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CLOSURE OPERATORS, MONOMORPHISMS AND EPIMORPHISMS IN CATEGORIES OF GROUPS

BY Gabriele CASTELLINI

RÉSUMÉ. On donne une caractérisation des épimorphismes dans une sous-catégorie $\underline{C}$ d'une catégorie concrète $(\underline{A}, U)$ sur une catégorie $\underline{X}$, en termes d'opérateurs de fermeture. On obtient aussi une caractérisation des monomorphismes dans $\underline{C}$ à l'aide de la notion duale d'opérateur de cofermeture. Les résultats sont illustrés par des exemples dans des catégories de groupes abéliens.

In many concrete categories, whose objects are structured sets (e.g., SET, TOP, GR, TG), the epimorphisms coincide with the surjective morphisms. However, this is not always the case. E.g., in the category TOP_2 of Hausdorff topological spaces, the epimorphisms are precisely the dense maps. The problem of characterizing the epimorphisms is still unsolved for many important categories such as, for example, the category of Hausdorff topological groups. In this paper a categorical approach to the above problem is presented.

In 1975, S. Salbany [14] introduced the concept of closure operator induced by a subcategory $\underline{C}$ of TOP. A similar idea had already appeared in 1965, in some papers by J.R. Isbell [11, 12, 13]. Salbany showed that in $TOP_{3\frac{1}{2}}$ (Tychonoff spaces) the closure induced by $TOP_{3\frac{1}{2}}$ is the usual closure and in TOP_0, the TOP_0-closure is the b-closure (cf. [1]), also called front-closure (cf. [15]). He also gave a characterization of epimorphisms in terms of TOP_0-closure. Other characterizations in TOP_1 and TOP_2 can be found in [2]. In 1980, E. Giuli [7] provided a characterization of epimorphisms in terms of $\underline{C}$-closure for any epireflective subcategory $\underline{C}$ of TOP. Many results and examples about the $\underline{C}$-closure for specific subcategories of TOP can be found in [4, 5, 8].

In §1, we define the concept of closure operator over an object of a category, together with the concept of $\underline{C}$-closure for a subcategory $\underline{C}$ of a concrete category $(\underline{A}, U)$ over a category $\underline{X}$. This allows us to show a relationship between epimorphisms in $\underline{C}$ and $\underline{C}$-closure. Such a characterization is less restrictive than the one in TOP that can be found in [7], since we do not require $\underline{C}$ to be epireflective. Moreover, this general formulation allows us to use such results in categories that are not necessarily concrete over SET.

In §2, we introduce the concept of coclosure operator, which is the exact dual of closure operator, and we formulate duals of results in §1. Thus, we get a characterization of the monomorphisms in $\underline{C}$, in terms of $\underline{C}$-coclosure. This idea is quite new, since the problem of characterizing the monomorphisms does not arise in TOP ; the reason being that in every full subcategory of TOP, the monomorphisms are injections. However, in other categories, such as AB, one can easily find interesting subcategories, where the monomorphisms are not necessarily injections.

Applications of the theory (and of the dual theory) are provided in §3.

1. CLOSURE OPERATORS AND EPIMORPHISMS.

Notations.

Given a category $\underline{X}$, throughout the paper $\tilde{M}$ (resp., $\tilde{E}$) will denote a class of $\underline{X}$-monomorphisms (resp. $\underline{X}$-epimorphisms) satisfying:

(1) $\tilde{M}$ (resp., $\tilde{E}$) is closed under pullbacks (resp., pushouts),

(2) $\tilde{M}$ (resp., $\tilde{E}$) contains all (arbitrary) intersections of regular subobjects (resp., cointersections of regular quotients).

For example the class of all $\underline{X}$-monomorphisms (resp., $\underline{X}$-epimorphisms) and the class of all strong monomorphisms (resp., strong epimorphisms) in $\underline{X}$, defined below, both satisfy the above conditions for $\tilde{M}$ (resp., $\tilde{E}$).

Given $X \in \underline{X}$, $\tilde{M}(X)$ denotes the class of all $\tilde{M}$-subobjects of X, i.e., $(M, m) \in \tilde{M}(X)$ means that $m : M \to X$ belongs to $\tilde{M}$. $\tilde{E}(X)$ denotes the class of all $\tilde{E}$-quotients of X, i.e., $(q, Q) \in \tilde{E}(X)$ means that $q : X \to Q$ belongs to $\tilde{E}$. Given (M, m) and (N, n) belonging to $\tilde{M}(X)$, we will write

$$(M, m) \tilde{\leq} (N, n) \quad \text{iff there exists a morphism } t : M \to N \text{ such that } nt = m.$$

Similarly, given (q, Q) and (p, P) belonging to $\tilde{E}(X)$, we will write

$$(q, Q) \leq^{o} (p, P) \quad \text{iff there exists a morphism } e : P \to Q \text{ such that } ep = q .$$

Definition 1.1. Let $\underline{X}$ be any category and let $X \in \underline{X}$. By a closure operator over X, we mean a function $[\ ]^{X} : \tilde{M}(X) \to \tilde{M}(X)$ satisfying for every (M, m) and (N, n) belonging to $\tilde{M}(X)$:

(a) $(M, m) \tilde{\leq} [(M, m)]^{X}$;

(b) $[[(M, m)]^{X}]^{X} \simeq [(M, m)]^{X}$;

(c) If $(N, n) \tilde{\leq} (M, m)$, then $[(N, n)]^{X} \tilde{\leq} [(M, m)]^{X}$.

The $\tilde{M}$-subobject (M, m) is called $[\ ]^X$-*closed* provided that

$$(M, m) \simeq [(M, m)]^X .$$

We observe that the concept of closure operator is given only up to isomorphism, i.e., two closure operators $[\ ]_1^X$ and $[\ ]_2^X$ are considered to be essentially the same if for every $(M, m) \in \tilde{M}(X)$,

$$([M]_1^X , [m]_1^X) \simeq ([M]_2^X, [m]_2^X),$$

in other words, there exists an isomorphism

$$i : [M]_1^X \to [M]_2^X$$

such that its composition with $[m]_2^X$ is equal to $[m]_1^X$.

We will often simply write M instead of (M, m) whenever no confusion is likely.

Throughout the remainder of the paper $\underline{X}$ will be a category with equalizers and arbitrary intersections of regular subobjects and $(\underline{A}, U)$ will be a concrete category over $\underline{X}$, i.e., $U : \underline{A} \to \underline{X}$ is faithful, and amnestic (*). Furthermore, all the subcategories are assumed to be full and isomorphism-closed.

Definition 1.2. Let $\underline{C}$ be a subcategory of $\underline{A}$ and let $X \in \underline{A}$. For every $(M, m) \in \tilde{M}(UX)$ we define:

$[M]_{\underline{C}}^X = \bigcap$ equ(Uf, Ug) such that $f, g \in \underline{A}(X, Y)$, $Y \in \underline{C}$ and $Ufm = Ugm$,

where $\bigcap$ denotes the intersection and equ(Uf, Ug) the equalizer of Uf and Ug.

When no confusion is likely, we simply will write $[M]_{\underline{C}}$ instead of $[M]_{\underline{C}}^X$.

Proposition 1.3. $[M]_{\underline{C}} \in \tilde{M}(UX)$ *and* $M \tilde{\leq} [M]_{\underline{C}}$.

Proposition 1.4. *Let us assume that U preserves monosources. Let M_0 be a class of monosources. If $X \in \underline{A}$ and $(M, m) \in \tilde{M}(UX)$, then for every subcategory $\underline{C}$ of $\underline{A}$, we have*

$$[M]_{\underline{C}} \simeq [M]_{M_0(\underline{C})}$$

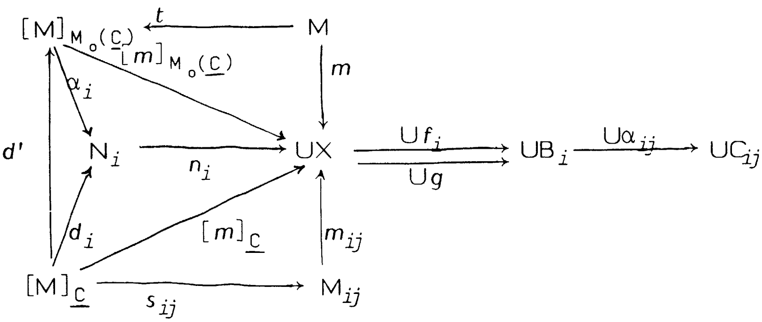
(*) Amnesticity means that every $\underline{A}$-isomorphism f such that Uf is an $\underline{X}$-identity is an $\underline{A}$-identity.

(where $M_0(\underline{C})$ is defined as follows: $X \in M_0(\underline{C})$ iff either $X \in \underline{C}$ or there exists a source $(m_i : X \to X_i)_I$ belonging to M_0 with $X_i \in \underline{C}$ for every $i \in I$).

Proof. Since $\underline{C}$ is contained in $M_0(\underline{C})$, there exists a monomorphism

$$d : [M]_{M_0(\underline{C})} \longrightarrow [M]_{\underline{C}} \text{ such that } [m]_{\underline{C}}\, d = [m]_{M_0(\underline{C})}$$

Let us consider the diagram



with:

$$B_i \in M_0(\underline{C}), \quad C_{ij} \in \underline{C},$$

$$M_{ij} = \mathrm{equ}(U\alpha_{ij}Uf_i\,,\ U\alpha_{ij}Ug_i), \qquad N_i = \mathrm{equ}(Uf_i,\ Ug_i)$$

and α_{ij} is either an M_0-source or the identity of B_i (in the case of $B_i \in \underline{C}$). $Uf_i m = Ug_i m$ implies

$$U\alpha_{ij}Uf_i m = U\alpha_{ij}\ Ug_i m.$$

Since $[M]_{\underline{C}} = \cap M_{ij}$, for fixed i, we get

$$U\alpha_{ij}U\ f_i[m]_{\underline{C}} = U\alpha_{ij}\ Ug_i[m]_{\underline{C}}$$

for every $j \in J$. Since $(\alpha_{ij})_J$ is a monosource, for fixed i, and U preserves monosources, $(U\alpha_{ij})_J$ is a monosource. So

$$Uf_i[m]_{\underline{C}} = Ug_i[m]_{\underline{C}}\,.$$

Since $N_i = \mathrm{equ}(Uf_i,\ Ug_i)$, for every $i \in I$, there exists a unique

$$d_i : [M]_{\underline{C}} \to N_i \text{ such that } [m]_{\underline{C}} = n_i d_i \quad \text{for every } i \in I.$$

Now

$$[M]_{M_0(\underline{C})} = \cap N_i\,,$$

so there exists

$$d' : [M]_{\underline{C}} \to [M]_{M_0(\underline{C})} \text{ such that } [m]_{M_0(\underline{C})}d' = [m]_{\underline{C}}\,.$$

This together with

$$[m]_{\underline{C}}\, d = [m]_{M_0(\underline{C})}$$

gives

$$[m]_{\underline{C}}\, dd' = [m]_{\underline{C}}\,, \qquad \text{so} \qquad dd' = 1_{[M]_{\underline{C}}}\,,$$

i.e., d is an isomorphism. ◊

Corollary 1.5. *If we add, in Proposition 1.4, the hypothesis that $\underline{A}$ is an (E, M_0)-category, with M_0 a class of monosources, we get*

$$[M]_{\underline{C}} \simeq [M]_{E(\underline{C})}$$

(where $E(\underline{C})$ denotes the E-reflective hull of $\underline{C}$).

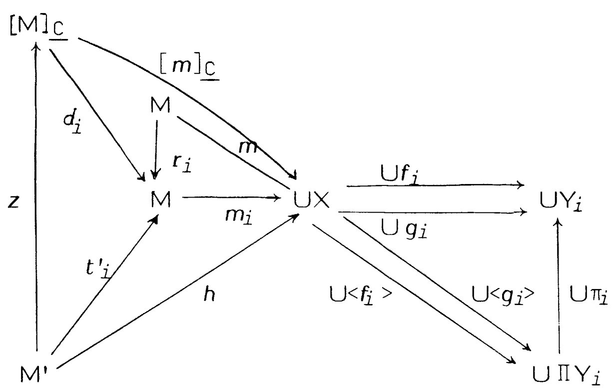
Proof. From [9], Proposition 1.2, we have that $X \in E(\underline{C})$ iff there exists an M_0-source from X into $\underline{C}$. Thus, we can apply Proposition 1.4. ◊

Proposition 1.6. *Let $\underline{A}$ be a category with products and let U carry products to monosources. Suppose that $\underline{X}$ is regular well-powered. If $\underline{C}$ is a subcategory of $\underline{A}$, $X \in \underline{A}$ and $(M, m) \in \tilde{M}(UX)$, then there exists $Y \in \underline{A}$ and $f, g : X \to Y$ such that*

$$[M]_{\underline{C}} = \text{equ}(Uf, Ug)\,.$$

Moreover, if $\underline{C}$ is closed under products, then $Y \in \underline{C}$.

Proof. First we observe that, since $\underline{X}$ is regular well-powered, we do not really need $\underline{X}$ to have arbitrary intersections of regular subobjects, but just intersections of set-indexed families of regular subobjects. Let us consider the following diagram:



where $M_i = \text{equ}(Uf_i, Ug_i)$ and $[M]_{\underline{C}}$ is defined (as in 1.2) relative to the pairs (f_i, g_i). We want to show that

$$[M]_{\underline{C}} = \text{equ}\,(U\langle f_i\rangle, U\langle g_i\rangle).$$

In order to construct ΠY_i , we observe that, since $\underline{X}$ is regular well-powered, we can restrict our attention to only a set of Y_i 's to obtain $[M]_{\underline{C}}$. Now

$$U\pi_i \, U\langle f_i\rangle [m]_{\underline{C}} = U f_i \, [m]_{\underline{C}} = U g_i \, [m]_{\underline{C}} = U\pi_i \, U\langle g_i\rangle [m]_{\underline{C}} \, .$$

From the hypothesis, we get that $(U\pi_i)_I$ is a monosource and so

$$U\langle f_i\rangle \, [m]_{\underline{C}} = U\langle g_i\rangle \, [m]_{\underline{C}} \, .$$

Suppose there exists h: M' → UX such that $U\langle f_i\rangle h = U\langle g_i\rangle h$, then

$$U f_i h = U\pi_i \, U\langle f_i\rangle h = U\pi_i \, U\langle g_i\rangle h = U g_i h \quad .$$

So there exists t_i' : M' → M_i such that

$$[m]_i t'_i = h \qquad \text{for every} \quad i \in I \, .$$

Since $[M]_{\underline{C}} = \cap M_i$, we get a unique

$$z : M' \to [M]_{\underline{C}} \qquad \text{such that} \quad [m]_{\underline{C}} \, z = h,$$

i.e.,

$$[M]_{\underline{C}} = \mathrm{equ}\,(U\langle f_i\rangle \, , \, U\langle g_i\rangle) \, .$$

Clearly, if $\underline{C}$ is closed under products, $\Pi Y_i \in \underline{C}$. ◊

Proposition 1.7. *If* $X \in \underline{A}$ *and* $(M, m) \in \tilde{M}(UX)$, *then*

$$[[M]_{\underline{C}}]_{\underline{C}} \simeq [M]_{\underline{C}}$$

(i.e., $[M]_{\underline{C}}$ *is* $\underline{C}$*-closed).*

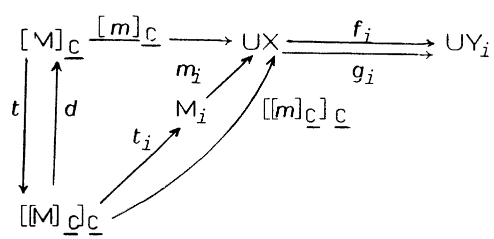
Proof. From Proposition 1.3, there exists a monomorphism

$$t : [M]_{\underline{C}} \to [[M]_{\underline{C}}]_{\underline{C}} \quad \text{such that} \quad [[m]_{\underline{C}}]_{\underline{C}} t = [m]_{\underline{C}}$$

with

$$[m]_{\underline{C}} : [M]_{\underline{C}} \to UX \qquad \text{and} \qquad [[m]_{\underline{C}}]_{\underline{C}} : [[M]_{\underline{C}}]_{\underline{C}} \to UX$$

both belonging to $\tilde{M}$(X). Let us consider the following diagram



with

$$M_i = \mathrm{equ}(Uf_i, Ug_i), \quad Uf_i m = Ug_i m \quad \text{and} \quad Y_i \in \underline{C}$$

for every $i \in I$. Since $Uf_i m = Ug_i m$ implies

$$Uf_i [m]_{\underline{C}} = Ug_i [m]_{\underline{C}}$$

by the definition of $[[M]_{\underline{C}}]_{\underline{C}}$ there exists a unique

$t_i : [[M]_{\underline{C}}]_{\underline{C}} \to M_i$ such that $m_i t_i = [[m]_{\underline{C}}]_{\underline{C}}$ for every $i \in I$.

However $[M]_{\underline{C}} = \cap M_i$ and so, there exists a unique

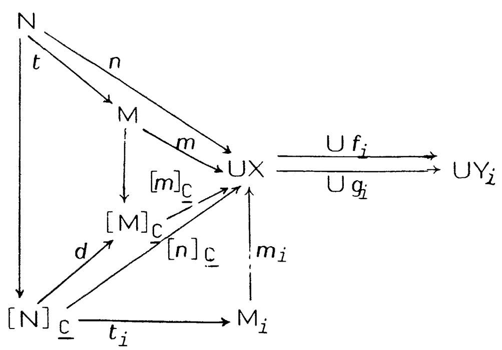
$$d : [[M]_{\underline{C}}]_{\underline{C}} \to [M]_{\underline{C}} \quad \text{such that} \quad [m]_{\underline{C}} d = [[m]_{\underline{C}}]_{\underline{C}} .$$

This, together with $[[m]_{\underline{C}}]_{\underline{C}} t = [m]_{\underline{C}}$ gives $[M]_{\underline{C}} \simeq [[M]_{\underline{C}}]_{\underline{C}}$. ◊

Proposition 1.8. *If* (N, n) *and* (M, m) *are* $\tilde{M}$*-subobjects of* UX *such that* $(N, n) \tilde{\leqq} (M, m)$, *then*

$$([N]_{\underline{C}}, [n]_{\underline{C}}) \tilde{\leqq} ([M]_{\underline{C}}, [m]_{\underline{C}}).$$

Prooof. Let us consider the diagram



with

$$M_i = \mathrm{equ}(Uf_i, Ug_i), \quad Uf_i m = Ug_i m, \quad Y_i \in \underline{C} \quad \text{for every } i \in I,$$

and $t : N \to M$ satisfying $m t = n$.
$Uf_i m = Ug_i m$ implies

$$Uf_i m t = Ug_i m t,$$

which implies $Uf_i n = Ug_i n$ and so, by definition of $[n]_{\underline{C}}$,

$$Uf_i [n]_{\underline{C}} = Ug_i [n]_{\underline{C}} .$$

Thus, for every $i \in I$, there exists a unique morphism

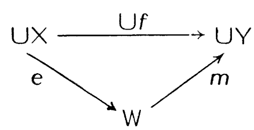
$$t_i : [N]_C \to M_i \quad \text{such that} \quad m_i t_i = [n]_C .$$

Since $[M]_C = \cap M_i$, there exists a unique morphism

$$d : [N]_C \to [M]_C \quad \text{such that} \quad [m]_C\, d = [n]_C . \qquad \Diamond$$

Remark 1.9. From Propositions 1.3, 1.7 and 1.8, we get that $[\ ]_C$ defined in 1.2 is a closure operator in the sense of Definition 1.1.

Definition 1.10. Let X be an (epi, $\tilde{M}$)-category and let C be a subcategory of A. A C-morphism $f : X \to Y$ is C-dense iff given the (epi, $\tilde{M}$)-factorization

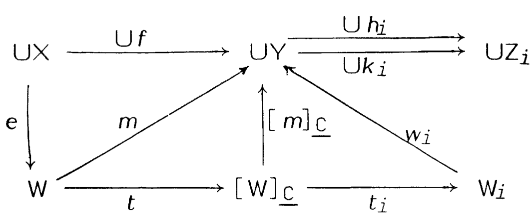


of Uf , one has $[W]_C \simeq UY$.

Notice that the above definition makes sense, because $m \in \tilde{M}$.

Theorem 1.11. *Let X be an (epi, $\tilde{M}$)-category and let C be a subcategory of A . A C-morphism $f : X \to Y$ is an epimorphism in C iff f is C-dense.*

Proof. (=>). Let us consider the following diagram:



with (e, m) the (epi, $\tilde{M}$)-factorization of Uf ,

$$(W_i, w_i) = \mathrm{equ}(Uh_i , U k_i), \qquad Uh_i m = U k_i m$$

and $Z_i \in C$ for every $i \in I$. $U h_i m = Uk_i m$ implies

$$U(h_i f) = Uh_i\, Uf = Uh_i m e = Uk_i m e = Uk_i\, Uf = U(k_i f),$$

which implies $h_i f = k_i f$, since U is faithful, and so $h_i = k_i$, because f is an epimorphism in C. Thus w_i is an isomorphism, for every $i \in I$, and

and so is d, since it is an intersection of isomorphisms.

(<=). Now, we can use the same diagram by simply dropping the subscript i. From $hf = kf$, we get

$$Uh\, Uf = Uk\, Uf ,$$

which implies

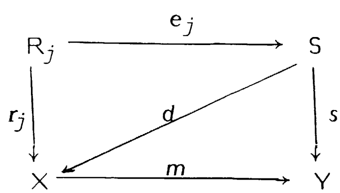
$$U hme = Ukme , \quad \text{i.e.,} \quad Uhm = Ukm,$$

because e is an epimorphism. By the definition of $[W]_{\underline{C}}$, we get

$$Uh\, [m]_{\underline{C}} = Uk[m]_{\underline{C}}$$

which implies $Uh = Uk$ since $[m]_{\underline{C}}$ is an isomorphism. Thus $h = k$ because U is faithful. ◊

Definition 1.12. Given a category $\underline{X}$, an $\underline{X}$-monomorphism $m : X \to Y$ is called a *strong monomorphism* iff, for every episink $(e_j : R_j \to S)_J$, sink $(r_j : R_j \to X)_J$ and morphism $s : S \to Y$ such that the diagram



commutes, there exists a morphism $d: S \to X$ such that

$$md = s \quad \text{and} \quad de_j = r_j \quad \text{for every} \quad j \in J .$$

For $\tilde{M} = \{\text{Strong Monomorphisms}\}$, we get the following result:

Proposition 1.13. *Let $\underline{X}$ be a balanced (epi, regular mono)-category and let $U : \underline{A} \to \underline{X}$ be topological. If (M, m) is a strong subobject of $UX \in \underline{X}$ and $\underline{B}$ is bireflective in $\underline{A}$, then $[M]_{\underline{B}} \simeq M$.*

Proof. We observe that $U(\underline{B}) = \underline{B}'$ is bireflective in $\underline{X}$ and since $\underline{X}$ is balanced, $\underline{B}' = \underline{X}$.
From the assumptions on $\underline{X}$, we get that every strong monomorphism is regular ([10], Prop. 17.18) for (E, M) = (epi, regular mono)), so there exist

$$h, k : UX \to Y \quad \text{such that} \quad (M, m) = \text{equ}\,(h, k).$$

Since $\underline{B}$ is bireflective, the indiscrete object Y' such that UY' = Y belongs to $\underline{B}$ (cf. [3], Theorem 1.5) and h and k can be lifted to morphisms $\tilde{h}, \tilde{k} : X \to Y'$. Thus $M \simeq [M]_{\underline{B}}$. ◊

2. COCLOSURE OPERATORS AND MONOMORPHISMS.

In the previous section, we have characterized the epimorphisms in a subcategory $\underline{C}$ in terms of $\underline{C}$-closure. The duality principle in category theory suggests that we could get a characterization of monomorphisms in $\underline{C}$ in terms of a concept dual to the $\underline{C}$-closure. We need to observe that such an idea does not appear in previous papers, for the simple reason that most of the work in this area has been done in the category TOP, and the monomorphisms in all full subcategories of TOP are injections. A similar result for a full subcategory $\underline{C}$ of AB is the following: if the additive group of integers Z belongs to $\underline{C}$, then the monomorphisms in $\underline{C}$ are injections. However there are many interesting subcategories of AB which do not contain Z. So, the problem of characterizing the monomorphisms in subcategories of AB is not trivial. With the above motivation, we are going to present in this section the basic definitions of the dual theory and we will state the two main results.

Definition 2.1. Given any category $\underline{X}$, by a coclosure operator over $X \in \underline{X}$, we mean a function

$$[\ ]_X : \tilde{E}(X) \to \tilde{E}(X)$$

satisfying for every (q, Q), (p, P) belonging to $\tilde{E}(X)$:

(a) $(q, Q) \leqq^o [(q, Q)]_X$.

(b) $[[(q, Q)]_X]_X \simeq [(q, Q)]_X$.

(c) If $(p, P) \leqq^o (q, Q)$, then $[(p, P)]_X \leqq^o [(q, Q)]_X$.

The $\tilde{E}$-quotient (q, Q) is called $[\ ]_X$-coclosed provided that

$$(q, Q) \simeq [(q, ,Q)]_X .$$

As in § 1, whenever we do not need to specify the morphism q, we will simply write Q instead of (q, Q).

Let $\underline{X}$ be a category with coequalizers and arbitrary cointersections of regular quotients, and let $(\underline{A}, U)$ be concrete over $\underline{X}$.

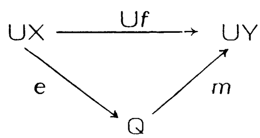
Definition 2.2. Let $X \in \underline{A}$ and let $\underline{C}$ be a subcategory of $\underline{A}$. For every $(q, Q) \in \tilde{E}(UX)$, we define

$$[Q]^{\underline{C}}_X = \cup \{\text{coequ}(Uf, Ug) \text{ such that } f, g : Y \to X,\ Y \in \underline{C},\ qUf = qUg\},$$

where $\cup$ denotes the cointersection and coequ(Uf, Ug) the coequalizer of Uf and Ug .

When no confusion is likely to arise, we simply will write $[Q]^{\underline{C}}$ instead of $[Q]^{\underline{C}}_X$.

Definition 2.3. Let $\underline{X}$ be an $(\tilde{E}$, mono)-category, and let $\underline{C}$ be a subcategory of $\underline{A}$. A $\underline{C}$-morphism $f : X \to Y$ is $\underline{C}$-codense iff given the (E, mono)-factorization



of Uf, we have $[Q]^{\underline{C}} \simeq UX$.

Theorem 2.4. *Let $\underline{X}$ be an $(\tilde{E}$,mono)-category and let $\underline{C}$ be a subcategory of $\underline{A}$. Then a $\underline{C}$-morphism $f : X \to Y$ is a monomorphism in $\underline{C}$ iff f is $\underline{C}$-codense.*

For $\tilde{E} = \{$Strong Epimorphisms$\}$, we get the following result:

Proposition 2.5. *Let $\underline{X}$ be a balanced (regular epi, mono)-category, and let $U : \underline{A} \to \underline{X}$ be topological. If $(q, \underline{Q})$ is a strong quotient of $UX \in \underline{X}$ and $\underline{B}$ is bicoreflective in $\underline{A}$, then $[Q]^{\underline{B}} \simeq Q$.*

3. APPLICATIONS TO CONCRETE CATEGORIES.

In this section we will see some applications of the results of the previous two sections in the categories AB (Abelian Groups), ATG (Abelian Topological Groups), GR (Groups) and TG (Topological Groups). For this purpose, AB and GR will be considered as concrete categories over themselves, via the identity functor, and ATG and TG will be considered as concrete over AB and GR respectively, via the usual forgetful functors. In order to use the preceding results, we will assume $\tilde{M}$ and $\tilde{E}$ to be the class of all monomorphisms and the class of all epimorphisms, respectively.

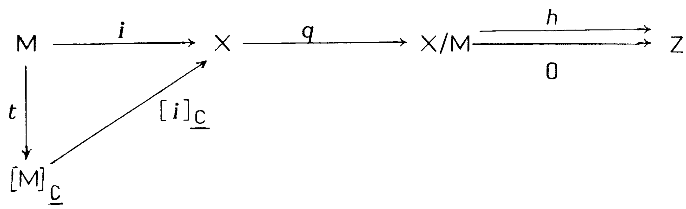
Remark 3.1. In AB and GR, strong monomorphisms coincide with monomorphisms and strong epimorphisms coincide with epimorphisms. In general, we have that regular monomorphism implies strong monomorphism, implies extremal monomorphism. Since ATG and TG are (epi, regular mono)-factorizable, then from [10], Proposition 17.18, dual, these three concepts agree. The same conclusion can be drawn about strong epimorphisms, since ATG and TG are (regular epi, mono)-factorizable.

Proposition 3.2. *Let $\underline{C}$ be a subcategory of* AB *and* M *a subgroup of*

X ϵ AB. M *is* $\underline{C}$*-dense in* X *iff*

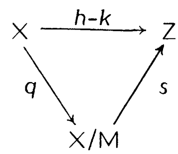
$$\mathrm{Hom}(X/M, Z) = \{0\} \qquad \textit{for every} \quad Z \in \underline{C}\,.$$

Proof. (=>). Let us consider the following diagram



$hqi = 0qi$ implies $hq[i]_{\underline{C}} = 0q[i]_{\underline{C}}$ and $[M]_{\underline{C}} \simeq X$ implies $hq = 0q$, i.e., $h = 0$ because q is an epimorphism.

(<=). Suppose $h, k : X \to Z$ are such that $hi = ki$, $i : M \to X$; then the following diagram



commutes, because M is contained in ker($h-k$). By the hypothesis, we get $s = 0$, so $h = k$, i.e., $[M]_{\underline{C}} \simeq X$. ◊

Corollary 3.3. *Let* $\underline{C}$ *be a subcategory of* AB, *and let* $f : X \to Y$ *be a* $\underline{C}$*-morphism.* f *is an epimorphism in* $\underline{C}$ *iff*

$$\mathrm{Hom}(Y/f(X), Z) = \{0\} \qquad \textit{for every} \quad Z \in \underline{C}\,.$$

Proof. From Theorem 1.11, f is an epimorphism in $\underline{C}$ iff $[f(X)]_{\underline{C}} \simeq Y$ and this, by Proposition 3.2, is equivalent to

$$\mathrm{Hom}(Y/f(X), Z) = \{0\} \qquad \text{for every} \quad Z \in \underline{C}\,.$$

Corollary 3.4. *Epimorphisms in the category* $\underline{TF}$ *of torsion free abelian groups are not surjective.*

Proof. Let $\tilde{Z}$, $2\tilde{Z}$ and $\tilde{Z}_{(2)}$ denote the group of integers, the group of even integers and the group $\tilde{Z}/2\tilde{Z}$, respectively. $i : 2\tilde{Z} \to \tilde{Z}$ is a morphism in $\underline{TF}$ which satisfies Corollary 3.3 (notice that

$$\mathrm{Hom}(\tilde{Z}_{(2)}, H) = \{0\} \qquad \text{for} \quad H \in \underline{TF}\ ).$$

Thus $i : 2\tilde{Z} \to \tilde{Z}$ is an epimorphism in $\underline{TF}$, which is clearly not surjective.

We observe that the above result can be found in [10] (Examples 6.10). We have just reproved it with this new approach.

Corollary 3.5. *In the category* $\underline{R}$ *of abelian reduced groups, the epimorphisms are not surjective.*

Proof. Let $\tilde{Q}$, $\tilde{Z}$ and $F\tilde{Q}$ denote the additive group of rationals, the group of integers and the free abelian group over the underlying set of $\tilde{Q}$. $\tilde{Q}$ is divisible and $F\tilde{Q}$ is reduced. If $e : F\tilde{Q} \to \tilde{Q}$ is the induced surjection, then $F\tilde{Q}/\ker(e) \simeq \tilde{Q}$. Thus $i : \ker(e) \to F\tilde{Q}$ is a morphism in $\underline{R}$ and satisfies Corollary 3.3, because $\tilde{Q}$ is divisible. Hence it is an epimorphism in $\underline{R}$, that is not surjective. ◊

Corollary 3.6. *Epimorphisms in the category* $\underline{F}$ *of free abelian groups are not surjective.*

Proof. The same example as in Corollary 3.5 can be used here. ◊

Lemma 3.7. Let $\underline{C}$ be a subcategory of AB (ATG, GR, TG). If every subgroup of a $\underline{C}$-object is $\underline{C}$-closed, then the epimorphisms in $\underline{C}$ are surjective.

Proof. Let $f : X \to Y$ be an epimorphism in $\underline{C}$. From Theorem 1.11, f is $\underline{C}$-dense, i.e., $[f(X)]_{\underline{C}} \simeq Y$. By hypothesis $[f(X)]_{\underline{C}} \simeq f(X)$. Since $f(X)$ is a subset of Y, we get $f(X) = Y$, i.e., f is surjective. ◊

Proposition 3.8. *If* $\underline{B}$ *is bireflective in* AB (ATG, GR, TG), *then the epimorphisms in* $\underline{B}$ *are surjective.*

Proof. From Remark 3.1 and Proposition 1.13, we get that every subgroup of a $\underline{B}$-object is $\underline{B}$-closed. So we can apply Lemma 3.7. Notice that AB, ATG, GR and TG satisfy the hypotheses of Proposition 1.13. ◊

Proposition 3.9. *Let* $\underline{C}$ *be a subcategory of* AB (ATG), *closed under extremal epimorphisms and let* $X \in \underline{C}$. *Then every subgroup of a* $\underline{C}$-*object* X *is* $\underline{C}$-*closed.*

Proof. If M is a subgroup of X, then

$$M \simeq \mathrm{equ}(q, 0), \quad q, 0 : X \to X/M \quad \text{and } X/M \in \underline{C}.$$

So M is $\underline{C}$-closed. Notice that if $X \in$ ATG, then X/M carries the quotient topology. ◊

Corollary 3.10. *If $\underline{C}$ is a subcategory of AB (ATG), that is closed under extremal epimorphisms, then the epimorphisms in $\underline{C}$ are surjective.*

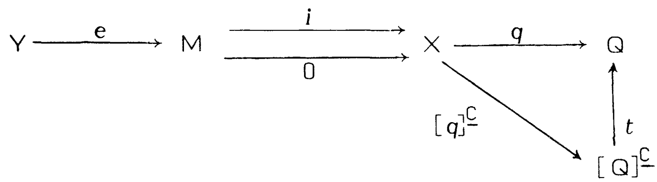
Corollary 3.11. *The epimorphisms in AB, ATG, GR and TG are surjective.*

Corollary 3.12. *If $\underline{C}$ is monocoreflective in AB (ATG), then the epimorphisms in $\underline{C}$ are surjective.*

Corollary 3.13. *In the following subcategories of AB the epimorphisms are surjective: divisible, torsion, bounded, cyclic, cotorsion.*

Proposition 3.14. *Let $\underline{C}$ be a subcategory of AB and let (q, Q) be a quotient of $X \in$ AB . (q, Q) is $\underline{C}$-codense in X iff for every subgroup M of $ker(q)$, $M \in Q(\underline{C})$ implies $M = \{0\}$, where $Q(\underline{C})$ denotes the quotient hull of $\underline{C}$.*

Proof. (=>). Let us consider the commutative diagram :

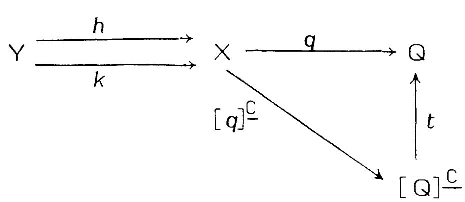


with $e : Y \to M$ an epimorphism, $Y \in \underline{C}$. Since M is a subgroup of $ker(q)$, we get $qie = q0e$ and by the definition of $[Q]^{\underline{C}}$,

$$[q]^{\underline{C}}\, ie = [q]^{\underline{C}}\, 0e \; .$$

Hence $ie = 0e$, because $[q]^{\underline{C}}$ is an isomorphism. Thus $i = 0$ since e is an epimorphism, i.e., $M = \{0\}$.

(<=). Let us consider the commutative diagram:



with $Y \in \underline{C}$. Suppose $qh = qk$, then $q(h-k) = 0$, which implies that $(h-k)(Y)$ is a subgroup of $ker(q)$. Moreover

$$(h\text{-}k)(Y) \in Q(\underline{C}) \quad \text{and so} \quad (h\text{-}k)(Y) = \{0\} .$$

This implies $h = k$. Hence $[q]^{\underline{C}}$ is an isomorphism, because the cointersection of a family of isomorphisms is an isomorphism. ◊

Corollary 3.15. *Let $\underline{C}$ be a subcategory of AB. A $\underline{C}$-morphism $f : X \to Y$ is a monomorphism in $\underline{C}$ iff for every subgroup $\overline{M}$ of ker(f), $M \in Q(\underline{C})$ implies $M = \{0\}$.*

Proof. Let $\bar{f}$ denote the restriction of f to $f(X)$. From Theorem 2.4, f is a monomorphism in $\underline{C}$ iff $(\bar{f}, f(X))$ is $\underline{C}$-codense and from Proposition 3.14, $(\bar{f}, f(X))$ is $\underline{C}$-codense iff for every subgroup M of $ker(\bar{f}) = ker(f)$, $M \in Q(\underline{C})$ implies $\overline{M} = \{0\}$. ◊

Corollary 3.16. *In the category $\underline{D}$ of abelian divisible groups, monomorphisms are not necessarily injective.*

Proof. Let us consider the $\underline{D}$-morphism $q : \tilde{Q} \to \tilde{Q}/\tilde{Z}$, where $\tilde{Q}$ and $\tilde{Z}$ denote the additive group of rationals and the group of integers, respectively, and $\tilde{Q}/\tilde{Z}$ is the quotient group. Clearly q is not injective. Now, if M is a subgroup of $\ker(q) = \tilde{Z}$ and $M \in Q(\underline{D})$, then M is divisible and so it must be equal to $\{0\}$, because $\tilde{Z}$ is reduced. Thus from Corollary 3.15, q is a monomorphism in $\underline{D}$. ◊

We observe that the above result can be found in [10] (Examples 6.3). We have just reproved it with this new approach.

Corollary 3.17. *In the category $\underline{AC}$ of algebraically compact abelian groups, monomorphisms are not necessarily injective.*

Proof. Let us consider $q : \tilde{Q} \to \tilde{Q}/\tilde{Z}$ as in Corollary 3.16. We observe that q is an $\underline{AC}$-morphism because divisible groups are algebraically compact ([6], Theorem 21.2 and also Chapter 38). If M is a subgroup of $\ker(q) = \tilde{Z}$ and $M \in Q(\underline{AC})$, then M is free and a cotorsion subgroup of $\ker(q)$ (notice that $Q(\underline{AC})$ is the category of cotorsion abelian groups, [6], Proposition 54.1). So, there exists $A \in \underline{AC}$ such that $q : A \to M$ is a surjective homomorphism. This implies

$$A = M \oplus \ker(q) ,$$

because M is free. M is algebraically compact (cf. [6], Chapter 38). Hence from [6], Chapter 38, Exercise 1, we get $M = \{0\}$. Thus, from Corollary 3.15, q is a monomorphism, that is clearly not injective. ◊

Corollary 3.18. *In the category $\underline{COT}$ of cotorsion abelian groups, monomorphisms are not necessarily injective.*

Proof. The same example as in Corollary 3.17 can be used here. ◊

Lemma 3.19. Let $\underline{C}$ be a subcategory of AB (ATG, GR, TG). If every quotient of a $\underline{C}$-object is $\underline{C}$-coclosed, then the monomorphisms in $\underline{C}$ are injective.

Proof. If $f : X \to Y$ is a $\underline{C}$-monomorphism, then from Theorem 2.4, f is $\underline{C}$-codense, i.e. $[f(X)]^{\underline{C}} \simeq X$. From the hypothesis, $f(X)$ is $\underline{C}$-coclosed, so

$$f(X) \simeq [f(X)]^{\underline{C}} \simeq X ,$$

hence f is injective. ◊

Proposition 3.20. *If* $\underline{B}$ *is bicoreflective in* AB (ATG, GR, TG) , *then the monomorphisms in* $\underline{B}$ *are injective.*

Proof. From Proposition 2.5, every quotient of a $\underline{B}$-object is $\underline{B}$-coclosed, so we can apply Lemma 3.19. ◊

Proposition 3.21. *If* $\underline{C}$ *is a subcategory of* AB (ATG, GR, TG), *closed under extremal monomorphisms, then every quotient of a* $\underline{C}$*-object* X *is* $\underline{C}$*-coclosed.*

Proof. If $q : X \to Q$ is an epimorphism, then

$$(q, Q) \simeq \text{coequ}(i, 0), \qquad i, 0 : \ker(q) \to X ,$$

i.e., Q is $\underline{C}$-coclosed, because $\ker(q) \in \underline{C}$. Notice that if X is a topological group, then $\ker(q)$ carries the relative topology. ◊

Corollary 3.22. *The monomorphisms in* AB, ATG, GR *and* TG *are injective.*

Corollary 3.23. *If* $\underline{C}$ *is a subcategory of* AB (ATG, GR, TG) *that is closed under extremal monomorphisms, then the monomorphisms in* $\underline{C}$ *are injective.*

Corollary 3.24. *If* $\underline{C}$ *is epireflective in* AB (ATG, GR, TG), *then the monomorphisms in* $\underline{C}$ *are injective.*

Corollary 3.25. *In the following subcategories of* AB *the monomorphisms are injective: torsion, reduced, torsion-free, free, bounded, cyclic, locally cyclic.*

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