CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 27, nº 3 (1986), p. 3-22
<http://www.numdam.org/item?id=CTGDC_1986__27_3_3_0>

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ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

Vol. XXVII-3 (1986)

ARCHIMEDIAN LOCAL C^∞-RINGS AND MODELS OF SYNTHETIC DIFFERENTIAL GEOMETRY

by Marta BUNGE and Eduardo J. DUBUC

RÉSUMÉ. Cet article considère les anneaux-C^∞ dans les topos de Grothendieck, et a pour but de les présenter de manière plus géométrique, et de rendre explicite leur rôle dans la construction de modèles de la GDS (Géométrie Différentielle Synthétique). En particulier, on étudie essentiellement ici les anneaux-C^∞ locaux et archimédiens, pour montrer qu'ils sont "la même chose" que les modèles bien adaptés de la GDS. En passant, on donne aussi une autre caractérisation des anneaux-C^∞ locaux et archimédiens dans un topos de Grothendieck; ce sont précisément les anneaux-C^∞ dans le topos qui possèdent, de plus, un morphisme d'anneaux locaux dans l'objet des réels de Dedekind du topos. En tant qu'étude intrinsèque des anneaux-C^∞, l'article ne présuppose aucune connaissance approfondie de la Géométrie Différentielle Synthétique.

0. INTRODUCTION.

In this note we establish, or rather clarify, the fact that Archimedian local C^∞-rings and well adapted models of SDG are essentially "the same thing". At the same time, a novel (geometric) approach to C^∞-rings is presented. This not only leads directly to the well adapted models among the Archimedian local C^∞-rings, but also makes explicit the basic principles employed in their construction. No specific acquaintance with the theory of SDG is presupposed.

This paper was written while the second author visited McGill University in the winter of 1986, as a guest of the Centre Interuniversitaire en études cétégoriques, for whose financial support he is grateful. The first author acknowledges support from the National Sciences and Engineering Research Council of Canada, and from the Ministère de l'Education du Québec.

1. C^∞-RINGS IN CATEGORIES WITH FINITE LIMITS.

By definition (cf. [7]) a C^∞-ring is a model (in Sets) of the algebraic theory whose n-ary operations are the C^∞-maps $\mathbf{R}^n \to \mathbf{R}$, with $\mathbf{R}$ the real numbers, and whose equations are all identities holding between them.

Denote by A the category of C^∞-rings and morphisms of C^∞-rings; A_{FT} and A_{FP} will denote - respectively - the full subcategories of A whose objects are the C^∞-rings of finite type, and the C^∞-rings which are of finite presentation among those of finite type.

By $\boldsymbol{M}$ we will denote the category of (paracompact) C^∞-manifolds and C^∞-maps. Let us recall a couple of *definitions*.

A pullback diagram

$$\begin{tikzcd} S \arrow[r, "k"] \arrow[d, "h"'] & M \arrow[d, "f"] \\ N \arrow[r, "g"] & T \end{tikzcd}$$

in $\boldsymbol{M}$ is called a *transversal pullback* if and only if for all $p \in S$, the images of the tangent spaces of M at $x = k(p)$ and of N at $y = h(p)$ generate the tangent space of T at $z = f(x) = g(y)$.

Let N be an n-dimensional C^∞-manifold; $h_1, \ldots, h_k$: N $\to$ R C^∞-maps. The functions $h_1, \ldots, h_k$ are said to be *independent* if and only if for all $p \in Z(h_1, \ldots, h_k)$ - with $Z(h_1, \ldots, h_k)$ the set of common zeros of the $h_1, \ldots, h_k$ in N - the rank of the Jacobian matrix $[\partial h_i/\partial x_j]$ at p equals k.

The following is the key result about C^∞-rings in the context of building models of SDG. The construction of all (known) models using C^∞-rings follows from it (cf. [3]).

1.1. THEOREM. *Let* N *be an n-dimensional C^∞-manifold;* $h_1, \ldots, h_k$: N $\to$ R *independent C^∞-functions. Then,* $M = Z(h_1, \ldots, h_k)$ *is an (n-k)-dimensional C^∞-manifold of* N, *and the restriction map*

$$C^\infty(N) \to C^\infty(M)$$

is a quotient in A, with kernel the ideal $(h_1, \ldots, h_k)$ *generated by the* h_i, $i = 1, \ldots, k$.

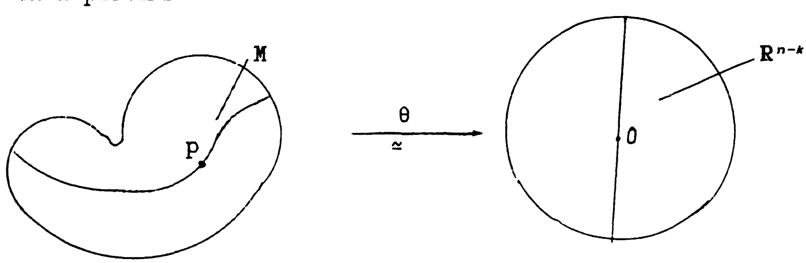
PROOF. In order to prove the theorem we will show that the above is an instance of a quotient of C^∞-rings for which "comuting it locally" agrees with "computing it globally". Three basic facts in the theory of C^∞-functions are essential for this purpose. Namely, the *implicit functions Theorem (I.F.T.)*, the *(local) Hadamard's Lemma (L.H.L)*, and *the existence of C^∞-partitions of unity (P.U.)*.

We recall the first:

(I.F.T.) For every $p \in M$, there exists an open $U \subset N$, $p \in U$, an open $V \subset \mathbf{R}^n$, $0 \in V$, and a diffeomorphism $\theta\colon U \to V$, $\theta(p) = 0$, such that the following diagram commutes.

$$\begin{tikzcd} M \cap U \arrow[r, hook] \arrow[d] & U \arrow[r, "(h_1, \ldots, h_k)"] \arrow[d, "\wr\ \theta"] & \mathbf{R}^k \arrow[d, equal] \\ \mathbf{R}^{n-k} \cap V \arrow[r] & V \arrow[r, "(x_{n-k+1}, \ldots, x_n)"'] & \mathbf{R}^k \end{tikzcd}$$

In a picture



This defines a structure of an $(n-k)$-dimensional C^∞-manifold for M (actually, a closed submanifold of N). It follows trivially that the restriction map $C^\infty(N) \to C^\infty(M)$ is locally surjective. That is, for all $p \in M$, $C^\infty{}_p(N) \twoheadrightarrow C^\infty{}_p(M)$ is surjective, where by $C^\infty{}_p$ we indicate the ring of germs at p of C^∞-functions.

The problem is to show now that the kernel of this map is the ideal $(h_1|_p, \ldots, h_k|_p)$ generated by the germs at p of the functions $h_1, \ldots, h_k$. For this, we need the *(local) Lemma of Hadamard*, in the form:

(L.H.L.) Let $U = U_1 \times U_2 \times \ldots \times U_n \subset \mathbf{R}^n$ be open, with the U_i, $i = 1, \ldots, n$, open intervals of $\mathbf{R}$, and let $f : U \to \mathbf{R}$ be any C^∞-function. Then there exist (unique) C^∞-functions

$$g_i : U \times U \to \mathbf{R}, \qquad i = 1, 2, \ldots, n,$$

such that for all $(\bar{x}, \bar{y}) \in U \times U$, $\bar{x} = (x_1, \ldots, x_n)$, $\bar{y} = (y_1, \ldots, y_n)$,

$$f(\bar{x}) - f(\bar{y}) = \Sigma_{i=1} \ (x_i - y_i) . \ g_i(\bar{x}, \bar{y}).$$

If we put

$$\bar{x} = (x_1, \ldots, x_n) \quad \text{and} \quad \bar{y} = (x_1, \ldots, x_{n-k}, 0, \ldots, 0)$$

above, we have the following form of the theorem:

Let $U = U_1 \times U_2 \times \ldots \times U_n \subset \mathbf{R}^n$ be as above, and let $f : U \to \mathbf{R}$ be a C^∞-function such that

$$f(x_1, \ldots, x_{n-k}, 0, \ldots, 0) = 0$$

for all $x_i \in \mathbf{R}$, $i = 1,2, \ldots, n-k$. Then there exist (unique) C^∞-functions $g_i : U \to \mathbf{R}$, $i = n-k+1, \ldots, n$, such that for all $\bar{x} \in U$, $\bar{x} = (x_1, \ldots, x_n)$,

$$f(\bar{x}) = \Sigma_{i=n-k+1} \ x_i . \ g_i(x).$$

We now return to the restriction map $C^\infty{}_p(N) \twoheadrightarrow C^\infty{}_p(M)$. We can assume that the open set V (as in (I.F.T.)) is a product of intervals (since these form a basis for the topology of $\mathbf{R}_n$). Let $f : U \to \mathbf{R}$ be such that $f|_{M \cap U} = 0$. It follows then that for any $\bar{x} = (x_1, \ldots, x_n)$ in V,

$$f\ (\theta^{-1}(x_1, \ldots, x_{n-k}, 0, \ldots, 0)) = 0.$$

Thus for every $\bar{x} \in V$,

$$f\, \theta^{-1}(\bar{x}) = \Sigma_{i=n-k+1} \ x_i . g_i(\bar{x}).$$

Therefore, for every $\bar{x} \in U$,

$$f(\bar{x}) = \Sigma_{i=1} \ h_i(\bar{x}) . \ g_{n-k+i}(\theta(\bar{x})),$$

which shows that the germ of f at $p \in U$ is in the ideal generated by the germs at p of the functions $h_1, \ldots, h_k$. Thus, for all $p \in M$ we have:

$$\begin{tikzcd}
C^\infty(N) \arrow[r, two heads] \arrow[d] \arrow[rr, bend left, "\text{restriction}"] & C^\infty(N)/(h_1, \dots, h_k) \arrow[r] \arrow[d] & C^\infty(M) \arrow[d] \\
C^\infty{}_p(N) \arrow[r, two heads] & C^\infty{}_p(N)/(h_1|_p, \dots, h_k|_p) \arrow[r] & C^\infty{}_p(M) \ .
\end{tikzcd}$$

This says that the restriction map is locally a quotient map.

To globalize we now use partitions of unity (P.U.). Let f be a C^∞-function defined on M. Take open sets $U_\alpha \subset N$ which cover M and smooth functions $f_\alpha \colon U_\alpha \to R$, such that

$$\text{for any } p \in U_\alpha \cap M, \qquad f_\alpha(p) = f(p) \ .$$

A standard (P.U.) argument permits to patch all the f_α together to obtain an extension of f to the whole of N. Let now $h : N \to R$ be any function such that $h|_M = 0$. Take open sets $U_\alpha \subset N$ such that they cover M and such that

$$h|U_\alpha \in (h_1|U_\alpha, \dots, h_k|U_\alpha) \subset C^\infty(U_\alpha).$$

Let

$$U_i = \{x \mid h_i(x) \neq 0\}, \qquad \text{for} \qquad i = 1, \dots, k.$$

Clearly, the U_α together with the U_i cover the whole space N and also

$$h|U_i \in (h_1|U_i, \dots, h_k|U_i),$$

trivially, for each $i = 1, \dots, k$. Let $\{U_\beta\}$ indicate the open covering $\{U_\alpha\} \cup \{U_i\}$. We then have, for all β,

$$h|U_\beta \in (h_1|U_\beta, \dots, h_k|U_\beta).$$

We can assume $\{U_\beta\}$ to be locally finite, with an associated partition of unity $\{\phi_\beta\}$. Let $f_i{}^\beta : U_\beta \to R$ be such that

$$h|U_\beta = \Sigma_{i=1}\, g_\beta .\ h_i|U_\beta \ .$$

The functions $\phi_\beta .\ g_i{}^\beta$ are all defined globally, and

$$\phi_\beta .\ h = \Sigma_{i=1}\, \phi_\beta .\ g_i{}^\beta .\ h_i .$$

Thus

$$h = \Sigma_B\ \phi_B .\ h = \Sigma_B\ \Sigma_{i=1}^{k}\ \phi_B .\ g_i^{\,B} .\ h_i = \Sigma_{i=1}^{k}\ \Sigma_B\ \phi_B\ .\ g_i^{\,B} .\ h_i = \Sigma_{i=1}^{k}\ g_i .\ h_i,$$

where

$$g_i = \Sigma_B\ \phi_B .\ g_i^{\,B} .$$

(Remark that for each $x \in$ N, the sums above are finite in a neighborhood of x.) This shows that $h \in (h_1, \ldots, h_k)$. Hence, the theorem is proved. •

In what follows, we derive some useful consequences of the above.

1.2. COROLLARY. a) For any U ⊂ $\mathbf{R}^n$ *open,* C^∞(U) *is a finitely presented* C^∞*-ring, and the restriction map* $C^\infty(\mathbf{R}^n) \to C^\infty(U)$ *is an epimorphism in* A.

b) For any U ⊂ $\mathbf{R}^n$ *and* V ⊂ $\mathbf{R}^m$ *both open,*

$$C^\infty(U) \otimes_\infty C^\infty(V) \simeq C^\infty(U \times V),$$

where $\otimes_\infty$ *denotes the coproduct in* A.

c) For U ⊂ $\mathbf{R}^n$, V ⊂ $\mathbf{R}^m$ *both open, and* f, g : U → V *independent functions,*

$$C^\infty(E) \simeq C^\infty(\mathbf{R}^n)/(f-g) ,$$

where E → U *is the equalizer of* f, g.

PROOF. a) Let $\phi : \mathbf{R}^n \to \mathbf{R}$ be a smooth characteristic function of U ⊂ $\mathbf{R}^n$, i.e., let ϕ be such that

$$U = \phi^{-1}(\mathbf{R}^*), \qquad \mathbf{R}^* = \mathbf{R}\setminus\{0\} .$$

The restriction map $C^\infty(\mathbf{R}^n) \to C^\infty(U)$ is the universal solution to the problem of making the element ϕ invertible in the category of C^∞-rings and, as such, is an epimorphism. Indeed, a simple direct proof of this may be given as follows. Let $\gamma : U \to \mathbf{R}^{n+1}$ be given by

$$\gamma(\bar{x}) = (\bar{x},\ 1/\phi(\bar{x})).$$

This map is injective, and identifies U with the set of zeroes of the function

$$1 - \phi(\bar{x}).y : \mathbf{R}^{n+1} \to \mathbf{R}.$$

This function is independent, thus $Z(1 - \phi(x).y)$ carries the structure of a closed submanifold of $\mathbf{R}^{n+1}$, and γ is a diffeomorphism of it with U. It follows from 1.1 that γ establishes an isomorphism

$$\gamma^* : C^\infty(\mathbf{R}^{n+1})/(1-\phi(\bar{x}).y) \simeq C^\infty(U).$$

Clearly, the following diagram is commutative:

$$\begin{tikzcd}
C^\infty(\mathbf{R}^n) \arrow[rr] \arrow[d] & & C^\infty(\mathbf{R}^{n+1}) \arrow[d] \arrow[dll, "\rho"'] \\
C^\infty(U) \arrow[rr, "\gamma^*"] & & C^\infty(\mathbf{R}^{n+1})/(1-\phi(x).y)
\end{tikzcd}$$

where ρ is induced by γ. We now show that the restriction map induced by the inclusion $U \subset \mathbf{R}^n$ is an epimorphism in the category of C^∞-rings. Let $h, h' : C^\infty(U) \to A$ be equal on the image of the restriction map. Then, $h(x_i) = h'(x_i)$, $i = 1, \ldots, n$. Also, $h(\phi) = h(\phi')$. Thus, also $h(1/\phi) = h(1/\phi)$. This shows that $h\rho(y) = h'\rho(y)$, since $\gamma^*(y) = 1/\phi$. But $C^\infty(\mathbf{R}^{n+1})$ is freely generated by $x_1, \ldots, x_n$ and y. Since h and h' coincide already on $x_1, \ldots, x_n$, it follows that they are equel on the whole of $C^\infty(\mathbf{R}^{n+1})$. Now, since ρ is a quotient map (by the commutativity of the diagram and γ^* iso), h and h' are equel on the whole of $C^\infty(U)$. This finishes the proof.

b) Let ϕ and ψ be smooth characteristic maps of U and V, respectively. Using (a) we see that

$$C^\infty(U) \otimes_\infty C^\infty(V) \simeq C^\infty(\mathbf{R}^{n+1+m+1})/(1-\phi(\bar{x}).\bar{y},\ 1-\psi(\bar{u}).\bar{v}).$$

Since

$$U \simeq Z(1-\phi(\bar{x}).\bar{y}) \quad \text{and} \quad V \simeq Z(1-\psi(\bar{u}).\bar{v}),$$

it follows that

$$U \times V \simeq Z(1-\phi(\bar{x}).\bar{y},\ 1-\psi(\bar{u}).\bar{v}).$$

Since the functions $1-\phi(x).y$, $1-\psi(u).v$ are independent, 1.1 gives that

$$C^\infty(U) \otimes_\infty C^\infty(V) \simeq C^\infty(U \times V).$$

c) Notice that $E = Z(f-g)$ and that the function $f-g$ is independent (since the functions f, g are assumed to be independent). Apply now 1.1 to get

$$C^\infty(E) \simeq C^\infty(\mathbf{R}^n)/(f-g). \qquad \bullet$$

Denote by C the category of opens of the $\mathbf{R}^n$, $n > 0$, and *smooth* maps. The following is easily shown (cf [5]).

1.3. LEMMA. *Let* F : $C \to E$ *be a functor, where* E *is any category with finite limits. Then, the following are equivalent:*

(i) F *preserves transversal pullbacks (i.e., takes any transversal pullback in* C *into a pullback in* E*) and the terminal object* 1 *;*

(ii) F *preserves open inclusions, finite products and equalizers of independent functions.* •

1.4. THEOREM. *The functor* $\overline{C^\infty(-)} : C \to A^{op}$*, defined on objects by the rule :* $U \mapsto \overline{C^\infty(U)}$ *(where the bar merely indicates that the object* $C^\infty(U)$ *of* A *is to be regarded in the opposite category), factors through the inclusion* $A_{FP}{}^{op} \hookrightarrow A^{op}$ *and preserves transversal pullbacks and* 1.

PROOF. By **1.2** (a), $C^\infty(U)$ is finitely presented. Also by **1.2** (a), $\overline{C^\infty(-)}$ preserves open inclusions of the form $U \subset \mathbf{R}^n$: this is enough to show that $\overline{C^\infty(-)}$ preserves arbitrary open inclusions in C, as if $U \subset V \subset \mathbf{R}^n$, if the composite

$$C^\infty(\mathbf{R}^n) \longrightarrow C^\infty(V) \longrightarrow C^\infty(U)$$

is an epimorphism, also $C^\infty(V) \to C^\infty(U)$ is an epimorphism in A.

By **1.2** (b), $\overline{C^\infty(-)}$ preserves finite products and, by **1.2** (c), it preserves equalizers of independent functions. Finally, 1.3 gives that $\overline{C^\infty(-)}$ preserves transversal pullbacks and 1. •

Given a C^∞-ring A in E (a category with finite limits), and any finitely presented C^∞-ring B, say

$$B = C^\infty(\mathbf{R}^n)/(h_1, \ldots, h_k),$$

denote by $\mathrm{Spec}_A(B) \in E$ the object $Z(h_1, \ldots, h_k)$ of E defined as in the equalizer

$$Z(h_1, \ldots, h_k) \hookrightarrow A^n \overset{(h_1, \ldots, h_k)}{\underset{0}{\rightrightarrows}} A^k$$

in E (the notation $h_i : A^n \to A$, for $h_i : \mathbb{R}^n \to \mathbb{R}$, indicates the corresponding n-ary operation on A given by its structure of a C^∞-ring in E). This association gives a functor

$$\mathrm{Spec}_A(-) : \mathcal{A}_{FP}{}^{op} \to E .$$

We now recall from the general and well established theory of algebraic theories, the following fact.

1.5. THEOREM. *Let E be a category with finite limits.*

(a) There is a bijection between :

(i) finite limit preserving functors $F : \mathcal{A}_{FP}{}^{op} \to E$ *;*

and (ii) C^∞*-rings* A *in E.*

The bijection is given by assigning, to a C^∞*-ring* A *in E the functor* $F_A = \mathrm{Spec}_A(-)$ *; under this bijection* $A = F_A(\overline{C^\infty(\mathbb{R})})$.

b) The above extends to a bijection between morphisms $A \to B$ *of* C^∞*-rings and natural transformations* $F_A \to F_B$. ♦

The following is now a characterization of C^∞-rings in categories with finite limits which renders explicit the geometric nature of the ("algebraically" defined notion of a) C^∞-ring.

1.6. THEOREM. *Let E be a category with finite limits.*

a) There is a bijection between:

(i) functors $F : C \to E$ *preserving transversal pullbacks and* 1 *;*

and (ii) C^∞*-rings* A *in E.*

b) This bijection extends to one between morphisms, as in **1.5.**

PROOF. For A a C^∞-ring in E, the composite

$$F_A = (C \xrightarrow{\overline{C^\infty(-)}} \mathcal{A}_{FP}{}^{op} \xrightarrow{\mathrm{Spec}_A(-)} E)$$

preserves transversal pullbacks and 1, by **1.4**, **1.5**.

In the opposite direction, if F : $C \to E$ preserves transversal pullbacks and 1, then A = F(R) is a C^∞-ring in E (since F preserves finite products, by **1.3**). Moreover, F = F_A. To see this, just observe that a transversal pullback and 1 preserving functor is totally determined by its value at $R \in C$. This follows from considering the diagrams:

$$\begin{tikzcd} R^* \arrow[r, tail, "{(x,\, 1/x)}"] \arrow[d] & R\times R \arrow[d, "\cdot"] \\ 1 \arrow[r, "\ulcorner 1 \urcorner"] & R \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U \arrow[r] \arrow[d] & R^n \arrow[d, "\phi"] \\ R^* \arrow[r] & R \end{tikzcd}$$

which are transversal pullbacks, where ϕ is - as in the proof of **1.2** (a), any smooth characteristic function of U. Part (b) is straightforward from **1.5**.b •

NOTATION. In order to stress the fact that a C^∞-ring A is the "same thing" as a functor $C \to E$ (preserving transversal pullbacks and 1), as established in **1.6**, we shall denote this functor by the same letter A. Thus, in this notation, A = A(R) and, for any open set $U \subset R^n$, there is a subobject $A(U) \hookrightarrow A^n$ such that any C^∞-function $f : U \to R$ has an interpretation $f : A(U) \to A$. Thus, for a C^∞-ring A, in addition to the n-ary operations $A^n \to A$ arising from the algebraic theory of C^∞-rings, viewing it as a transversal pullbacks preserving functor (in addition to a finite products preserving functor) makes explicit also those partially defined n-ary operations with domain $A(U) \hookrightarrow A^n$ corresponding to an open $U \subset R^n$, and a C^∞-function $f : U \to R$.

Very often, the functor is given more naturally than in the canonical construction utilized in the proof of **1.6**. Thus, for example, if M is a manifold, and $A = C^\infty(M)$, then $A(U) = C^\infty(M, U)$, and if $A = C^\infty{}_p(M)$, then $A(U) = C^\infty{}_p(M, U)$ (i.e., n-tuples of germs

$$(f_1, \ldots, f_n) \in C^\infty{}_p(M),$$

such that the n-tuple of their values at p is in U,

$$(f_1(p), \ldots, f_n(p)) \in U \quad).$$

For the remainder of this Section (and article), E will be assumed to be a Grothendieck topos. We will make use then of the internal language and logic of E (cf. [7], or [1]).

1.7. PROPOSITION. *Let E be a category with finite limits. For* A *a* C^∞-*ring in* E, *denote by* A^* *the subobject of its invertible elements, i.e.,*

$$A^* = [\![x \in A \mid \quad y \in A \ (x.y = 1)]\!].$$

a) Then, $A(R^*) = A^*$, where $R^* = R\setminus\{0\}$.

b) Moreover, for any $U \subset R^n$ *open, with* ϕ *a smooth characteristic map of* U, $A(U) = \phi^{-1}(A^*)$. *That is,* A(U) *can be described as*

$$A(U) = [\![\bar{x} \in A^n \mid \phi(\bar{x}) \text{ is invertible in } A]\!].$$

PROOF. It follows immediately from considering the two diagrams in the proof of 1.6. •

2. ARCHIMEDIAN LOCAL C∞-RINGS IN GROTHENDIECK TOPOSES.

Let A be a C^∞-ring in E, E a Grothendieck topos. Then A possesses a strict order $>$, compatible with the ring operations, as well as a strict order $<$, also, compatible with the ring operations. Indeed, define

$$A_{>0} = A(R_{>0}) \hookrightarrow A(R) = A.$$

Compatibility with the ring operations follows from the functoriality of A. E.g., the commutativity of

$$\begin{tikzcd} R_{>0} \times R_{>0} \arrow[r, "+"] \arrow[d, hook] & R_{>0} \arrow[d] \\ R\times R \arrow[r, "+"] & R \end{tikzcd}$$

implies that of

$$\begin{array}{ccc} A_{>0} \times A_{>0} & \xrightarrow{\quad + \quad} & A_{>0} \\ \downarrow & & \downarrow \\ A \times A & \xrightarrow{\quad + \quad} & A . \end{array}$$

Similarly, let

$$A_{<0} = A(R_{<0}) \hookrightarrow A = A(\mathbf{R}).$$

Since

$$R_{<0} = \{x \in \mathbf{R} \mid -x \in R_{>0}\},$$

we have, for $x \in A$, that $x \in A_{<0}$ iff $-x \in A_{>0}$.

Denote by $\mathbf{N}$ the natural numbers object of E. A C^∞-ring A in E is said to be *local* if and only if

$$\forall\ x \in A\ [x \in A^* \vee (1-x) \in A^*]$$

holds in E. It is said to be *archimedian* if and only if

$$\forall\ x \in A\ [\ \bigvee_{n \in N} (-n<x \vee x<n)]$$

holds in E.

The next proposition follows immediately from **1.7**. In it, the expression

" A: *$C \to E$ preserves the open covering* $\{U_\alpha \subset U\}_\alpha$ "

is to be interpreted as saying:

" A takes the open covering $\{U_\alpha \subset U\}_\alpha$ into an *epimorphic family* $\{A(U_\alpha) \to A(U)\}_\alpha$ *in* E ".

2.1. PROPOSITION. *Let* A *be a C^∞-ring in a Grothendieck topos E. Then, the following are equivalent:*

(i) A *is archimedian local as a C^∞-ring in E;*

and

(ii) A : *$C \to E$ preserves the open coverings*

$$(a) \qquad R = R^* \cup (1-R^*), \text{ where}$$
$$1 - R^* = \{x \in R \mid 1-x \in R^*\},$$
$$(b) \quad R = \bigcup_{n\in N} (-n, n). \quad \bullet$$

The following holds; the proof given here was obtained in collaboration with A. Joyal.

2.2. PROPOSITION. *Every open covering of C is generated (by pullback composition and refinement) by the two basic coverings (a) and (b) of 2.11.*

PROOF. We may restrict ourselves to open coverings of R^n, $n > 0$. Let $\{U_\alpha \subset R^n\}_{\alpha\in I}$ be any such open covering. If J is the set of finite parts of I and, for $\beta = \{\alpha_1, \ldots, \alpha_k\} \in J$ we let

$$V_\beta = U_{\alpha 1} \cup \ldots \cup U_{\alpha k},$$

then the given open covering may be obtained, by composition, from the covering $\{W_\beta \subset R^n\}_{\beta\in J}$ and the coverings

$$\{U_\alpha \subset W_\beta)_{\alpha\in\beta}\ , \qquad \text{for each} \quad \beta \in J.$$

Any finite covering can be reduced to the case $R^n = U \cup V$, with U, $V \subset R^n$, open. In turn, if $\phi, \psi : R^n \to R$ are smooth characteristic functions of U, V, we may assume that $\phi + \psi = 1$ (replacing, if necessary, ϕ, ψ by

$$\phi^2/(\phi^2 + \psi^2),\ \psi^2/(\phi^2 + \psi^2)\ ).$$

Thus, while $U = \phi^{-1}(R^*)$, $V = \phi^{-1}(1-R^*)$. Hence

$$U \cup V = \phi^{-1}(R^* \cup (1-R^*)),$$

a covering of type (a). This applies to each of the $\{U_\alpha \subset W_\beta)_{\alpha\in\beta}$, $\beta \in J$.

Consider now the smooth map $\sigma : R^n \to R$ given by

$$(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2\ .$$

Let

$$U_m = \sigma^{-1}(-m, m), \qquad R^m = \bigcup_{m\in N} U_m.$$

For $\{W_\beta \subset \mathbf{R}^n\}_{\beta\in J}$, a refinement can be given by the $\{U_m \subset \mathbf{R}^n\}_{m\in \mathbf{N}}$. Indeed, since for each m, $\overline{U}_m$ (the closure in $\mathbf{R}^n$) is compact, and since $\{U_\alpha \subset \mathbf{R}^n\}_{\alpha\in I}$ is an open covering, for some finite

$$\beta = \{\alpha_1, \ldots, \alpha_k\} \subset I, \; U_m \subset \overline{U}_m \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_k} = W_\beta.$$

Thus, $\{W_\beta \subset \mathbf{R}^n\}_{\beta\in J}$ is obtained from a covering of type (b) by refinement from $\{U_m \subset \mathbf{R}^n\}_{m\in \mathbf{N}}$. •

2.3. ***THEOREM.*** *Let E be a Grothendieck topos. There is a bijection between:*

(i) functors $C \to E$ preserving transversal pullbacks (and 1) and arbitrary open coverings,

(ii) archimedian local C^∞-rings in E.

PROOF. Immediately from 2.1 and 2.2. •

Let $\mathbf{R}_E$ be the object of Dedekind reals in E. It is the object of Dedekind cuts, each of which is a model of the theory determined by the lattice (locale) $\sigma(\mathbf{R})$ of open subsets of the real numbers $\mathbf{R}$ (in Sets). This can also be presented as a theory of open intervals with rational end points (Joyal and Joyal-Tierney, see [4]). Thus, a Dedekind cut is a functor $D : \sigma(\mathbf{R}) \to E$, which sends 1 ($1 = \mathbf{R}$) to $\mathbf{R}_E$, and which preserves finite intersections and arbitrary unions. It is not difficult to see that this extends to a functor $D : O \to E$, from the *category O* of open sets of the $\mathbf{R}^n$, $n > 0$, and *continuous maps*, which preserves all finite limits and arbitrary unions; with $\mathbf{R}^n$ sent to $\mathbf{R}_E$. (Consider the fact that if U is open and f continuous, then $f^{-1}(U)$ is open.)

Since $C \hookrightarrow O$, it follows, in particular, that $\mathbf{R}_E$ *is an archimedian local C^∞-ring in E.*

As we will show below, the existence of a so-called *standard map* (cf. [8]) $\pi : A \to \mathbf{R}_E$ is automatic for any archimedian local C^∞-ring A in E, and is, in a sense, characteristic of the latter. This requires a definition.

2.4. ***DEFINITION.*** Let $f : A \to B$ be a morphism of C^∞-rings in E. Call f *a local map of C^∞-rings* if and only if for each $U \subset W$ in C, the commutative diagram (by naturality of f)

$$\begin{tikzcd} A(U) \arrow[r, "f_U"] \arrow[d] & B(U) \arrow[d] \\ A(W) \arrow[r, "f_W"] & B(W) \end{tikzcd}$$

is a pullback.

2.5. ***PROPOSITION.*** *Given any morphism* $f : A \to B$ *of* C^∞*-rings in* E*, the following statements are equivalent:*

a) f *is a local map of* C^∞*-rings;*

b) for every open inclusion $U \subset \mathbb{R}^n$ *(and* ϕ *any smooth characteristic map of* U*), the commutative diagram*

$$\begin{tikzcd} A(U) \arrow[r, "f_U"] \arrow[d, hook] & B(U) \arrow[d, hook] \\ A^n \arrow[r, "f^n"] & B^n \end{tikzcd}$$

is a pullback; equivalently, in terms of $\phi : \mathbb{R}^n \to \mathbb{R}$ (recall **1.7**), *the statement*

$$\forall\ \bar{x} \in A^n\ [\ \phi(\bar{x}) \in A^* \Leftrightarrow \phi(f^n(\bar{x})) \in B^*]$$

holds in E *;*

c) f *is a local map of rings (in the usual sense), i.e., it satisfies:* $\forall\ x \in A\ [x \in A^* \Leftrightarrow f(x) \in B^*]$.

PROOF. (a) $\Leftrightarrow$ (b) follows immediately from general properties of pullback diagrams on account of the fact that

$$U \subset W \subset \mathbb{R}^n \quad \text{for some} \quad n > 0 .$$

Clearly (b) $\Leftrightarrow$ (c). It remains to be seen that (c) $\Leftrightarrow$ (b). Consider the diagram

$$\begin{tikzcd}[row sep=small, column sep=small] & A^* \arrow[rr] \arrow[dd, hook] & & B^* \arrow[dd, hook] \\ A(U) \arrow[ur] \arrow[rr, crossing over, "f_U"] \arrow[dd, hook] & & B(U) \arrow[ur] & \\ & A \arrow[rr, "f"] & & B \\ A^n \arrow[ur, "\phi"] \arrow[rr, "f_n"] & & B^n \arrow[ur, "\phi"'] \arrow[from=uu, crossing over, hook] & \end{tikzcd}$$

where the lateral faces are pullbacks (by preservation of transversal pullbacks) and the back face is a pullback by assumption; it follows from general properties of pullbacks that also the front face is a pullback. •

2.6. LEMMA. Let $f : A \to B$ be a local map of C^∞-)rings in E. Let $\{U_\alpha \subset U\}_\alpha$ be any open covering in C which is preserved by B. Then such an open covering is also preserved by A.

PROOF. In any Grothendieck topos, epimorphic families are pullback stable. •

2.7. COROLLARY. Let $f : A \to B$ be a local map of C^∞-rings in E. Then, if B is archimedian local, so is A.

PROOF. Immediate from 2.1, 2.6. •

2.8. THEOREM. Let E be a Grothendieck topos, with R_E the object of Dedekind reals in it. Then, there is a bijection between:

(i) archimedian local C^∞-rings A in E;

(ii) C^∞-rings A in E, together with a local map $\pi : A \to R_E$ of C^∞-rings;

(iii) C^∞-rings A in E, together with a map $\pi : A \to R_E$ of C^∞-rings which is local as a map of rings.

PROOF. (i) ⇒ (iii). Clearly (by 2.3) an archimedian local C^∞-ring in E, viewed as a functor $C \to E$, determines a morphism of locales (observe that $O(\mathbf{R}) \hookrightarrow C$) :

$$O(\mathbf{R}) \longrightarrow P(A) \;:\; U \mapsto A(U),$$

where $P(A)$ is the locale of subobjects of A in E. In turn (cf. [4]), this induces a morphism $\pi : A \to R_E$ in E, with the property that for any $U \subset \mathbf{R}$ open, the diagram

$$\begin{array}{ccc} A(U) & \longrightarrow & R_E(U) \\ \downarrow & & \downarrow \\ A & \xrightarrow{\pi} & R_E \end{array}$$

is a pullback. To finish the proof, just take $U = \mathbf{R}^*$.

(iii) $\Rightarrow$ (ii). Immediate from 2.5.

(ii) $\Rightarrow$ (i). Since $\mathbf{R}_E$ is archimedian local (by a remark made earlier), this follows from 2.7. •

3. REMARKS CONCERNING WELL ADAPTED MODELS OF SDG.

Well adapted models of SDG were introduced in [2] (see also [6,7]). Theorem 2.3 above says that they are, essentially, just archimedian local C^∞-rings in some Grothendieck topos. These, in turn, by Theorem 2.8, are C^∞-rings in a Grothendieck topos, together with a local map of rings into the object of Dedekind reals in that topos. Thus, to construct a well adapted model of SDG amounts, basically, to construct a topos with a C^∞-ring in it, together with a local map into the Dedekind reals. Of course, a further *non-elementary* requirement is imposed; namely, the model, viewed as a functor, should be full and faithful on a sufficiently large category of (duals of) C^∞-rings.

Usually, also, a well adapted model is supposed to be defined, as a functor, on the category $\mathcal{M}$ of all (paracompact) C^∞-manifolds, rather than just on $\mathcal{C}$. This turns out to be equivalent because the inclusion $\mathcal{C} \hookrightarrow \mathcal{M}$ is "dense" in the sense that for any $M \in \mathcal{M}$, there exists an open cover $\{U_\alpha \subset M\}_{\alpha\in I}$ with $U_\alpha \approx V_\alpha \in \mathcal{C}$. In fact, we have:

3.1. THEOREM. *Let $\mathcal{E}$ be a Grothendieck topos. There is a bijection between:*

(i) functors $\mathcal{M} \to \mathcal{E}$ preserving transversal pullbacks and 1, and open coverings,

and (ii) archimedian local C^∞-rings A in $\mathcal{E}$.

PROOF. Clearly (i) $\Rightarrow$ (ii) since an archimedian local C^∞-ring A in $\mathcal{E}$ can be identified with a functor $\mathcal{C} \to \mathcal{E}$ preserving transversal pullbacks and 1, and open coverings (see Theorem 2.3), so that one such results from a functor $\mathcal{M} \to \mathcal{E}$ as in (i), by restricting to $\mathcal{C} \to \mathcal{M}$.

To see that (ii) $\Rightarrow$ (i), let $A : \mathcal{C} \to \mathcal{E}$ be an archimedian local C^∞-ring in $\mathcal{E}$, viewed as a functor. Consider the definition of a manifold $M \in \mathcal{M}$ as a collection of open sets $\{U_\alpha \subset \mathbf{R}^n\}_{\alpha\in I}$ together with patching data, that is, a construction of the manifold M as a quotient of the disjoint union (coproduct) of the U_α's. Then, A determines a collection of objects in $\mathcal{E}$,

$$\{A(U_\alpha) \longrightarrow A^n\}_{\alpha \in I} \; ,$$

together with (the same) patching data. Thus, the same construction performed to obtain M, when performed in E, determines an object A(M) of E. It is straightforward to show that this process defines a functor $M \to E$ which preserves transversal pullbacks and open covers and which restricts to $A : C \to E$ on C. •

3.2. REMARK. Let us recall, in connection with Theorem 2.1, the observation (cf. [3]) - due to Lawvere - that since a C^∞-manifold M, can be viewed as a retract of an open U of some $\mathbf{R}^n$, the C^∞-ring $C^\infty(M)$ is finitely presented, as $C^\infty(U)$ is finitely presented. Hence, there is an extension of $\overline{C^\infty(-)} : C \to A_{FP}{}^{op}$ to M, i.e.

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\overline{C^\infty(-)}"'] & M \arrow[dl, "\overline{C^\infty(-)}"] \\ A_{FP}{}^{op} & \end{tikzcd}$$

As in Theorem 1.4, it can be proved that this extension preserves transversal pullbacks and 1. If A is an archimedian local C^∞-ring in E, then the composite

$$M \xrightarrow{\overline{C^\infty(-)}} A_{FP}{}^{op} \xrightarrow{\mathrm{Spec}_A(-)} E$$

also preserves open covers. Having established this, it immediately follows from Theorem 1.6 (which says that $\mathrm{Spec}_A(\overline{C^\infty(U)}) = A(U)$ for all $U \subset \mathbf{R}^n$) that this composite must agree with the functor $M \to E$ associated with A in Theorem 3.1. Thus, these remarks constitute an alternative proof of the same theorem. However, it must be pointed out that this way of argueing relies on the Whitney embedding Theorem - a highly non elementary fact, also focusing attention on a secondary aspect of manifolds, namely, that of being closed subsets of some $\mathbf{R}^n$, whereas, by their very definition, manifolds are locally open subsets of some $\mathbf{R}^n$, which is what is used in the proof we give here of Theorem 3.1.

3.3. REMARK. The geometric approach to C^∞-rings presented here is particularly useful in order to establish directly many properties of the ring of line type R in a well adapted model. Some of these properties are postulated by the authors elsewhere (cf. [9]) in their axiomatic treatment of well adapted models. For example, the order in a C^∞-ring, as defined above, is *strict* $(\neg(x < x))$, and for an archimedian local C^∞-ring - although only the preservation of finite open covers is needed here - this order is *local*

$$(y < x \Rightarrow \forall z\ (y < x \vee y < z))$$

and *total on the invertibles*

$$(\neg(x_1, \ldots, x_n) = 0 \Rightarrow \bigvee_{i=1}^{n} x_i > 0 \vee x_i < 0),$$

where an additional assumption is that R is a field of fractions, i.e., that $R^* = \neg\{0\}$. Using similar techniques, the result proven in [10], which says that, in a topos, *any local C^∞-ring is separably real closed*, is immediately recovered using the coherent axiomatization of the notion of separably real closed local ring, once the appropriate transversal pullback is identified, and applies, in particular, also to the object of Dedekind reals, a proof of which is included separately, in [10].

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