$\langle u, g\rangle = \langle ug_1, g_1{}^{-1}gg_1\rangle \quad \text{for any } g_1 \in G.$$

This contrasts with the notation for a produced principal bundle: if $\phi : G \to H$ is a Lie group morphism then the produced principal bundle is denoted by $P{\times}H/G\ (B, H)$. Here the elements are classes $\langle u, h\rangle$ where $u \in P$, $h \in H$ and

$$\langle u, h\rangle = \langle ug, \phi(g)^{-1}h\rangle \quad \text{for any } g \in G.$$

The action of H on $P{\times}H/G$ is

$$\langle u, h\rangle h' = \langle u, hh'\rangle.$$

Lastly, the Lie groupoid associated to $P(B, G)$ is denoted $P{\times}P/G$; here the elements are $\langle u_2, u_1\rangle$ where $u_2, u_1 \in P$ and

$$\langle u_2, u_1\rangle = \langle u_2g, u_1g\rangle \quad \text{for any } g \in G.$$

The canonical map $P{\times}G//G \rightarrowtail P{\times}P/G$ is $\langle u, g\rangle \mapsto \langle ug, u\rangle$. Compare [10].

For background on Lie groupoids we refer throughout to [10]. However for a Lie groupoid Ω on B we denote the inner group bundle $\bigcup_{x\in B}\Omega_x^x$ by $I\Omega$; not by $G\Omega$ as in [10].

DEFINITION 1.1. (i) An *extension of principal bundles* is a sequence

(1) $$N \overset{\iota}{\rightarrowtail} Q(B, H, q) \overset{\pi}{\twoheadrightarrow} P(B, G, p)$$

in which π denotes both a surjective submersion $Q \twoheadrightarrow P$ and a surjective Lie group, morphism $H \twoheadrightarrow G$ (necessarily a submersion), such that $\pi(id_B, \pi)$ is a morpàhism of principal bundles; i denotes an injective Lie group morphism $N \rightarrowtail H$, and where $N \overset{\iota}{\rightarrowtail} H \overset{\pi}{\twoheadrightarrow} G$ is an extension of Lie groups.

(ii) An *extension of Lie groupoids* is a sequence

(2) $$M \overset{\iota}{\rightarrowtail} \Phi \overset{\pi}{\twoheadrightarrow} \Omega$$

in which π is a base-preserving morphism of Lie groupoids and a surjective submersion, M is a Lie group bundle and i is a base-preserving groupoid morphism and an injective immersion; lastly the sequence is exact. //

Assume given (1). One can now construct an extension of Lie groupoids

(3) $$Q\times N//H \xrightarrow{i^\sim} Q\times Q/H \xrightarrow{\pi^\sim} P\times P/G.$$

Here $Q\times Q/H$ and $P\times P/G$ are the associated Lie groupoids and $Q\times N//H$ is the Lie group bundle associated to $Q(B, H)$ through the action of H on N by (the restrictions of) inner automorphisms. The map $i^\sim$ is $\langle v, n\rangle \mapsto \langle vn, v\rangle$ and the map $\pi^\sim$ is $\langle v_2, v_1\rangle \mapsto \langle \pi(v_2), \pi(v_1)\rangle$. It is easy to verify that (3) is an extension of Lie groupoids.

Assume given (2). Then for any chosen $b \in B$,

(4) $$M_b \xrightarrow{i_b} \Phi_b(B, \Phi_b{}^b) \xrightarrow{\pi_b} \Omega_b(B, \Omega_b{}^b)$$

is clearly an extension of principal bundles. The constructions (1) $\Rightarrow$ (3), (2) $\Rightarrow$ (4) are easily seen to be mutually inverse, modulo the usual attention to base-points.

DEFINITION 1.2. (i) Consider two principal bundle extensions of $P(B, G)$ by N:

(1) $$N \xrightarrow{i} Q(B, H) \xrightarrow{\pi} P(B, G),$$

(1') $$N \xrightarrow{i'} Q'(B, H') \xrightarrow{\pi'} P(B, G).$$

An *equivalence* of principal bundle extensions is an isomorphism

$$\phi(\mathrm{id}_B, \phi)\colon Q(B, H) \to Q'(B, H')$$

such that $\qquad \phi \circ i = i' \quad \text{and} \quad \pi' \circ \phi = \pi.$

(ii) Consider two Lie groupoid extensions of Ω by M:

(2) $$M \xrightarrow{i} \Phi \xrightarrow{\pi} \Omega,$$

(2') $$M \xrightarrow{i'} \Phi' \xrightarrow{\pi'} \Omega.$$

An *equivalence* of Lie groupoid extensions is an isomorphism $\phi: \Phi \to \Phi'$ such that $\phi \circ i = i'$ and $\pi' \circ \phi = \pi$. //

It is not immediately clear that these two concepts of equivalence do correspond; for the proof, see [11], §4.

In what follows we will work mainly with principal bundle extensions, since this case both requires more attention, and is the more established language. Now that the preparatory remarks have been made, our purpose is to show that principal bundle extensions are themselves equivalent to Lie groupoids together with an additional structure.

Consider, then, the extension of principal bundles (1). Observe that $Q(P, N, \pi)$ is a principal bundle; we call it the *transverse bundle*. Here N acts on Q via its embedding in H, but in practice we write simply vn for $vi(n)$.

Denote by T the associated groupoid $\frac{Q\times Q}{N}$ on P. We call T the *transverse groupoid* of (1). One may note that $Q(P, N)$ is a reduction of the inverse image principal bundle $p^*Q(P, H)$, and T accordingly is a reduction of the inverse image Lie groupoid $p^*(\frac{Q\times Q}{H})$ (see the description of the groupoid case which is given at the end of the section).

PROPOSITION 1.3. *IT $\simeq \frac{Q\times N}{N}$ is naturally isomorphic to $p^*(\frac{Q\times N}{H})$ under $\langle v, n\rangle^N \mapsto (\pi(v), \langle v, n\rangle^H)$.*

PROOF. Here the superscripts N and H indicate the group with respect to which the orbit is taken. We verify the surjectivity; the rest is similar. Take

$$(u, \langle v, n\rangle^H) \in p^*(\tfrac{Q\times N}{H}).$$

Then $p(u) = q(v)$ so there exists $g \in G$ such that $\pi(v) = ug$. Choose h in H with $\pi(h) = g$. Then

$$\pi(vh^{-1}) = u \quad \text{and} \quad \langle v, n\rangle^H = \langle vh^{-1}, hnh^{-1}\rangle^H;$$

thus $\langle vh^{-1}, hnh^{-1}\rangle^N$ is mapped to $(u, \langle v, n\rangle^H)$. //

Thus the groupoid T can now be presented in the following form, which we shall regard as canonical:

$$p^*(Q\times N//N) \rightarrowtail T \twoheadrightarrow P\times P. \tag{5}$$

Here the injection is

$$(u,\ \langle v,n\rangle^H) \mapsto \langle vnh^{-1},\ vh^{-1}\rangle$$

where the element $h \in H$ is chosen so that $\pi(v) = u\,\pi(h)$. The submersion is the anchor

$$\langle v_2,\ v_1\rangle \mapsto (\pi(v_2),\ \pi(v_1))$$

of T.

We now define a right action of G on T by

$$\langle v_2,\ v_1\rangle g = \langle v_2h,\ v_1h\rangle \qquad \text{where} \qquad \pi(h) = g.$$

That this formula gives a well-defined action is easy to check.

PROPOSITION 1.4. This action interacts with the algebraic structure of T *in the following ways:*

(i) $\alpha^\sim(\mu g) = \alpha^\sim(\mu)g\ ;\quad \beta^\sim(\mu g); = \beta^\sim(\mu)g\ ;$

(ii) $u^\sim g = (ug)^\sim\ ;$

(iii) $(\mu'\mu)g = (\mu'g)(\mu g)\ ;$

(iv) $(\mu g)^{-1} = \mu^{-1}g\ ;$

where $\mu,\ \mu' \in T$, $g \in G$, $u \in P$ *and* $\alpha^\sim(\mu') = \beta^\sim(\mu)$.

PROOF. Here $\alpha^\sim$, $\beta^\sim$ denote the source and target projections of T. For (iii), let $\mu = \langle v_2,\ v_1\rangle$; then μ' can be written in the form $\langle v_3,\ v_2\rangle$. Now

$$(\mu'\mu)g = \langle v_3,\ v_1\rangle g = \langle v_3h,\ v_1h\rangle$$

where $\pi(h) = g$, and

$$(\mu'g)(\mu g) = \langle v_3h,\ v_2h\rangle\langle v_2h,\ v_1h\rangle = \langle v_3h,\ v_1h\rangle.$$

The other parts are similar. //

PROPOSITION 1.5. The action of G *on* T *is free and principal with respect to the map*

$$\#\colon T \to Q\times Q/H,\quad \langle v_2,\ v_1\rangle^N \mapsto \langle v_2,\ v_1\rangle^H.$$

PROOF. Suppose that $\langle v_2,\ v_1\rangle^N g = \langle v_2,\ v_1\rangle^N$. Then $\langle v_2h, v_1h\rangle^N = \langle v_2, v_1\rangle^N$, where $\pi(h) = g$, so there exists $n \in N$ such that $v_2h = v_2n$ and

$v_1 h = v_1 n$. Since the action of H on Q is free, it follows that $h = n \in N$ and so $g = \pi(h) = 1$. Thus the action of G on T is free.

Suppose that $\langle v_2, v_1\rangle^N$ and $\langle v'_2, v'_1\rangle^N$ have $\langle v_2, v_1\rangle^H = \langle v_2, v_1\rangle^H$. Then $v'_2 = v_2 h$, $v'_1 = v_1 h$ for some $h \in$ H and so $\langle v'_2, v'_1\rangle^N = \langle v_2, v_1\rangle^N g$ where $g = \pi(h)$. The converse is obvious. //

Thus T(Q×Q/H, G, #) is a principal bundle.

From 1.4 (i) it follows that IT is stable under G. We now compute the action of G on IT in terms applicable to (5).

PROPOSITION 1.6. *(i) Regarding* IT *as* Q×N//N, *the action of* G *becomes*

$$\langle v, n\rangle^N g = \langle vh, h^{-1}nh\rangle^N \quad \text{where} \quad g = \pi(h).$$

(ii) Regarding IT *as* p^*(Q×N//H), *as in 1.3, the action of* G *becomes*

$$(u, \langle v, n\rangle^H)g = (ug, \langle v, n\rangle^H).$$

PROOF. (i) Given $\langle v, n\rangle^N$, the corresponding element of IT ⊂ Q×Q/N is $\langle vn, n\rangle$. The action of g on this gives $\langle vnh, vh\rangle$, where $\pi(h) = g$. Rewriting this as $\langle vh.h^{-1}nh, vh\rangle$, it is seen to correspond to $\langle vh, h^{-1}nh\rangle^N$.

(ii) Given $(u, \langle v, n\rangle^H)$, the corresponding element of Q×N//N is $\langle vj^{-1}, jnj^{-1}\rangle^N$, where $j \in$ H is such that $\pi(v) = u\pi(j)$. By (i), the action of g on this gives $\langle vj^{-1}h, h^{-1}jnj^{-1}h\rangle^N$, where $\pi(h) = g$. By 1.3, the corresponding element of p^*(Q×N//H) is

$$(\pi(vj^{-1}h), \langle vj^{-1}h, h^{-1}jnj^{-1}h\rangle^H).$$

Now $\pi(vj^{-1}h) = ug$ and since $j^{-1}h \in$ H, the second component reduces to $\langle v, n\rangle^H$. //

Thus the action of G on p^*(Q×N//H) induced from T is the natural action of G which exists on any inverse image bundle across p: P → B. The sequence (5) is now G-equivariant, with the actions of G on the first and last terms being given solely by the action of G on P.

In the abelian case there is a further simplification.

PROPOSITION 1.7. *Let* N *in (1) be an abelian Lie group* A. *Then* IT ≃ Q×A//A *is naturally isomorphic to the trivial Lie group bundle* P×A *under* $\langle v, a\rangle \mapsto (\pi(v), a)$, *and this isomorphism transports the action of* G *on* IT *to* $(u, a)g = (ug, g^{-1}a)$.

PROOOF. The action of G on A is that induced by the extension

$$A \rightarrowtail H \twoheadrightarrow G.$$

Represent $(u, a) \in P\times A$ by $\langle v, a\rangle \in Q\times A//A$ where $\pi(v) = u$. Then $\langle v, a\rangle g = \langle vh, h^{-1}ah\rangle$, where $\pi(h) = g$. Now $\pi(vh) = ug$ and $h^{-1}ah = g^{-1}a$. //

We briefly describe the construction of the transverse groupoid from an extension of Lie groupoids (2). First choose $b \in B$ and write $P = \Omega_b$, $G = \Omega_b^b$, $p = \beta_b$.

Construct the inverse image Lie groupoid $p^*\Phi$ on P (see, for example, [10], I 2.11). Elements of $p^*\Phi$ have the form (u_2,λ,u_1) *where*

$$u_2,\ u_1 \in P,\ \lambda \in \Phi \text{ and } \quad p(u_2) = \beta'(\lambda), \quad p(u_1) = \alpha'(\lambda).$$

Here α', β' are the source and target projections of Φ. Define

$$d\colon p^*\Phi \to G \quad \text{by} \quad (u_2,\ \lambda,\ u_1) \mapsto u_2^{-1}\pi(\lambda)u_1,$$

and denote $d^{-1}(1)$ by T. It is easy to verify that T is a reduction of $p^*\Phi$. If (2) is in fact of the form (3), then $T = d^{-1}(1)$ is isomorphic to $Q\times Q/N$ under

$$\langle v_2,\ v_1\rangle^N \mapsto (\pi(v_2),\ \langle v_2,\ v_1\rangle^H,\ \pi(v_1)).$$

The inner group bundle IT of $T = d^{-1}(1)$ is evidently

$$\{\ (u,\ m,\ u) \in P\times M\times P \mid p(u) = p^M(m)\ \}$$

and it is now immediate that $IT \simeq p^*M$.

PROPOSITION 1.8. *The action of* G *on* $T = d^{-1}(1)$ *which corresponds to the action of* G *on* $Q\times Q/N$ *defined above (1.4) is*

$$(u_2,\ \lambda,\ u_1)g = (u_2g,\ \lambda,\ u_1g).$$

PROOF. Assume that (2) is of the form (3). Given $\langle v_2,\ v_1\rangle^N \in Q\times Q/N$ and $g \in G$, one has $\langle v_2,\ v_1\rangle^N g = \langle v_2h,\ v_1h\rangle^N$ where $\pi(h) = g$. The corresponding element of $d^{-1}(1)$ is $(\pi(v_2h),\ \langle v_2h,\ v_1h\rangle^H,\ \pi(v_1h))$ and this equals $(\pi(v_2)g,\ \langle v_2,\ v_1\rangle^H,\ \pi(v_1)g)$. //

The properties of 1.4 are now immediate. So too is the fact that the action of G on $IT = p^*M$ is the natural action on an inverse image bundle.

REMARK. The inverse image groupoid $p^*\Phi$ is naturally isomorphic to $T\times^{\sim}G$, the semi-direct product of T with G as given in Brown [4] §2. Here $T\times^{\sim}G$ has the source and target projections $\alpha^{\wedge}(\mu, g) = \alpha^{\sim}(\mu)g$, $\beta^{\wedge}(\mu, g) = \beta^{\sim}(\mu)$ and composition

$$(\mu_2, g_2)(\mu_1, g_1) = (\mu_2(\mu_1 g_2^{-1}), g_2 g_1),$$

and $p^*\Phi \to T\times^{\sim}G$ is

$$(u_2, \lambda, u_1) \mapsto ((u_2, \lambda, \pi(\lambda)^{-1}u_2), u_2^{-1}\pi(\lambda)u_1). \quad //$$

Now suppose given two Lie groupoid extensions (2) and (2'), equivalent under $\phi\colon \Phi \to \Phi'$ as in 1.2 (ii). Then the obvious induced map $\phi^{\sim}\colon T \to T'$ is an equivalence of groupoids

$$(6) \qquad \begin{array}{ccccc} p^*M & \rightarrowtail & T & \twoheadrightarrow & P\times P \\ \| & & \downarrow{\scriptstyle \phi^{\sim}} & & \| \\ p^*M & \rightarrowtail & T & \twoheadrightarrow & P\times P \end{array}$$

and is G-equivariant. This behaviour is formalized in 2.6.

2. PBG-GROUPOIDS.

1.5 suggests that it is possible to recover the original extensions from their transverse groupoid. We first abstract the properties of transverse groupoids into the concept of PBG-groupoid. Throughout this section we work with a fixed principal bundle $P(B, G, p)$.

DEFINITION 2.1. A *PBG-groupoid* on P(B, G) is a Lie groupoid T on base P together with a right action of G on the manifold T such that:

(i) $\alpha^{\sim}\colon T \to P$ and $\beta^{\sim}\colon T \to P$ are G-equivariant;
(ii) $\epsilon^{\sim}\colon P \to T$ is G-equivariant;
(iii) $(\mu'\mu)g = (\mu'g)(\mu g)$ for $(\mu', \mu) \in T{*}T$ and $g \in G$;
(iv) $(\mu g)^{-1} = \mu^{-1}g$ for $\mu \in T$, $g \in G$. //

Thus a PBG-groupoid is a Lie groupoid object in the category of principal bundles. Actions of groups on groupoids, and indeed of groupoids on groupoids, were defined in Brown [4]. Concerning terminology, the expression "PBG-groupoid" is intended to suggest that T is both a G-groupoid and a principal bundle. Perhaps a portmanteau word such as "grundleoid" would be better.

THEOREM 2.2. *Let* T *be a PBG-groupoid on* P(B, G). *Then the action of* G *on* T *is free and the quotient manifold* T/G *exists.*

PROOF. That G acts freely on T follows easily from (i). By the criterion of Godement (see, for example, [6], 16.10.3) it suffices to show that

$$\Gamma = \{ (\mu, \mu g) \in T\times T \mid \mu \in T, g \in G \}$$

is a closed submanifold of T×T. Firstly consider the map

$$\phi: T\times T \to P^4, \quad (\mu', \mu) \mapsto (\beta^\sim\mu', \beta^\sim\mu, \alpha^\sim\mu', \alpha^\sim\mu).$$

This is a surjective submersion, being a rearrangement of the square of the anchor of T. So $\phi^{-1}((P\times_p P)\times(P\times_p P))$ is a closed submanifold of T×T; call it R. Clearly $\Gamma \subset R$.

Next define

$$\psi: R \to G\times G, \quad (\mu', \mu) \mapsto (\delta(\beta^\sim\mu, \beta^\sim\mu'), \delta(\alpha^\sim\mu, \alpha^\sim\mu'))$$

where $\delta: P\times_p P \to G$ is the division map $\delta(ug, u) = g$. From the fact that δ is a surjective submersion, it follows that ψ is also, and hence $\psi^{-1}(\Delta_G)$ is a closed submanifold of R; call it S. Clearly $\Gamma \subset S$.

Lastly define

$$\chi: S \to IT \quad \text{by} \quad \chi(\mu', \mu) = \mu'(\mu g)^{-1}$$

where $g = \delta(\beta^\sim\mu', \beta^\sim\mu) = \delta(\alpha^\sim\mu', \alpha^\sim\mu)$. Then $\chi^{-1}(P) = \Gamma$, where P denotes the set of unities $\{u^\sim \mid u \in P\} \subset T$. So it suffices to prove that χ is a surjective submersion. Now χ is the composition of the map $S \to C \subset T\times T$, $(\mu', \mu) \mapsto (\mu', \mu g)$, where g is as above, and

$$C = T\times_{(\beta,\alpha)}T = \{ (\mu', \mu) \in T\times T \mid \alpha^\sim\mu' = \alpha^\sim\mu, \beta^\sim\mu' = \beta^\sim\mu \},$$

and the map $C \to IT$, $(\mu', \mu) \mapsto \mu'\mu^{-1}$. Now $S \to C$ is clearly a diffeomorphism and $C \to IT$ is easily seen to be a submersion by working locally: using a section-atlas for T with respect to an open cover $\{U_i\}$ of P, the map $C \to IT$ is

$$((u_2, n', u_1), (u_2, n, u_1)) \mapsto (u_2, n'n^{-1}, u_2),$$

where n, n' are elements of some vertex-group N of T. //

This result seems somewhat surprising. If all the data are real-analytic a simpler argument suffices, since in this case it suffices to verify that Γ is a closed subset of T×T and that

$$T\times G \to T\times T,\ (\mu, g) \mapsto (\mu g, \mu)$$

is a homeomorphism onto Γ ([12], 2.9.10). Returning to the smooth case, other properties of the action of G on P lift automatically to the action of G on T:

PROPOSITION 2.3. *Let* T *be a FBG-groupoid on* P(B, G).
(i) If $P\times G \to P$ *is properly discontinuous, then* $T\times G \to T$ *is also properly discontinuous.*
(ii) If $P\times G \to P$ *is proper, then* $T\times G \to T$ *is also proper.*

PROOF. We prove (i); the proof of (ii) is similar. Let $\{\sigma_i\colon U_i \to T_{u*}\}$ be a section-atlas for T with respect to some reference-point $u_* \in P$ and open cover $\{U_i\}$ of P. Without loss of generality, we can assume that

$$U_i g \cap U_i \neq \emptyset \Rightarrow \ g = 1 \quad \text{for all indexes } i.$$

Now, given any element μ_0 of T, there is an open neighborhood U of μ_0 which is diffeomorphic to $U_j\times N\times U_i$, where $N = T_{u*}^{u*}$. If we now have $\mu \in Ug \cap U$ for some $g \in G$, and some element $\mu \in T$, then $\alpha^\sim\mu \in (U_i)g \cap U_i$ and so $g = 1$. //

From 2.2 it follows that T(T/G, G) is a principal bundle. We denote elements of T/G by $\langle\mu\rangle$ and the projection $T \to T/G$ by #.

PROPOSITION 2.4. T/G *has a natural Lie groupoid structure with base* B, *in such a way that* #: $T \to T/G$ *is a morphism of Lie groupoids over* $p\colon P \to B$.

PROOF. Since $\alpha^\sim$, $\beta^\sim$: $T \to P$ are G-equivariant they induce maps α', β': $T/G \to B$; since $\alpha^\sim$, $\beta^\sim$, # and p are all surjective submersions, it follows that α', β' are also.

Take μ_1, $\mu_2 \in T$ and suppose that $\alpha'(\langle\mu_1\rangle) = \beta'(\langle\mu_2\rangle)$. Then there exists $g \in G$ such that $\alpha^\sim(\mu_1) = \beta^\sim(\mu_2)g$, and so it is meaningful to define

$$\langle\mu_1\rangle\langle\mu_2\rangle = \langle\mu_1(\mu_2 g)\rangle.$$

It is straightforward to verify that this is a well-defined operation $T/G * T/G \to T/G$, and that it makes T/G a groupoid on B. Note that the unities of T/G are $x^{\sim} = \langle u^{\sim}\rangle$ where $x \in B$ and $u \in P$ has $p(u) = x$. Also, $\langle\mu\rangle^{-1} = \langle\mu^{-1}\rangle$. The proof that the multiplication is smooth $T/G * T/G \to T/G$ follows as in [10], II 1.12. //

Note that there is no subgroupoid of T with respect to which T/G is a quotient groupoid.

Now consider the anchor $[\beta^{\sim}, \alpha^{\sim}]\colon T \to P\times P$ of T. It also is equivariant and so induces a map $\pi\colon T/G \to P\times P/G$; as with α' and β', it follows that π is a smooth submersion. It is clear that π is a groupoid morphism over B, and that its kernel is IT/G. Thus we have proved the following:

***PROPOSITION* 2.5.** *If* T *is a PBG-groupoid on* P(B, G), *then*

(1) $$IT/G \rightarrowtail T/G \overset{\pi}{\twoheadrightarrow} P\times P/G$$

is an extension of Lie groupoids over B. //

The corresponding extension of principal bundles is

$$IT/G|_b \rightarrowtail T/G|_b(B,\ T/G|_b^b) \twoheadrightarrow P(B,\ G)$$

where $b \in B$. Choose $u_* \in P$ with $p(u_*) = b$. Then

$$T_{u*} \to T/G|_b\ , \qquad \mu \mapsto \langle\mu\rangle$$

is a diffeomorphism (see 3.3). The preimage of $T/G|_b^b$ under this map is $\cup_{g\in G}T_{u*}^{u*g}$, and this acquires a group structure from $T/G|_b^b$. Namely, given $\lambda \in T_{u*}^{u*g}$, $\lambda' \in T_{u*}^{u*g'}$, the multiplication of $\langle\lambda\rangle$ with $\langle\lambda'\rangle$ gives $\langle\lambda(\lambda' g^{-1})\rangle$ or, equivalently, $\langle(\lambda g')\lambda'\rangle$. Now $(\lambda g')\lambda' \in T_{u*}^{u*gg'}$ and it can be checked directly that this operation $(\lambda, \lambda') \mapsto (\lambda g')\lambda'$ makes $\cup_{g\in G}T_{u*}^{u*g}$ into a group isomorphic to $T/G|_b^b$.

Lastly, $T_{u*}^{u*} \to IT/G|_b^b$, $\lambda \mapsto \langle\lambda\rangle$ is clearly an isomorphism of Lie groups. Thus the principal bundle extension corresponding to T can be presented as

(2) $$T_{u*}^{u*} \rightarrowtail T_{u*}(B,\ \cup_{g\in G}T_{u*}^{u*g}) \twoheadrightarrow P(B,\ G).$$

DEFINITION 2.6. Let T and T' be two PBG-groupoids on $P(B, G)$. Then a *morphism* of PBG-groupoids $T \to T'$ is a morphism of Lie groupoids over P which is G-equivariant.

If $I \rightarrowtail T \twoheadrightarrow P\times P$ and $I \rightarrowtail T' \twoheadrightarrow P\times P$ are two PBG-groupoids on $P(B, G)$ with the same inner group bundle I, then T and T' are *equivalent* PBG-groupoids if there exists an isomorphism $\phi\colon T \to T'$ of PBG-groupoids such that

$$\begin{array}{ccccc} I & \rightarrowtail & T & \twoheadrightarrow & P\times P \\ \| & & \phi\downarrow & & \| \\ I & \rightarrowtail & T' & \twoheadrightarrow & P\times P \end{array}$$

commutes. //

It is clear from 2.5 that two equivalent PBG-groupoids

$$I \rightarrowtail T \twoheadrightarrow P\times P \quad \text{and} \quad I \rightarrowtail T' \twoheadrightarrow P\times P$$

induce equivalent extensions of Lie groupoids. Conversely, the content of (6) at the end of §1 is that equivalent Lie groupoid extensions have PBG-groupoids which are equivalent. In summary then, there is the following:

THEOREM 2.7. *Let* M *be a Lie group on* B. *Then there is a bijective correspondence between the set of equivalence classes of Lie groupoid extensions*

$$M \rightarrowtail \Phi \twoheadrightarrow \Omega = P\times P/G$$

and the set of equivalence classes of PBG-groupoids

$$p^* M \rightarrowtail T \twoheadrightarrow P\times P. \quad //$$

In [11], §4, it was shown that the concept of coupling allows one to establish a bijective correspondence between equivalence classes of principal bundle extensions and equivalence classes of Lie groupoid extensions. Thus 2.7 may also be stated in terms of principal bundle extensions. We do not need the concept of coupling in §4 and so we omit the precise details.

EXAMPLE 2.8. Assume that P is connected. There is a natural action of G on the fundamental Lie groupoid $\Pi(P)$ of P (for which see, for example, [10], I 1.8, II 1.14), namely $\langle c\rangle g = \langle R_g \circ c\rangle$, where $c: [0, 1] \to P$ is any path in P and $R_g: P \to P$ is the right-translation within the principal bundle. It is immediate that $\Pi(P)$ is then a PBG-groupoid on P(B, G).

A comparison with [10], II §6, shows that $\Pi(P)/G$ is precisely the monodromy Lie groupoid $M\Omega$ of $\Omega = P{\times}P/G$. Using $M\Omega = \Pi(P)/G$ as the definition, it is easy to see that $\pi: M\Omega \to \Omega$ is étale, by a dimension count, using the facts that $\Pi(P) \twoheadrightarrow P{\times}P$ is étale, and that G acts freely on both $\Pi(P)$ and $P{\times}P$. Further, one can easily show that the anchor $\Pi(P) \twoheadrightarrow P{\times}P$ admits a G-equivariant local right-inverse morphism, by adapting the proof of [10], II 6.8, and it then follows immediately that $\pi: M\Omega \twoheadrightarrow \Omega$ admitzs a local right-inverse morphism ([10], II 6.11). //

It may perhaps be emphasized that the concept of PBG-groupoid is a genuinely groupoid concept, although it arises in the analysis of extensions of principal bundles. There is no natural formulation of the PBG-groupoid concept entirely in terms of principal bundles.

The geometric interest of the transverse bundle was noted by the author some time ago ([9], pp. 146-8), but the approach referred to was cumbersome and details were not given. The introduction of the group action on the transverse groupoid was suggested by Kumjian's concept of G-groupoid ([8], §2), but despite the evident similarity between the two concepts, it is not clear that there is any general connection between them. Actions of groups on fundamental groupoids were treated by Higgins and Taylor [7], who were primarily concerned with the relationship between the homotopy properties of a G-space and those of its orbit-space. Their map q_* (op. cit. §2 (*)), in the context of our Example 2.8, is the map $\Pi(P)/G \twoheadrightarrow \Pi(B)$ obtained by lifting the anchor of $\Pi(P)/G$ across the anchor of $\Pi(B)$; it is a surjective submersion. As was mentioned already, actions of groups on groupoids go back at least to Brown [4].

REMARK. Consider an extension of Lie groups $N \rightarrowtail H \overset{\pi}{\twoheadrightarrow} G$. The transverse groupoid $T = H{\times}H/N$ over (the manifold) G is naturally isomorphic to the action groupoid $H{\times}G$ with respect to the left action of H on (the manifold) G via π. The isomorphism is

$$\langle h_1, h_2\rangle \mapsto (h_2h_1^{-1}, \pi(h_1))$$

(see, for example, [10], II 1.12) and the groupoid structure on H×G is

$$(h', g')(h, g) = (h'h, g) \quad \text{defined iff} \quad g' = \pi(h)g.$$

Under this identification, the action of G on T is carried to

$$(h, g)g' = (h, gg'),$$

and T becomes $N\times G \rightarrowtail H\times G \twoheadrightarrow G\times G$.

This construction was given (in the setting of discrete groups) by Brown and Danesh-Naruie [5], §4. Conversely, they associate to the action of any group on any (discrete transitive) groupoid, an extension of groups; applied to the action of G on any PBG-groupoid T this yields $N \rightarrowtail H \twoheadrightarrow G$. //

3. PBG-LIE ALGEBROIDS.

Throughout this section we work with a single principal bundle P(B, G, p). Let T be a PBG-groupoid on P(B, G). Denote by R_g: P → P and R_g^-: T → T the right translations by $g \in$ G. Then Definition 2.1 states that R_g^- is an automorphism of the Lie groupoid T over R_g. Denote the induced automorphism of the Lie algebroid AT by $(R_g^-)_*$ (see, for example, [10], III §4). Then the $(R_g^-)_*$ make AT into a PBG-Lie algebroid in the sense of the following

DEFINITION 3.1. A *PBG-Lie algebroid* on P(B, G) is a transitive Lie algebroid A on base P together with a right action $(X, g) \mapsto Xg = R_g^{\sim}(X)$ of G on A such that each $R_g^{\sim}$: A → A is a vector bundle automorphism of A over R_g: P → P, and a Lie algebroid automorphism. //

As in [10], we also denote by $R_g^{\sim}$ the map ΓA → ΓA defined by

$$(R_g^{\sim}(X))(u) = R_g^{\sim}(X(ug^{-1})), \qquad u \in P.$$

Then the conditions in 3.1 entail the following ([10], III §4):
(i) the diagram

$$\begin{array}{ccc} A & \xrightarrow{\;q^{\sim}\;} & TP \\ {\scriptstyle R_g^{\sim}}\downarrow & & \downarrow{\scriptstyle T(R_g)} \\ A & \xrightarrow{\;q^{\sim}\;} & TP \end{array}$$

commutes for all $g \in G$ where $q^{\sim}$ is the anchor of A; and

(ii) $[R_g^{\sim}(X), R_g^{\sim}(Y)] = R_g^{\sim}([X, Y])$ for $X, Y \in \Gamma A$ and $g \in G$.

PROPOSITION 3.2. *Let* A *be a PBG-Lie algebroid on* P(B, G) *for which the quotient manifold* A/G *exists. Then there is a natural structure of transitive Lie algebroid on* A/G *with base* B, *and* A/G *is an extension*

$$L/G \rightarrowtail A/G \twoheadrightarrow TP/G = A\Omega$$

of the Lie algebroid of $\Omega = P{\times}P/G$ *by* L/G.

PROOF. Observe firstly that the vector bundle structure on A quotients to A/G, once it is assumed that the quotient manifold exists. Thus $\#: A \twoheadrightarrow A/G$, $X \mapsto \langle X \rangle$, is a pullback of vector bundles over $p: P \to B$. Next, since $q^{\sim}: A \twoheadrightarrow TP$ is G-equivariant, it quotients to a map $\pi: A/G \to TP/G$, which is a surjective submersion since $q^{\sim}$ and both the projections are. Now let $q: A\Omega = TP/G \twoheadrightarrow TB$ denote the anchor of $A\Omega$, and define $q' = q \circ \pi: A/G \twoheadrightarrow TB$.

For the bracket on $\Gamma(A/G)$, notice first that $\Gamma(A/G)$ can be naturally identified with the module $\Gamma^G A$ of G-invariant sections of A, by the same argument as in [10], A 2.4. Now $\Gamma^G A$ is closed under the bracket on ΓA, by (ii) above, and so the ΓA bracket, restricted to $\Gamma^G A$, can be transferred to $\Gamma(A/G)$.

We verify that

$$[X, fY] = f[X, Y] + q'(X)(f)Y$$

for $X, Y \in \Gamma(A/G)$ and $f \in C(B)$, the ring of smooth functions on B. Let X^{-}, etc., denote the element of $\Gamma^G A$ corresponding to X, etc. Then

$$[X, fY]^{-} = [X^{-}, (f \circ p)Y^{-}] = (f \circ p)[X^{-}, Y^{-}] + q^{\sim}(X^{-})(f \circ p)Y^{-}$$

and it evidently suffices to verify that

$$q^{\sim}(X^{-})(f \circ p) = q'(X)(f) \circ p.$$

Now

$$q' = q \circ \pi \quad \text{and} \quad q(\pi X)(f) \circ p = (\pi X)^{-}(f \circ p),$$

where $(\pi X)^{-}$ is the element of $\Gamma^G TP$ corresponding to $\pi X \in \Gamma(TP/G)$ (see, for example, [10], A §3). But the element of $\Gamma^G TP$ corresponding to πX is manifestly $q^{\sim}(X)$.

This completes the proof that A/G is a transitive Lie algebroid on B. That the kernel of A/G $\longrightarrow\!\!\!\!\rightarrow$ TP/G is L/G is clear. //

For the definition of an extension of Lie algebroids see [10], IV 3.4. It follows incidentally from 3.2 that the adjoint bundle of A/G is an extension of $L\Omega = P\times\mathbf{g}/G$ by L/G.

The proof of 3.2 generalizes the construction of TP/G given in [10], A §§2, 3. It is fairly evident that #: A $\longrightarrow\!\!\!\!\rightarrow$ A/G is entitled to be regarded as a morphism of Lie algebroids; for the actual definition of a morphism of Lie algebroids over varying bases see [1].

One may prove that the quotient manifold A/G does exist in several important cases: Firstly, note that the action of G on A is always free, for the same reason as in §2. Hence if G acts properly on A then the quotient exists. (See, for example, [3], III §1.5.) Further, this is so whenever G acts properly on P, by the same argument as for 2.3 (using merely vector bundle charts for A, not the Lie algebroid structure). Alternatively, if G acts properly discontinuously on P, then, by the same reasoning, it acts properly discontinuously on A. For discrete groups acting freely on smooth manifolds, proper discontinuity is equivalent to propriety, and so the quotient exists in this case also. Lastly, the quotient will also exist if A admits equivariant charts in the sense of [10] A §2.

This ensures the existence of the quotient whenever G is compact or discrete. It seems fairly certain that the quotient does not always exist, but this does not affect the arguments in §4.

PROPOSITION 3.3. *Let* T *be a PBG-groupoid on* P(B, G). *Then the quotient manifold* AT/G *exists and is naturally isomorphic to* A(T/G).

PROOF. This is routine once one has established that the map $\#_*$: AT $\to$ A(T/G) induced by #: T $\longrightarrow\!\!\!\!\rightarrow$ T/G is a surjective submersion. This point, however, may need some explanation.

Denote T/G by Φ. The projection #: T $\longrightarrow\!\!\!\!\rightarrow$ Φ is a surjective submersion and a Lie groupoid morphism over p: P $\to$ B. The construction of $\#_*$: AT $\to$ AΦ is given in, for example, [10], III §4 and [1]. It suffices to prove that

$$(\#_*)_u\colon AT|_u \to A\Phi|_x\,, \quad x = p(u),$$

is surjective and this is $T(\#_u)_{u^\sim}\colon T(T_u)_{u^\sim} \to T(\Phi_x)_{x^\sim}$.

Consider the diagram

$$\begin{array}{ccccc}
 & & T(T_u)_{u^\sim} & \xrightarrow{T(\#_u)_{u^\sim}} & T(\phi_x)_{x^\sim} \\
 & & \downarrow & & \downarrow \\
\ker T(\#)_{u^\sim} & \rightarrowtail & T(T)_{u^\sim} & \xrightarrow{T(\#)_{u^\sim}} & T(\phi)_{x^\sim} \\
\vdots & & T(\alpha^\sim)_{u^\sim} \downarrow & & \downarrow T(\alpha')_{x^\sim} \\
\ker T(p)_u & \rightarrowtail & T(P)_u & \xrightarrow[T(p)_u]{} & T(B)_x
\end{array}$$

The existence of the dotted map follows from the arrangement of the other data. A standard diagram-chase shows that to prove that $T(\#_u)_{u^\sim}$ is surjective, it is sufficient to show that the dotted map is surjective. But

$$\ker T(\#)_{u^\sim} = T(\{u^\sim g \mid g \in G\})_{u^\sim} \quad \text{and} \quad \ker T(p)_u = T(\{ug \mid g \in G\})_u,$$

and it is easy to see that the relevant map is that induced by $\alpha^\sim$, and is $u^\sim g \mapsto ug$. Thus the result. //

A slight extension of this argument shows that $\#_u\colon T_u \to T/G|_{pu}$ is a surjective submersion; since it is certainly bijective it follows that it is a diffeomorphism. This fact will be used later.

Clearly $AT/G \to A(T/G)$ is now an isomorphism of transitive Lie algebroids over B.

The following result is crucial for the main result of §4.

***THEOREM* 3.4.** *Let* T *be an* α*-simply connected Lie groupoid on a simply-connected base* P, *which is the total space of a principal bundle* P(B, G). *Suppose that for all* $g \in G$ *there is given a Lie algebroid automorphism* $R_g^\sim\colon AT \to AT$ *over* $R_g\colon P \to P$, *which defines the structure of a PBG-Lie algebroid on* AT. *Then there is a natural structure of a PBG-groupoid on* T *for which* $(R_g^-)_* = R_g^\sim$ *for all* $g \in G$.

PROOF. First consider $R_g^\sim\colon \Gamma AT \to \Gamma AT$ for a particular g. As with any morphism of Lie algebroids over a diffeomorphism, $R_g^\sim$ may be considered to be a morphism over P from AT to the inverse image Lie algebroid $R_g^*(AT)$. Here $R_g^*(AT)$ is the inverse image vector bundle

$$\{(u, X) \in P\times AT \mid X \in AT|_{ug}\}$$

with anchor

$$q^g\colon R_g^*(AT) \twoheadrightarrow TP, \quad (u, X) \mapsto T(R_{g^{-1}})(qX)$$

and bracket

$$[X, Y] = [X \circ R_{g^{-1}}, Y \circ R_{g^{-1}}] \circ R_g$$

for X, Y $\in \Gamma(R_g^*(AT))$, where the bracket on the RHS is the bracket on ΓAT.

Denote this map AT $\to R_g^*(AT)$ by $R_g^\vee$. Explicitly,

$$R_g^\vee(X) = (u, R_g^\sim(X)), \text{ where } X \in AT|_u.$$

Since $R_g^\vee$ is a base-preserving isomorphism from AT to $R_g^*(AT) = A(R_g^*T)$, and since T and R_g^*T are α-simply connected, there is a (unique) isomorphism of Lie groupoids T $\to R_g^*T$, which we denote by $R_g^\wedge$, such that $(R_g^\wedge)_* = R_g^\vee$ (see, for example, [10], III §6). Now $R_g^\wedge$ defines a map R_g^-: T $\to$ T by the requirement that

$$R_g^\wedge(\mu) = (\beta^\sim(\mu), R_g^-(\mu), \alpha^\sim(\mu)).$$

It is straightforward to check that R_g^-: T $\to$ T is an isomorphism of Lie groupoids over, R_g: P $\to$ P, and that $(R_g^-)_* = R_g^\sim$.

It now remains to prove that T×G $\to$ T, $(\mu, g) \mapsto R_g^-(\mu)$ is smooth, and this is covered by the following lemma. //

LEMMA 3.5. *Let* T *be an* α*-simply connected Lie groupoid on a simply-connected base* P *which is the total space of a principal bundle* P(B, G). *For each* $g \in$ G let R_g^-: T $\to$ T *be a Lie groupoid automorphism over* R_g: P $\to$ P. *Then if the induced map* AT×G $\to$ AT, $(X, g) \mapsto (R_g^-)_*(X)$ *is smooth, it follows that* T×G $\to$ T, $(\mu, g) \mapsto R_g^-(\mu)$ *is smooth.*

PROOF. It suffices to work locally. Assume therefore that T is a trivial Lie groupoid P×N×P where we can assume that N is connected, in view of the connectivity assumptions on T and P. (However, P, after localization, is no longer assumed to be simply-connected.) Fix $u_0 \in$ P.

Now each R_g^-: P×N×P $\to$ P×N×P has the form

$$(u_2, n, u_1) \mapsto (u_2 g, \theta^g(u_2) f^g(n) \theta^g(u_1)^{-1}, u_1 g)$$

where θ^g: P $\to$ N has $\theta^g(u_0) = 1$ and f^g: N $\to$ N is an automorphism of Lie groups. It suffices to prove that the maps θ: P×G $\to$ N, $(u, g) \mapsto \theta^g(u)$ and f: N×G $\to$ N, $(n, g) \mapsto f^g(n)$ are smooth.

Now

$$(R_g^-)_*(X \oplus V) = T(R_g)(X) \oplus (\omega^g(X) + \phi^g(V))$$

where $\omega^g = D(\theta^g)$ is the right Darboux derivative of $\theta^g: P \to N$, and $\phi^g = Ad(\theta^g) \circ (f^g)_*$ (see, for example, [10], III 3.21). By assumption,

$$TP\times G \to P\times \mathfrak{N}, \quad (X, g) \mapsto (u, \omega^g(X)),$$

where $X \in T(P)_u$ and $\mathfrak{N}$ is the Lie algebra of N, is smooth. It follows that θ is smooth, ultimately from the result that a linear first-order system of equations, whose right-hand sides depend smoothly on auxiliary parameters, has solutions which depend smoothly on these parameters, providing that the initial conditions vary smoothly. Now f can be regarded as a map $G \to Aut(N)$ and the corresponding map $G \to Aut(\mathfrak{N}) = Aut(\tilde{N}) \geq Aut(N)$ is $g \mapsto f^g$, and hence f, is smooth. //

The assumption in 3.4 and 3.5 that P is simply-connected is introduced only to ensure that N in the proof of 3.5 is connected, so that Aut(N) will be a subgroup of $Aut(\mathfrak{N})$. There is presumably a result to the effect that for any α-simply connected Lie groupoid T, any Lie group of Lie algebroid automorphisms of AT acts smoothly on T in the manner of 3.4. However we do not wish to develop here the apparatus necessary for such a result.

4. THE INTEGRABILITY CRITERIA.

First of all, consider an extension of transitive Lie algebroids

(1) $$K \rightarrowtail A' \xrightarrow{\pi} A\Omega$$

in which Ω is an α-connected Lie groupoid on a (connected) base B and K is a Lie algebra bundle on B (see, for example, [10], IV 3.4). We call (1) *integrable* if there is a Lie groupoid Φ on B such that AΦ ≅ A', and a Lie groupoid morphism $\Pi: \Phi \to \Omega$ over B, necessarily a surjective submersion ([10] III 3.14 and III 1.31), such that $\Pi_* = \pi$. It then follows that $M = \ker \Pi$ is a Lie group bundle on B with $M_* \simeq K$ and that

(2) $$M \rightarrowtail \Phi \xrightarrow{\Pi} \Omega$$

is an extension of Lie groupoids which differentiates to (1).

Choose $b \in B$ and write $P = \Omega_b$, $G = \Omega_b^b$, $p = \beta_b$. We will construct an inverse image Lie algebroid $p^*K \rightarrowtail p^*A' \twoheadrightarrow TP$ on base P and with the structure of a PBG-Lie algebroid. Here p^*A' and p^*K are inverse image vector bundles, namely

$$p^*A' = \{(u, X') \in P \times A' \mid X' \in A'_{p(u)}\}$$

and likewise for p^*K.

Recall that $\Gamma(p^*A') \cong C(P) \otimes_{C(B)} \Gamma A'$, by a standard result on pullback vector bundles. Here $\phi \otimes X'$ corresponds to $\phi(X' \circ p)$. Define a map $q^\sim\colon A' \to TP$ by $q^\sim(u, X') = T(R_u)(\pi X')$ or, on the section-level, by

$$q^\sim(\phi \otimes X')(u) = \phi(u)T(R_u)(\pi(X'(pu))).$$

To see that $q^\sim\colon A' \to TP$ is a surjective submersion, recall ([10] III 3.3) that $TP \cong p^*A\Omega$ under the map $X_u \mapsto (u, T(R_{u^{-1}})(X))$. With this identification, $q^\sim$ corresponds to $p^*(\pi)\colon p^*A' \to p^*A\Omega$, and this is a surjective submersion since π is.

Now define a bracket on $\Gamma(p^*A')$ by

$$[\phi \otimes X', \psi \otimes Y'] = \phi\psi \otimes [X', Y'] + \phi\pi(X')^\rightarrow(\psi) \otimes Y' - \psi\pi(Y')^\rightarrow(\phi) \otimes X'$$

for $X', Y' \in \Gamma A'$ and $\phi, \psi \in C(P)$. Here $\pi(X')^\rightarrow$ denotes the right-invariant vector field on P corresponding to $\pi(X') \in \Gamma A\Omega \cong \Gamma(TP/G)$. It is routine to check that this is well-defined with respect to $\otimes$, extends by linearity, and makes p^*A' into a transitive Lie algebroid on P with adjoint bundle p^*K', the inverse image Lie algebra bundle.

Now define an action of G on p^*A' by

$$R_g^\sim(u, X') = (u, X')g = (ug, X') \;;$$

the standard action of G on an inverse image bundle across p. One checks easily that $q^\sim \circ R_g^\sim = T(R_g) \circ q^\sim$ for $g \in G$. To verify that the induced map $R_g^\sim\colon \Gamma(p^*A') \to \Gamma(p^*A')$ preserves the bracket, note first that $R_g^\sim(\phi \otimes X') = (\phi \circ R_{g^{-1}}) \otimes X'$; and secondly that

$$\pi(X')^\rightarrow(\psi \circ R_{g^{-1}}) = \pi(X')^\rightarrow(\psi) \circ R_{g^{-1}},$$

by the right invariance of $\pi(X')^\rightarrow$. The bracket-preservation equation now follows easily. We thus have the following result.

PROPOSITION 4.1. $p^*K' \rightarrowtail p^*A' \xrightarrow{q^\sim} TP$, *as constructed above, is a PBG-Lie algebroid on* P. //

Of course in this case the quotient manifold p^*A'/G always exists, and p^*A'/G is naturally isomorphic to A'. The following procedural result is worth recording because it is *not* the case that $T \cong p^*\Phi$.

PROPOSITION 4.2. *Let* $M \rightarrowtail \Phi \twoheadrightarrow \Omega$ *be an extension of Lie groupoids, and let* T *be the transverse PBG-groupoid. Then* $AT \cong p^*A\Phi$.

PROOF. In the notation of §1, it has to be proved that $TQ/N \cong p^*(TQ/H)$. For $\langle X\rangle^N \in TQ/N$ with $X \in T(Q)_v$, define

$$\phi(\langle X\rangle^N) = (\pi(v), \langle X\rangle^H).$$

It is easy to see that ϕ is an isomorphism of vector bundles $TQ/N \to p^*(TQ/H)$; compare the proof of 1.3. The proof that ϕ preserves the Lie algebroid structures is routine. //

The following result is the key to the integrability results.

THEOREM 4.3. *Continue the above notation. If* Ω *is* α*-simply connected, then the extension (1) is integrable iff the Lie algebroid* p^*A' *is integrable.*

PROOF. Suppose (1) is integrable and gives (2). Then the transverse groupoid T of (2) has $AT \cong p^*A\Phi$, by 4.2, and so $p^*A' = p^*A\Phi$ is integrable.

Conversely, suppose that $p^*A' = AT$ for some Lie groupoid T on P. By taking the monodromy groupoid MT, if necessary, we can assume that T is α-simply connected. So, by 3.4, T is a PBG-groupoid. Let $\Phi = T/G$, and let $M = IT/G$. Then

$$A\Phi \cong AT/G \underset{\text{(by 3.3)}}{\cong} p^*A'/G \cong A',$$

the last having been noted beneath 4.1. Now Φ is α-simply connected (by the remark following 3.3) and so $A\Phi \cong A' \xrightarrow{\pi} A\Omega$ integrates to $\Pi: \Phi \to \Omega$. Since Ω is α-connected, it follows that Π is surjective and a submersion (see, for example, [10] III 3.14, 1.31). //

Taken together with [10] V 1.2) this gives the following.

THEOREM 4.4. *An extension of Lie algebroids* $K \rightarrowtail A' \twoheadrightarrow A\Omega$ *in which* Ω *is* α*-simply connected is integrable iff the integrability obstruction* $e \in \check{H}^2(P, ZN^\sim)$ *of* p^*A' *lies in* $\check{H}^2(P, D)$ *for some discrete subgroup*

D *of* $ZN^{\sim}$. (Here $N^{\sim}$ is the simply-connected Lie group corresponding to the fibre-type of K and $ZN^{\sim}$ is its centre.) //

We also recover the following generalization, due to Almeida and Molino [2], §III.4, of the basic result on the integrability of Lie algebroids over simply-connected bases ([10] V 1.2).

PROPOSITION 4.4. *Let* $L \rightarrowtail A \xrightarrow{q} TB$ *be a transitive Lie algebroid on a connected base* B. *Let* $p\colon B^{\sim} \to B$ *be the universal covering of* B. *Then* A *is integrable iff the integrability obstruction* $e \in H^2(B^{\sim},ZN^{\sim})$ *of* p^*A *lies in* $\check{H}^2(B^{\sim}, D)$ *for some discrete subgroup* D *of* $ZN^{\sim}$. (Here $N^{\sim}$ is the simply-connected Lie group corresponding to the fibre-type of L.)

PROOF. Regard TB as $A\Pi(B)$, the Lie algebroid of the fundamental groupoid of B. Then, since $\Pi(B)$ is α-simply connected, we can apply 4.3. It only remains to show that the existence of an integrated extension

$$M \rightarrowtail \Phi \xrightarrow{\pi} \Pi(B)$$

is equivalent to the integrability of A. Given Φ, we certainly have $A\Phi \simeq A$ and since $\Pi(B) \twoheadrightarrow B\times B$ differentiates to the identity $TB \to TB$, the composite $\Phi \twoheadrightarrow \Pi(B) \twoheadrightarrow B\times B$ differentiates to $q\colon A \twoheadrightarrow TB$. Conversely, if A is integrable to Φ, which can be assumed to be α-simply connected, then $q\colon A\Phi \twoheadrightarrow A\Pi(B)$ integrates to $\Pi\colon \Phi \to \Pi(B)$, which must be surjective and a submersion by the same argument as before, and so gives an extension $\ker \Pi \rightarrowtail \Phi \twoheadrightarrow \Pi(B)$. //

Putting these two cases together, we have the following encompassing result.

THEOREM 4.5. *Let* $K \rightarrowtail A' \twoheadrightarrow A\Omega$ *be an extension of Lie algebroids with* Ω *an* α*-connected Lie groupoid. The extension is integrable to an extension of Lie groupoids* $M \rightarrowtail \Phi \twoheadrightarrow \Omega$ *iff the integrability obstruction* $e \in \check{H}^2(P^{\sim}, ZN^{\sim})$ *of the inverse image Lie algebroid* $p^*K \rightarrowtail p^*A' \twoheadrightarrow TP^{\sim}$*, where* $P^{\sim}$ *is the universal cover of the* α*-fibre type of* Ω*, lies in* $H^2(P^{\sim}, D)$ *for some discrete subgroup* D *of* $ZN^{\sim}$. //

REMARK. Given an extension $M \rightarrowtail \Phi \twoheadrightarrow \Omega$ of Lie groupoids with Ω and Φ α-connected, one can of course lift the composite $M\Phi \twoheadrightarrow \Phi \to \Omega$ to $M\Phi \twoheadrightarrow M\Omega$ and obtain an extension of $M\Omega$. Conversely, an extension $M \rightarrowtail \Phi \twoheadrightarrow M\Omega$ induces an extension of Ω by taking the composite $\Phi \twoheadrightarrow M\Omega \twoheadrightarrow \Omega$ and its kernel.

From this description it is possible to see how the kernels of extensions of Ω and extensions of $M\Omega$ may be related. This argument however is not a replacement for the proof of 4.4. //

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