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COHERENT EUCLIDEAN GEOMETRY

by Rosanna Succi CRUCIANI *

Résumé. Nous donnons des axiomes pour une théorie de la Géométrie Euclidienne considérée comme une catégorie logique *E* : nous présentons les objets et les morphismes qui "engendrent" *E* et nous en décrivons les propriétés. Nous indiquons une théorie d'anneaux appropriée et, à partir d'un modèle de cette théorie, nous construisons un modèle de la géométrie ; nous montrons que la catégorie E/S_2 (S_2 = "objet des couples de points séparés") fournit un modèle de la théorie d'anneaux et donc un modèle de la géométrie ; celui-ci est isomorphe au foncteur S_2x ( ) : $E \to E/S_2$.

INTRODUCTION.

Following the foundational, philosophical and didactic motivations expounded by F.W. Lawvere since 1978 in view of a new foundation of Euclidean Geometry, we showed in [16] how Euclidean Geometry can be regarded as a logical category *E* : with our axioms, we produced the objects and morphisms that "generate" *E* and gave their properties.

In [13] it is shown how logical categories are the "same" as theories in a finitary coherent logic : roughly speaking, one can say that, for all practical purposes, using one or the other framework is equivalent in the sense that each axiom or inference can be stated either in a logical category or in a coherent theory, and the passage from one setting to the other is always possible. (The coherent logic is a logic that we can call "positive" because the formulas are built by means of $\wedge$, $\vee$, $\exists$ and the rest of the logical operators cannot be used at all.)

The categorical aspect is of great significance for us, because the mathematical processes of Euclidean Geometry (geometric constructions, algebraic operations, etc.) can be unified under the general concept of mapping (for example, the fundamental map $\pi: S_2 x L x L \to P$ of our axiomatization represents the process of drawing the perpendicular to a line through one of its points) ; moreover the composition of maps and other processes (images, cartesian products, etc.) yield further maps, thus representing the development of our geometrical thinking.

In Section 1 we describe the axiomatization we gave in [16]. The choice of axioms has been guided by the proposal of obtaining, by means of coherent logic, the properties of order, parallelism and metric properties of Euclidean Geometry. On the other hand, we had

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in mind significant models for our axioms, such as Grothendieck toposes : this is why we did not introduce strong properties as, for example, total order on the line. In our approach, there are not only Space-objects, as the *line* L, the *plane* P and the object $S_2 \hookrightarrow P \times P$ of pairs of points that "lie apart", but also Quantity-objects : the semiring Q of *pure quantities*, , the Q-semimodules Λ object of *lengths* and the object of *areas* A ; this corresponds to the philosophical and didactic motivations of Lawvere (according to him, we have to accept the two aspects, geometric and algebraic, and to analize the interactions between the two : he claims that, in this way, the axiomatic for Euclidean Geometry can be made simpler) but its significance will become more evident as soon as one exhibits a model in which those quantity-objects are not canonically isomorphic. We also remark that the object A will enable us to complete the axiomatic with a "measure theory" in the plane.

In our axiomatics two points have a distance only if they lie on a line ; we didn't need more than that in our context, but the main motivation is the fact that, by removing the distance map on the line and the axioms that imply

$$\forall \; x \in Q : x^2 = 0 \; \Rightarrow x = 0,$$

we would like to describe an "Euclidean Geometry with infinitesimals" and so to obtain significant models in those sheaf toposes that arise in Synthetic Differential Geometry (see [8]) (for this we will have to leave the world of coherent logic).

In Section 2 we describe a theory (i.e. a logical category S) of local rings that are ordered, pythagorean and normed, and we construct a S-model M of the theory outlined in Section 1.

In Section 3 we consider the category E of Section 1 and the functor $S_2 \times (\ ) : E \longrightarrow E/S_2$; we show that $S_2 \times (\ )$ "transforms the theory of Euclidean Geometry of Section 1 into the theory of rings described in Section 2" ; indeed we show that the object

$$R = (S_2 \times L \xrightarrow{\;p_1\;} S_2)$$

of E/S_2 is a ring object which satisfies all the properties of that theory ; moreover, if we construct from R a model $\bar{M} : E \to E/S_2$, as well as we did in Section 2, we prove that there is an isomorphism of models $S_2 \times (\ ) \simeq \bar{M}$.

We often write (particularly in proofs) as if the category in which we work were the category of sets, however it is clear that it is possible to express everything in a logical category.

We want to thank A. Joyal, A. Kock and F.W. Lawvere for their stimulating discussions.

1. In this section we describe the "Plane Euclidean Geometry Theory" regarded as a logical category E (i.e. a category with finite limits, stable finite sups and stable images) : with our axioms we produce the objects and the morphisms that "generate" E and give their properties.

As we shall specify later, we assume that there are in E "Space objects" (as the "Line", the "Plane", their subobjects...) and "Quantity objects" (as the "object of lengths", the "object of areas", ...) ; the last are equipped with an algebraic structure ; we analize the interactions between such objects introducing the fundamental morphisms of E and their properties.

We begin giving the first axioms that define E.

Let us assume that there exists in E a "Line object" L, an "object of lengths" Λ and an "object of pure quantities" Q and the maps, between "quantity objects", representing the "algebraic operations":

$$\Lambda \times \Lambda \xrightarrow{\ +\ } \Lambda \qquad \text{and} \qquad Q \times Q \underset{\cdot}{\overset{+}{\rightrightarrows}} Q;$$

we assume that $(\Lambda, +)$ is a commutative monoid object and $(Q, +, .)$ a commutative semiring object with zero and unit ; furthermore a map $\delta : Q \times \Lambda \to \Lambda$ makes Λ a Q-semimodule.

Let us suppose that there exist in E maps

$$t_i : \Lambda \times L \to L \qquad (i = 1, 2) \qquad \text{and } d : L \times L \to \Lambda \text{ (distance map)}$$

that represent the interactions between the "line" L and the "monoid of lengths" Λ ; the maps t_i represent the actions of Λ on L : "each length gives translations $L \rightrightarrows L$ in two directions" ; on the other hand "each point of L, by means of t_i , gives rise to two maps $\Lambda \rightrightarrows L$ "; Lawvere calls "Coordinate system" this kind of map, from "Quantity objects" to "Space objects", and "Variable quantity" a map from "Space objects" to "Quantity objects", as the distance map $d : L \times L \to \Lambda$.

If $l \in \Lambda$ and $Q \in L$, we also denote

$$t_1(l, Q) = Q + l, \qquad t_2(l, Q) = Q - l$$

and, if $Q_1, Q_2 \in L$, $d(Q_1, Q_2) = \overline{Q_1 Q_2}$; let us assume that the maps t_i and d satisfy the following obvious axioms :

L_1. If $l_1, l_2 \in \Lambda$, $Q \in L$: $(Q \pm l_1) \pm l_2 = Q \pm (l_1 + l_2)$, $\quad Q \pm 0 = Q$.
L_2. If $l \in \Lambda$, $Q \in L$: $(Q + l) - l = (Q - l) + l = Q$.
L_3. If $Q_1, Q_2 \in L$: $d(Q_1, Q_2) = 0 \Rightarrow Q_1 = Q_2$.
L_4. If $l \in \Lambda$, $Q \in L$: $d(Q, Q \pm l) = l$.

Let us assume that there is in E an "object of lengths which are apart from zero" $\Lambda^{o} \hookrightarrow \Lambda$; by pullback we can define :

$$S_1 = [\![(Q_1, Q_2) \in L \times L \mid d(Q_1, Q_2) \in \;^{o}]\!]$$

(If $(Q_1, Q_2) \in S_1$ we say "Q_1 is apart from Q_2 ").

We assume the following axioms L_1^0 to L_5^0 ; from L_1^0 and L_5^0 it follows that the map

$$\Lambda^0 \times L \longrightarrow L \times L \qquad \text{given by} \qquad (1, Q) \mapsto (Q, Q+1)$$

defines a strict order relation on L and L_3^0 means that "any couple $(Q_1, Q_2) \in S_1$ individualizes either order" ; the axiom L_4^0, already mentioned by Heyting in [3], is important because it lead to very employed Proposition 1.1 and then to the structure of local ring for the object $S_2 \times L \to S_2$ of E/S_2 (cf. § 3).

L_1^0. If $l \in \Lambda$: $l \in \Lambda^0 \wedge l = 0$ is false.
L_2^0. $\exists$ (Q_1, Q_2), $(Q_1, Q_2) \in S_1$.
L_3^0. If $Q_1, Q_2 \in L$: $(Q_1, Q_2) \in S_1 \Rightarrow$

$$\exists\ l \in \Lambda^0,\ Q_2 = Q_1 + l \ \vee\ \exists\ l' \in \Lambda^0,\ Q_2 = Q_1 - l'.$$

L_4^0. If $Q_1, Q_2, Q \in L$: $(Q_1, Q_2) \in S_1 \Rightarrow (Q_1, Q) \in S_1 \vee (Q_2, Q) \in S_1$.
L_5^0. If $l_1, l_2 \in \Lambda$: $l_1 \in \Lambda^0 \vee l_2 \in \Lambda^0 \Rightarrow l_1 + l_2 \in \Lambda^0$.

The following axioms Q_1 and Q_2 concern the actions of the semiring Q on Λ and the connection between Λ^0 and the object of "invertible elements of Q ".

Let us denote $(k, l) \mapsto kl$ the product $\delta : Q \times \Lambda \to \Lambda$.

Q_1. The map

$$Q \times \Lambda^0 \to \Lambda^0 \times \Lambda \qquad \text{given by} \qquad (k, l) \mapsto (l, kl)$$

is an isomorphism (the inverse will be denoted $(l, l') \mapsto (\rho(l, l'), l)$).

Q_2. If $k \in Q$, $l \in \Lambda^0$: $kl \in \Lambda^0 \iff k \in U(Q)$, where $U(Q)$ is the object of units of Q.

Let us define two maps

$$S_1 \times L \times Q \longrightarrow L \qquad \text{denoted by} \qquad ((Q_1, Q_2), Q, k) \mapsto Q\,(\pm)\,kQ_1Q_2\ ,$$

by L_3^0 and L_4 (this very employed notation is useful to have a fluent language ; "$Q(+)\,kQ_1Q_2$ $(Q(-)k\,Q_1Q_2)$ is the point, $k\,\overline{Q_1Q_2}$ distant from Q, that follows (precedes) Q in the order individualized by the couple (Q_1, Q_2) ").:

$$Q(+)k\,Q_1Q_2 = Q + k\overline{Q_1Q_2} \quad \text{if} \quad Q_2 = Q_1 + \overline{Q_1Q_2},$$

$$Q(+)k\,Q_1Q_2 = Q - k\overline{Q_1Q_2} \quad \text{if} \quad Q_2 = Q_1 - \overline{Q_1Q_2}\ ,$$

$$Q(-)k\,Q_1Q_2 = Q - k\overline{Q_1Q_2} \quad \text{if} \quad Q_2 = Q_1 + \overline{Q_1Q_2}\ ,$$

$$Q(-)k\,Q_1Q_2 = Q + k\overline{Q_1Q_2} \quad \text{if} \quad Q_2 = Q_1 - \overline{Q_1Q_2}\ .$$

Then in particular " $Q_1(+)\,kQ_1Q_2$ $(Q_2(-)\,kQ_1Q_2)$ is the point that lies in the same half-line of Q_2 (Q_1) with origin Q_1 (Q_2) and whose distance from Q_1 (Q_2) is $k\ \overline{Q_1Q_2}$ ".

From L_3^0 and L_4^0 it follows :

Proposition 1.1. *If* $(Q_1, Q_2) \in S_1$, $Q \in L$:

$$\exists h \in U(Q),\ Q = Q_1(+)\ hQ_1Q_2\ \vee\ \exists k \in U(Q),\ Q = Q_2\ (-)\ kQ_1Q_2 .$$

Definition 1.2. Let X be an object of E ; a map $f: X\text{x}L \to L$ is called X-*isometry* if :

i) If $x \in X$, $Q_1, Q_2 \in L$:

$$(f(x\ ,\ Q_1), f(x\ ,\ Q_2)) \in S_1 \iff (Q_1\ ,\ Q_2) \in S_1.$$

ii) If $x \in X$, $(Q_1, Q_2) \in S_1$: $d(f(x\ ,\ Q_1), f(x, Q_2)) = d(Q_1, Q_2)$.

In [16] we have proved the following proposition ; it states that "isometries are isomorphisms and preserve the order".

Proposition 1.3. *If* $f : X\text{x}S_1\text{x}L \to L$ *is an* $(X\text{x}S_1)$*-isometry :*

i) If $x \in X$, $k \in U(Q)$, $(Q_1, Q_2) \in S_1$:

$$f(Q_1)\ (+)\ kf(Q_1)f(Q_2) = f(Q_1(+)k\ Q_1Q_2)$$

where we wrote $f(x, (Q_1, Q_2), Q)$ *simply as* $f(Q)$.

ii) $\langle p_1, p_2, f\rangle: X\text{x}S_1\text{x}L \to X\text{x}S_1\text{x}L$ *is an isomorphism. (We denote* p_i *the projection* $X_1\text{x} \ldots \text{x}X_n \to X_i$, p_{ij} *the projection* $X_1\text{x}\ldots\text{x}X_n \to X_i \text{x} X_j$, *etc).*

The following axiom D is equivalent to "the isometries preserve the distance for all couples $(Q, Q') \in L\text{x}L$ " (we refuse this statement as a definition of isometry because many of our considerations are valid even if we remove the distance map d ; then they can be used, for example, to describe an "Euclidean Geometry with infinitesimals" in which a distance map is not opportune).

D. If $(Q_1, Q_2), (R_1, R_2) \in S_1$, $h \in U(Q)$:

$$d(Q_1, Q_2) = d(R_1, R_2) \Rightarrow d(Q_1(+)h\ Q_1Q_2,\ Q_2) = d(R_1(+)hR_1R_2,\ R_2).$$

With the following axioms we admit in E an "object of areas" A , a "Plane object" P and a subobject $S_2 \hookrightarrow P\text{x}P$ of "couples of points of P that are apart". The axioms $\alpha_1, \alpha_2, \alpha_3$ state the interactions between A and the objects Q and Λ by means of the "operations" β and α ; the "area map" $a : S_2\text{x}P \to A$ is a "variable quantity" and "for each

$(A_1, A_2, A) \in S_2 x P$, $a(A_1, A_2, A)$ can be interpreted as the area of the parallelogram given, in an evident way, by (A_1, A_2, A) ".

We assume that there is in E an "object of areas" A, which is a Q-semimodule (whose product $\beta: Q x A \to A$ will be denoted $(k, a) \mapsto ka$) together with a map $\alpha: \Lambda x \Lambda \to A$ such that

α_1. If $l_1, l_2 \in \Lambda$: $\alpha(l_1, l_2) = \alpha(l_2, l_1)$.
α_2. If $k \in Q$, $l_1, l_2 \in \Lambda$: $\alpha(kl_1, l_2) = k\alpha(l_1, l_2)$.
α_3. The map $\Lambda^o x \Lambda \to \Lambda^o x A$ given by $(l, l') \mapsto (l, \alpha(l, l'))$ is an isomorphism.

Let us assume that there is in E a "plane object" P and a subobject $S_2 \hookrightarrow P x P$ (if $(A_1, A_2) \in S_2$, we say that "A_1 is apart from A_2 ") so that $\exists (A_1, A_2)$, $(A_1, A_2) \in S_2$, and there is a map $a : S_2 x P \to A$ such that

a_1. If $A_1, A_2, A_3 \in P$:

$$(A_1, A_2) \in S_2 \wedge (A_1, A_3) \in S_2 \Rightarrow a(A_1, A_2, A_3) = a(A_1, A_3, A_2).$$

a_2. If $(A_1, A_2) \in S_2$, $A_3 \in P$: $a(A_1, A_2, A_3) = a(A_2, A_1, A_3)$,

Let us assume that there is in E a map $\pi : S_2 x L x L \to P$ such that, taking

$$r = (S_2 x \Delta_L)\pi : S_2 x L \to P ,$$

the following axioms π_1 to π_5 are satisfied ; the maps r and π, between "Space objects", are maps representing "geometric constructions" and their properties mean that r represents the S_2-indexed family of "lines connecting two points A_1 and A_2, $(A_1, A_2) \in S_2$ " and that π represents the $(S_2 x L)$-indexed family of the "lines perpendicular to the lines through A_1 and A_2 in the point $r(A_1, A_2, Q)$, $(A_1, A_2, Q) \in S_2 x L$" .

π_1. $\langle p_1, \pi\rangle : S_2 x L x L \to S_2 x P$ is an isomorphism.

We let

$$h' = \langle p_1, \pi\rangle^{-1} p_2 \quad \text{and} \quad h = \langle p_1, \pi\rangle^{-1} p_3 : S_2 x P \to L.$$

π_2. Let $r = (S_2 x \Delta_L)\pi : S_2 x L \to P$ and let $q = \langle q_1, q_2\rangle : S_2 \to L x L$ be the map given by

$$q_i(A_1, A_2) = h'(A_1, A_2, A_i) \qquad (i = 1, 2) :$$

If $(A_1, A_2) \in S_2$:

$$r(A_1, A_2, q_i(A_1, A_2)) = A_i \qquad (i = 1, 2).$$

Let

$$d_2 : S_2 \xrightarrow{q} L x L \xrightarrow{d} \Lambda ,$$

also denoted $d_2(A_1, A_2) = \overline{A_1 A_2}$.

Definition 1.4. Let X be an object of E ; a map $f : X x L \to P$ is called an

X-*isometry* if :

i) If $x \in X$, $Q_1, Q_2 \in L$: $(f(x, Q_1), f(x, Q_2)) \in S_2 \Leftrightarrow (Q_1, Q_2) \in S_1$.
ii) If $x \in X$, $(Q_1, Q_2) \in S_1$: $d_2(f(x, Q_1), f(x, Q_2)) = d(Q_1, Q_2)$.
iii) If $x \in X$, $(Q_1, Q_2) \in S_1$, $k \in U(Q)$, $A \in P$:

$$a(f(x, Q_1), f(x, Q_1(+)kQ_1Q_2), A) = ka(f(x, Q_1), f(x, Q_2), A).$$

π_3. r and π are an S_2-isometry and an $(S_2 x L)$-isometry respectively.

To understand the sense of axioms π_4 and π_5, let us consider the following elementary arguments : "two lines r_1 and r_2 through a point A give rise to a function assigning to each couple of lengths (say the positive real numbers a and b) an area $(ab \sin r_1 r_2)$ " ; the axioms π_4 and π_5 mean, among other things, that "the perpendicular to line r_1 is the unique line through $A \in r_1$, that with r_1 gives the map α, in the above mentioned sense".

π_4. If $(A_1, A_2) \in S_2$, $Q', Q \in L$:

$$a(A_1, A_2, \pi(A_1, A_2, Q', Q) = \alpha(\overline{A_1 A_2}, \overline{QQ'}).$$

If $f : XxL \to P$ is a map of E, if $x \in X$ and $A \in P$, we shall write $\exists Q \in L$, $A = f(x, Q)$ simply as $A \in f(x, -)$.

π_5. Let
$M = [\![(A_1, A_2) \in S_2, Q' \in L, A \in P \mid (A, r(A_1, A_2, Q')) \in S_2]\!]$;
If $(A_1, A_2, Q', A) \in M$:
$a(A_1, A_2, A) = \alpha(\overline{A_1 A_2}, \overline{A\, r(A_1, A_2, Q')}) \Rightarrow A \in \pi(A_1, A_2, Q', -)$.

If $(A_1, A_2) \in S_2$, $k \in Q$, $(Q_1, Q_2) = q(A_1, A_2)$, we shall write : $r(A_1, A_2, Q_1(+)kQ_1Q_2)$ simply as $A_1(+)kA_1A_2$ and for $A_2(-)kA_1A_2$ similarly

From 1.1 we can deduce :

Proposition 1.5. If $(A_1, A_2) \in S_2$, $A \in P$: $A \in r(A_1, A_2, -) \Rightarrow$

$$\exists h \in U(Q),\ A = A_1(+)hA_1A_2 \vee \exists k \in U(Q),\ A = A_2(-)kA_1A_2.$$

In [16] we have proved that, if $\pi' : S_2 x L x L \to P$ is a map satisfying π_1 to π_5 and such that it gives the same map $d_2 : S_2 \to L$ as π, then there are two maps $f : S_2 x L \to L$ and $g : S_2 x L x L \to L$, an S_2-isometry and an $(S_2 x L)$-isometry respectively, so that

$$\pi' = \langle p_1, p_{12} f, g \rangle \pi,$$

$\langle p_1, p_{12} f, g \rangle$ is isomorphic and, for all $(A_1, A_2, Q) \in S_2 x L$,

$$g(A_1 . A_2, Q, Q) = f(A_1, A_2, Q).$$

If $(A_1, A_2) \in S_2$, $A \in P$, $l \in \Lambda^o$, we shall write

$$\pi\,(A_1, A_2, h'(A_1, A_2, A), h(A_1, A_2, A) \pm l\,)$$

simply as $(A \pm l)^{\pi}_{A_1A_2}$ respectively ; then " $(A+l)^{\pi}_{A_1A_2}$ and $(A-l)^{\pi}_{A_1A_2}$ are the two points on the perpendicular to $r(A_1, A_2, -)$ through A, l distant from A ".

Let us assume the following axiom τ ; it means that "if A_1, A_2 are points of a line r and $(A_1, A_2) \in S_2$, the translations with direction perpendicular to r preserve apartness, the distance $\overline{A_1A_2}$ and the area of any parallelogram with two vertices A_1 and A_2 ".

τ. If $(A_1, A_2) \in S_2$, $(Q_1, Q_2) \in S_1$, $l \in \Lambda^o$, $A \in P$, let

$$B_i = r(A_1, A_2, Q_i) \qquad (i = 1, 2):$$

i) $((B_1 \pm l)^{\pi}_{A_1A_2}, (B_2 \pm l)^{\pi}_{A_1A_2}) \in S_2$.

ii) $d_2((B_1 \pm l)^{\pi}_{A_1A_2}, (B_2 \pm l)^{\pi}_{A_1A_2}) = d_2(B_1, B_2)$.

iii) $a(B_1, B_2, A) = a((B_1 \pm l)^{\pi}_{A_1A_2}, (B_2 \pm l)^{\pi}_{A_1A_2}, (A \pm l)^{\pi}_{A_1A_2})$.

If $(A_1, A_2) \in S_2$ and $A \in P$, we shall write $r(A_1, A_2, h'(A_1, A_2, A))$ simply as $\pi_r(A_1, A_2, A)$; then $\pi_r(A_1, A_2, A)$ is the "foot of the perpendicular to $r(A_1, A_2, -)$ through A ".

Now we define a map $p: S_2xPxL \to P$ to represent the (S_2xP)-indexed family of the "lines parallel to $r(A_1, A_2, -)$ through $A \in P$ ".

From 1.1. we have :

If $(A_1, A_2) \in S_2$, $A \in P$, if we let

$$A' = \pi_r(A_1, A_2, A) \quad \text{and} \quad B = (A' + \overline{A_1A_2})^{\pi}_{A_1A_2} :$$

$$\exists\, k \in U(Q),\ A = (A' + k\overline{A'B})^{\pi}_{A_1A_2} \ \vee \ \exists\, k' \in U(Q),\ A = (B - k'\overline{A'B})^{\pi}_{A_1A_2}$$

and, if $(A_1, A_2) \in S_2$, $A \in P$, $Q \in L$, we define $p: S_2xPxL \to P$ as follows :

$$p(A_1, A_2, A, Q) = (r(A_1, A_2, Q) + k\overline{A'B})^{\pi}_{A_1A_2} \quad \text{if } A = (A' + k\overline{A'B})^{\pi}_{A_1A_2},$$

$$p(A_1, A_2, A, Q) = (r(A_1, A_2, Q) + \overline{A'B} - k'\overline{A'B})^{\pi}_{A_1A_2} \quad \text{if } A = (B - k'\overline{A'B})^{\pi}_{A_1A_2}.$$

If $(A_1, A_2, A) \in S_2xP$, $d(h'(A_1, A_2, A), h(A_1, A_2, A))$ is "the distance between A and the line $r(A_1, A_2, -)$ " and we shall write it simply as $h^A_{A_1A_2}$.

We let
$\bar{S} = [\![(A_1, A_2, A_3) \in P^3 \mid (A_1, A_2) \in S_2 \wedge (A_1, A_3) \in S_2]\!]$.

We can prove

Proposition 1.6. *If* $(A_1, A_2, A_3) \in \bar{S}$, $k \in U(Q)$ *:*

$$h_{A_1 A_2}^{A_1 (\pm) k A_1 A_3} = k h_{A_1 A_2}^{A_3} .$$

Let us consider the subobject of $\bar{S}$:

$$T = [\![(A_1, A_2, A_3) \in \bar{S} \mid h_{A_1 A_2}^{A_3} \in \Lambda^o]\!] .$$

If $(A_1, A_2, A_3) \in T$, we will say "the line $r(A_1, A_2, -)$ is apart from the line $r(A_1, A_3, -)$ ".

We will use some propositions that, roughly speaking, we can enunciate as follows :

Proposition 1.7. *For all* $(A_1, A_2) \in S_2$, $k \in U(Q)$, $A \in r(A_1, A_2, -)$, *if* A'_1, A'_2, A' *are the points which correspond to* A_1, A_2, A *under the orthogonal projection i) on a line parallel to* $r(A_1, A_2, -)$ *or ii) on a line such that* $r(A_1, A_2, -)$ *is apart from the direction perpendicular to such a line :*

$$A = A_1 (\pm) k A_1 A_2 \iff A' = A'_1 (\pm) k A'_1 A'_2 .$$

Proposition 1.8. *For all* $(A_1, A_2, A_3) \in T$, $r(A_1, A_2, -)$ *intersects any parallel to* $r(A_1, A_3, -)$ *in a unique point.*

Proposition 1.9. *For all* $(A_1, A_2, A_3) \in \bar{S}$, $k \in U(Q)$, $A \in r(A_1, A_2, -)$, *if* A'_2 , A' *are the points which correspond to* A_2 , A *on* $r(A_1, A_3, -)$, *under a projection parallel to a line apart from* $r(A_1, A_2, -)$ *and from* $r(A_1, A_3, -)$:

$$A = A_1 (\pm) k A_1 A_2 \iff A' = A_1 (\pm) k A_1 A'_2 .$$

Let us assume the following two axioms S and P ; from them it follows Pythagoras' Theorem and the following Propositions 1.10 and 1.11.

S. If $(A_1, A_2) \in S_2$, $Q \in L$, $A \in r(A_1, A_2, -)$, $B \in \pi(A_1, A_2, Q, -)$, let $C = r(A_1, A_2, Q)$ (then "A, B, C are the vertices of a right triangle") :

$$(A, C) \in S_2 \vee (B, C) \in S_2 \Rightarrow (A, B) \in S_2 .$$

In the classical case, where the order on the line is total, the following axiom P expresses the "commutativity of the scalar product", that in our theory we can state as in Proposition 3.7 ; Axiom P is stronger because it implies that "in any right triangle (ABC), the foot of the perpendicular through C to the hypotenuse AB is inside the interval [AB] (defined in an obvious way by the order on the line)" ; it follows that in the ring object $S_2 x L \to S_2$ of E/S_2 (cf. § 3) "any square is positive".

P. If $(A_1, A_2, A_3) \in \overline{S}$, let $c = \rho(\overline{A_1 A_3}, h^{A_3}_{A_1 A_2})$:

$$\pi_r(A_1, A_3, \pi_r(A_1, A_2, A_3)) = A_3(+)c^2 A_3 A_1 .$$

Proposition 1.10. *If* $k_1, k_2 \in Q$:

$$k_1 \in U(Q) \vee k_2 \in U(Q) \Rightarrow \exists!\, h \in U(Q),\ h^2 = k_1^2 + k_2^2 .$$

Proposition 1.11. *If* $(A_1, A_2, A) \in \overline{S}$ *and* $A_2' = (A_1 + \overline{A_1 A_2})^{\pi}_{A_1 A_2}$:

$$(A_1, A_2, A) \in T \vee (A_1, A_2', A) \in T .$$

2. We obtain a model of the theory described in Section 1, i.e. a logical category S with a logical functor $M : E \to S$ (i.e. it preserves finite limits, regular epis and sup), if we consider a logical category S in which there is a commutative ring object R that is a local ring, i.e. so that :

$$0 = 1 \quad \text{is false.}$$

$$\text{If } x, y \in R : (x + y) \text{ inv} \Rightarrow x \text{ inv} \vee y \text{ inv}$$

(x inv means $\exists y \in R,\ xy = 1$);

and furthermore so that :

2.1. If $x \in R$: $x^2 = 0 \Rightarrow x = 0$.
2.2. If $x, y \in R$: x inv $\vee\, y$ inv $\Rightarrow \exists z \in R,\ z^2 = x^2 + y^2$.

Moreover, let R be an ordered ring object such that :

2.3. If $x, y \in R$: $x \geq 0 \wedge y \geq 0 \wedge (x \text{ inv} \vee y \text{ inv}) \Rightarrow (x + y)$ inv.
2.4. If $x \in R$: x inv $\Rightarrow x > 0 \vee -x > 0$
($x > 0$ means $x \geq 0 \wedge x$ inv).

Finally, if $R^{\geq 0} = [\![x \in R \mid x \geq 0]\!]$, suppose there exists a map $R \to R^{\geq 0}$ denoted $x \mapsto |x|$ such that :

2.5. If $x \in R$: $|x| = |-x|$.
2.6. If $x \in R$: $x \geq 0 \Rightarrow |x| = x$.
2.7. If $x, y \in R$: $x > 0 \Rightarrow |xy| = x|y|$.
2.8. If $x \in R$: $x^2 = |x|^2$.

From 2.2, 2.3, 2.4 and 2.8, it follows :

If $x, y \in R : x$ inv $\vee$ y inv $\Rightarrow \exists! z \in R, z^2 = x^2 + y^2 \wedge z > 0$, and in this case we will write $z = (x^2 + y^2)^{\frac{1}{2}}$.

From results of [13], we have that, to find a model M, it is enough to assign, for every object and map of E, objects and maps of S so that the axioms of Section 1 hold in S. We obtain this by taking the objects and the maps as follows :

$$M(L) = R, \qquad M(\Lambda) = (R^{\geq 0}, +),$$

and the maps $M(t_1) : R^{\geq 0} xR \to R$ are defined by means of addition and sub-traction ; the map $M(d) : RxR \to R^{\geq 0}$ is given by $(x, y) \mapsto |x - y|$;

$$M(\Lambda^o) = [\![x \in R \mid x > 0]\!]; \; M(Q) = (R^{\geq 0}, +, .), \; M(A) = (R^{\geq 0}, +)$$

and $M(\delta), M(\beta), M(\alpha) : R^{\geq 0} xR^{\geq 0} \to R^{\geq 0}$ are defined by means of multiplication ; $M(P) = RxR$;

$$M(S_2) = [\![((x_1, y_1), (x_2, y_2)) \in R^2 x R^2 \mid (x_1 - x_2) \text{ inv } \vee (y_1 - y_2) \text{ inv}]\!] ;$$

the map $M(a)$: $M(S_2)xRxR \to R^{\geq 0}$ is given by

$$((x_1, y_1), (x_2, y_2), (x, y)) \mapsto |x_1 y_2 + x_2 y + x y_1 - x y_2 - x_2 y_1 - x_1 y_1|.$$

If $((x_1, y_1), (x_2, y_2)) \in M(S_2)$, let

$$z = ((x_2 - x_1)^2 + (y_2 - y_1)^2)^{-\frac{1}{2}}$$

and let $\bar{\pi} : M(S_2)xRxR \to RxR$ be the map given by

$$((x_1, y_1), (x_2, y_2), t', t) \mapsto$$

$$(x_1 + zt'(x_2 - x_1) + z(t' - t)(y_2 - y_1), y_1 + zt'(y_2 - y_1) + z(t - t')(x_2 - x_1)) ;$$

let $\mathcal{J}$ be the set of maps

$$<p_1, p_{12} f, g> : M(S_2)xRxR \to M(S_2)xRxR$$

where $f : M(S_2)xR \to R$ is a $M(S_2)$-isometry and $g : M(S_2)xRxR \to R$ a $(M(S_2)xR)$-isometry (X-isometries are defined here as in 1.2 by means of $M(\Lambda^o)$ and $M(d)$) so that :

$$\text{if } ((x_1, y_1), (x_2, y_2), k) \in M(S_2)xR :$$

$$g((x_1, y_1), (x_2, y_2), k, k) = f((x_1, y_1), (x_2, y_2), k).$$

$<p_1, f>$ and $<p_1 \cdot p_2, g>$ are isomorphisms and then $<p_1, p_{12} f, g>$ is an isomorphism. The map $M(\pi)$ is given by the map $\bar{\pi}$ "up to isomorphisms of $\mathcal{J}$" in the sense that, for all $b \in \mathcal{J}$, the map $b\bar{\pi}$ satisfies the axioms of Section 1.

We don't have any difficulty in finding models with values in a topos. For instance, the properties of R are satisfied by the object of Dedekind reals in a topos with natural number object and, in particular, in the category of sheaves over a topological space X, where the object of Dedekind reals is the sheaf of continuous real-valued functions on X.

Furthermore we can obtain a model by considering a small full subcategory R of the category of rings in which all objects satisfy all the properties listed in this Section ; then (see [8]) a model is given by the category $S = \mathrm{Set}^{R}$ in which R is the forgetful functor.

3. Let E be the logical category described in Section 1. Let us consider the comma category E/S_2 and the functor $E \to E/S_2$, denoted S_2x( ), given by

$$A \mapsto p_1 : (S_2 \times A \to S_2)$$

and defined on the maps in a natural way ; it is easy to verify that E/S_2 is a logical category and that the functor S_2x( ) preserves finite limits, regular epis and sups, and hence preserves the "logical structure " of E (see [9]).

In this Section we show that the object $p_1 : S_2 \times L \to S_2$ of E/S_2, that we call R, is a ring object which satisfies all the properties of ring theory described in Section 2 ; moreover, if we construct from R, as we did in Section 2, a model $\overline{M} : E \to E/S_2$, we have an isomorphism of models $S_2\mathrm{x}(\ ) \simeq \overline{M}$.

If $(A_1, A_2) \in S_2$, in the following we will denote $(Q_1, Q_2) = q(A_1, A_2)$.

Proposition 3.1. *The object* $R = (S_2 \times L \to S_2)$ *of* E/S_2 *carries the structure of a commutative ring.*

Proof. Let us define the map + : RxR → R as the map

$$<p_1, \sigma>: S_2 \times L \times L \to S_2 \times L,$$

σ given by (see 1.1) :

$$(A_1, A_2, Q, Q') \mapsto Q(+)h'Q_1Q_2 \qquad \text{if } \exists\, h' \in U(Q),\ Q' = Q_1(+)h'Q_1Q_2$$

$$(A_1, A_2, Q, Q') \mapsto Q(+)Q_1Q_2(-)k'Q_1Q_2 \qquad \text{if } \exists\, k' \in U(Q),\ Q' = Q_2(-)k'Q_1Q_2.$$

Let us define the map 0 : 1 → R as the map

$$S_2 \to S_2 \times L \quad \text{given by} \quad (A_1A_2) \mapsto (A_1, A_2, Q_1)$$

and the map op : R → R as the map $<p_1, \mu>: S_2 \times L \to S_2 \times L$, μ given by :

$$(A_1, A_2, Q) \mapsto Q_1(-)hQ_1Q_2 \qquad \text{if } \exists\, h \in U(Q),\ Q = Q_1(+)hQ_1Q_2,$$

$(A_1, A_2, Q) \mapsto Q_1(-)Q_1Q_2(+)kQ_1Q_2$ if $\exists\ k \in U(Q),\ Q = Q_2(-)k\,Q_1Q_2$.

Let us define the map $\cdot : R{\times}R \to R$ as the map $<p_1, \nu> : S_2{\times}L{\times}L \to S_2{\times}L$, ν given by

$(A_1, A_2, Q, Q') \mapsto Q_1(+)hh'\,Q_1Q_2$ if
$\exists\ h \in U(Q),\ Q = Q_1(+)h\,Q_1Q_2 \wedge \exists\ h' \in U(Q),\ Q' = Q_1(+)h'Q_1Q_2$,

$(A_1, A_2, Q, Q') \mapsto Q_1(-)hk'\,Q_1Q_2(+)h\,Q_1Q_2$ if
$\exists\ h \in U(Q),\ Q = Q_1(+)h\,Q_1Q_2 \wedge \exists\ k' \in U(Q),\ Q' = Q_2(-)k'\,Q_1Q_2$,

$(A_1, A_2, Q, Q') \mapsto Q_1(-)\,kh'Q_1Q_2(+)h'Q_1Q_2$ if
$\exists\ k \in U(Q),\ Q = Q_2(-)k\,Q_1Q_2 \wedge \exists\ h' \in U(Q),\ Q' = Q_1(+)h'Q_1Q_2$,

$(A_1, A_2, Q, Q') \mapsto Q_1(-)(k{+}k')Q_1Q_2(+)(1{+}\,kk')Q_1Q_2$ if
$\exists\ k \in U(Q),\ Q = Q_2(-)k\,Q_1Q_2 \wedge \exists\ k' \in U(Q),\ Q' = Q_2(-)k'Q_1Q_2$,

and the map $1 : \mathbf{1} \to R$ as the map $S_2 \to S_2{\times}L$ given by

$$(A_1, A_2) \mapsto (A_1, A_2, Q_2).$$

We can prove all the ring properties, distinguishing the different cases we have by 1.1 ; the commutativity of R follows from the commutativity of the product in Q.

The following proposition characterizes the invertible elements of R. Let us write

$\exists\ Q' \in L,\ \nu(A_1, A_2, Q, Q') = Q_2$ simply as (A_1, A_2, Q) inv.

Proposition 3.2. *If* $(A_1, A_2) \in S_2,\ Q \in L$:

(A_1,A_2,Q) inv $\Leftrightarrow \exists\, a \in U(Q),\ Q = Q_1(+)\,aQ_1Q_2 \vee \exists\ b \in U(Q),\ Q = Q_1(-)bQ_1Q_2$

Proof. $\Rightarrow$: By 1.1 we have

$\exists\ h \in U(Q),\ Q = Q_1(+)h\,Q_1Q_2 \vee \exists\ k \in U(Q),\ Q = Q_2(-)k\,Q_1Q_2$;

the first case is trivial ; for the second, distinguishing the cases

$\exists\ h' \in U(Q),\ Q' = Q_1(+)h'Q_1Q_2$ and $\exists\ k' \in U(Q),\ Q' = Q_2(-)k'Q_1Q_2$

and recalling the definition of the product in R, we have, in the first case, $Q = Q_1(+)h'^{-1}\ Q_1Q_2$ and, in the second, $Q = Q_1(-)k'^{-1}k\,Q_1Q_2$.
$\Leftarrow$: If $Q = Q_1(+)a\ Q_1Q_2$, $Q' = Q_1(+)a^{-1}\,Q_1Q_2$ and if $Q = Q_1(-)b\,Q_1Q_2$, then $Q' = Q_1(-)b^{-1}Q_1Q_2$.

Proposition 3.3. R *carries the structure of a local ring, i.e. :*

i) If $(A_1, A_2) \in S_2$, $Q_1 = Q_2$ *is false.*

ii) If $(A_1, A_2) \in S_2$, $Q, Q' \in L$:

$$(A_1, A_2, \sigma(A_1, A_2, Q, Q')) \text{ inv } \Rightarrow (A_1, A_2, Q) \text{ inv } \vee (A_1, A_2, Q') \text{ inv}.$$

Proof. i) follows from L_1^o and L_3.
ii) : by 1.1 we have to calculate only in the case

$$\exists\, k \in U(Q),\ Q = Q_2(-)k\, Q_1Q_2 \ \wedge\ \exists\, k' \in U(Q),\ Q' = Q_2(-)k'Q_1Q_2$$

and, because $(Q_1, \sigma(A_1, A_2, Q, Q')) \in S_1$, in the case in which furthermore we have

$$(Q, \sigma(A_1, A_2, Q, Q') \in S_1 \wedge (Q', \sigma(A_1, A_2, Q, Q') \in S_1 ;$$

from here it follows (A_1, A_2, Q') inv $\wedge$ (A_1, A_2, Q) inv.

Proposition 3.4. *R is an ordered ring, i.e. there exists (in E/S_2) a relation $(\leqq) \hookrightarrow S_2 x L x L$ which is an order relation and which is compatible with the operations. Moreover the properties 2.3 and 2.4 are satisfied.*

Proof. Let us define $(\leqq)$ in the following manner :

$$(A_1, A_2, Q, Q') \in (\leqq) \Leftrightarrow \exists\, k \in Q,\ Q' = Q(+)k\, Q_1 Q_2 ;$$

then it is straightforward that $(\leqq)$ is an order relation and that it is compatible with the operations. Moreover 2.3 follows from 3.2 and L_5^o and 2.4 is 3.2, because the relation $(<)$ is given by

$$(A_1, A_2, Q, Q') \in (<) \Leftrightarrow \exists\, k \in U(Q),\ Q' = Q(+)k\, Q_1Q_2.$$

We observe that we have not yet used the distance and its properties (axioms L_3, L_4 and D) ; we used only the property :

$$\text{if } Q \in L,\quad l_1, l_2 \in L :\ Q \pm l_1 = Q \pm l_2 \Rightarrow l_1 = l_2$$

that follows from L_4.

The following proposition states that in R there is a norm satisfying 2.5 and 2.6.
Let $\|\ \| : S_2 x L \to Q$ the map given by

$$(A_1, A_2, Q) \mapsto \|A_1, A_2, Q\| = \rho(\overline{Q_1 Q_2}, \overline{Q_1 Q})$$

and $|\ | : S_2 x L \to L$ the map given by

$$(A_1, A_2, Q) \mapsto |A_1, A_2, Q| = Q_1(+)\|A_1, A_2, Q\|\, Q_1Q_2.$$

Proposition 3.5. *i) If* $(A_1, A_2) \in S_2$, $Q \in L$:

$$|A_1, A_2, Q| = |A_1, A_2, \mu(A_1, A_2, Q)| .$$

ii) If $(A_1, A_2) \in S_2$, $Q \in L$:

$$(A_1, A_2, Q_1, Q) \in (\leqq) \Rightarrow |A_1, A_2, Q| = Q.$$

Proof. We can prove i distinguishing the different cases we have by 1.1 and using the axiom D. ii is straightforward.

We observe that $R^{\geqq 0}$ is isomorphic to the subobject $S_2 \times Q \hookrightarrow S_2 \times L$ of R given by

$$(A_1, A_2, k) \mapsto (A_1, A_2, Q_1 (+) k\, Q_1 Q_2)$$

and that the norm in R is defined by the map $<p_1, \|\ \|> : S_2 \times L \to S_2 \times Q$.

For the following proposition we use the plane axioms ; taking account of the definition of product in R, it means that "vertical projection of a line $r(A_1, A, -)$ onto a line $r(A_1, A_2, -)$ preserves ratios".

Proposition 3.6. *If* $(A_1, A_2, A) \in \bar{S}$, $B \in P$, $h, k \in U(Q)$, let

$$A' = \pi_r(A_1, A_2, A) \quad \textit{and} \quad B' = \pi_r(A_1, A_2, B) :$$

i) $A' = A_1(+)h A_1A_2 \wedge B = A_1(+)k A_1A \Rightarrow B' = A_1(+)hk A_1A_2$.

ii) $A' = A_1(+)h A_1A_2 \wedge B = A(-)k A_1A$
$\Rightarrow B' = A_1(+)h A_1A_2(-)hk A_1A_2$.

iii) $A' = A_2(-)h A_1A_2 \wedge B = A_1(+)k A_1A$
$\Rightarrow B' = A_1(+) k A_1A_2(-)hk A_1A_2$.

iv) $A' = A_2(-) h A_1A_2 \wedge B = A(-) k A_1A$
$\Rightarrow B' = A_1(+)(1+ hk)A_1A_2(-)(h+k) A_1A_2$.

Proof. i and ii follow from 1.7 because $A' = A_1(+)h A_1A_2$ entails that " $r(A_1, A, -)$ is apart from the direction perpendicular to $r(A_1, A_2, -)$ ". For iii and iv we distinguish the two cases (Proposition 1.11)

$$(A_1, A_2, A) \in T \quad \text{and} \quad (A_1, A'_2, A) \in T .$$

Let $(A_1, A_2, A) \in T$; by Axiom S, $(A, A_2) \in S_2$ and moreover the lines $r(A_1, A, -)$ and $r(A_1, A_2, -)$ are both apart from $r(A, A_2, -)$; if we consider the projection parallel to $r(A, A_2, -)$ of $r(A_1, A, -)$ on $r(A_1, A_2, -)$ and if we call C the projection of B, from $B = A_1(+) k A_1 A$ it follows (Proposition 1.9)

$$C = A_1 (+)k A_1A_2 \quad \text{and} \quad B' = C(-)hk A_1A_2 ;$$

from here it follows iii. We obtain iv "projecting on $p(A_1, A_2, A, -)$ " and applying iii. - Let $(A_1, A'_2, A) \in T$; we can prove iii and iv using 1.7 and distinguishing the two cases

$\exists a \in U(Q), \ A' = A_1 (+) a A_1A_2$ and $\exists b \in U(Q), \ A' = A_1 (-) b A_1 A_2.$

Using 3.6 i, iii and Axiom P we can prove the following proposition ; it states the "symmetry of the vertical projection ratio of two lines".

Proposition 3.7. *If* $(A_1, A_2, A_3) \in \overline{S}$, $h \in U(Q)$, *let*

$k = \rho(\overline{A_1 A_3}, \overline{A_1 A_2}), \quad A'_3 = \pi_r(A_1, A_2, A_3), \quad A'_2 = \pi_r(A_1, A_3, A_2) :$

i) $A'_3 = A_1(+) h A_1 A_2 \Rightarrow A'_2 = A_1(+) hk^2 A_1 A_3.$

ii) $A'_3 = A_2(-) h A_1 A_2 \Rightarrow A'_2 = A_1(+) k^2 A_1 A_3 (-) hk^2 A_1 A_3.$

The following proposition states that R satisfies 2.7 and 2.8.

Proposition 3.8. *i) If* $(A_1, A_2) \in S_2$, $Q, Q' \in L$:

$(A_1, A_2, Q_1, Q) \in (<) \Rightarrow |A_1, A_2, \nu(A_1, A_2, Q, Q')| = \nu(A_1, A_2, Q, |A_1, A_2, Q'|)$

ii) If $(A_1, A_2) \in S_2$, $Q \in L$:

$\nu(A_1, A_2, |A_1, A_2, Q|, |A_1, A_2, Q|) = \nu(A_1, A_2, Q, Q).$

Proof. i) We have

$\exists h \in U(Q), \ Q' = Q_1(+) h Q_1 Q_2 \ \vee \ \exists k \in U(Q), \ Q' = Q_2(-) k Q_1 Q_2$;

in the first case i follows from 3.5, ii ; in the second, we can prove it by using 1.6 and 3.6 iii.

ii) let us distinguish the cases

$\exists h \in U(Q), \ Q' = Q_1(+) h Q_1 Q_2$ and $\exists k \in U(Q), \ Q = Q_2(-) k Q_1 Q_2$;

the first is trivial by 3.5 ii ; in the second we apply 3.7, 3.6 iv and Axiom P to the configuration

$(A_1, A_2, A_3) \in \overline{S}$, with $A_3 = (A'_3 + \overline{A_1 A_2})^{\pi}_{A_1 A_2}$ and $A'_3 = A_2(-) k A_1 A_2.$

The following proposition states that R satisfies 2.1 and 2.2.

Proposition 3.9. *i) If* $(A_1, A_2) \in S_2$, $Q \in L$: $\nu(A_1, A_2, Q, Q) \models Q_1 \Rightarrow Q = Q_1.$

ii) If $(A_1, A_2) \in S_2$, $Q, Q' \in L$: (A_1, A_2, Q) inv $\vee$ (A_1, A_2, Q') inv

$\Rightarrow \exists S \in L, \nu(A_1, A_2, S, S) = \sigma(A_1, A_2, \nu(A_1, A_2, Q, Q), \nu(A_1, A_2, Q', Q'))$

Proof. i) Distinguishing the two cases

$$\exists\, h \in U(Q),\ Q = Q_1(+)h\,Q_1Q_2 \quad \text{and} \quad \exists\, k \in U(Q),\ Q = Q_2(-)k\,Q_1Q_2,$$

only the second one is possible ; in this case, if we let

$$A' = A_2(-)k\,A_1A_2 \quad \text{and} \quad A = (A' + \overline{A_1A_2})^{\pi}_{A_1A_2},$$

using 3.8 ii and Pythagoras' Theorem, we have $\overline{A_1A} = \overline{AA'}$ and, by π_5, $A' = A_1$ and then $Q = Q_1$.

ii) We have

$$(A_1, A_2, Q) \text{ inv} \Rightarrow \|A_1, A_2, Q\| \in U(Q)$$

and then by 1.10,

$$\exists\, h \in U(Q),\ h^2 = \|A_1, A_2, Q\|^2 + \|A_1, A_2, Q'\|^2 ;$$

if we take $S = Q_1(+)h\,Q_1Q_2$, by 3.8 ii we obtain ii.

We observe that, if $\pi' : S_2 \times L \times L \to P$ is a map satisfying π_1 to π_5 and which gives the same map $d_2 : S_2 \to L$ as π, we obtain a ring R' and a map $R \to R'$ given by a map $f : S_2 \times L \to L$ that is an S_2-isometry (cf. § 1) ; from 1.3 it follows that $< p_1, f > : S_2 \times L \to S_2 \times L$ is a ring isomorphism and that it preserves ($\leqq$) ; moreover, from the Axiom D, it follows that it preserves $|\ |$.

We proved that the object R of E/S_2 satisfies all the properties mentioned in Section 2. The following considerations show that there is an isomorphism of models $S_2 \times (\ ) \simeq \bar{M}$, as we said at the beginning of this Section.

We know that there is a canonical isomorphism $(S_2 \times Q \to S_2) \simeq R^{\geqq 0}$ and it is easy to verify that there are canonical isomorphisms

$$(S_2 \times \Lambda \to S_2) \simeq R^{\geqq 0}, \quad (S_2 \times \Lambda^0 \to S_2) \simeq R^{\geqq 0}, \quad (S_2 \times A \to S_2) \simeq R^{\geqq 0},$$

given by Q_1, Q_2 and α_3 ; moreover "under these isomorphisms" the maps $S_2 \times t_I : S_2 \times \Lambda \times L \to S_2 \times L$ "are" $R \times R^{\geqq 0} \overset{+}{\underset{-}{\rightrightarrows}} R$ (in the sense that

$S_2 \times <t_1, p_2> \vee S_2 \times <t_2, p_2> \hookrightarrow S_2 \times L \times L$ "is" $<p_1, +> \vee <p_1, -> \hookrightarrow R \times R$),

$S_2 \times d : S_2 \times L \times L \to S_2 \times \Lambda$ "is" the map $R \times R \to R^{>0}$ given by $(x, y) \mapsto |x - y|$,

$S_2 \times \delta : S_2 \times Q \times \Lambda \to S_2 \times \Lambda$, $S_2 \times \beta : S_2 \times Q \times A \to S_2 \times A$, $S_2 \times \alpha : S_2 \times \Lambda \times \Lambda \to S_2 \times A$

"are" the multiplication $R^{\geqq 0} \times R^{\geqq 0} \xrightarrow{\cdot} R^{\geqq 0}$.

Now we define a map $<X, Y> : S_2 \times P \to L \times L$ to prove that, in E/S_2,

$$(S_2 \times P \to S_2) \simeq R \times R ;$$

"<X, Y> assigns to each point of P its coordinates".

For all $(A_1, A_2, A) \in S_2 x P$ let us denote (leaving out the notation A_1A_2)

$A_X = \pi_r(A_1, A_2, A)$ and $A_Y = \pi_r(A_1, A'_2, A)$ $(A'_2 = (A_1 + \overline{A_1A_2})^{\pi}_{A_1A_2})$;

X is defined by (see 1.5) :

$$X(A_1, A_2, A) = Q_1(+)h\, Q_1Q_2 \quad \text{if } \exists\, h \in U(Q),\ A_X = A_1(+)h\, A_1A_2,$$
$$X(A_1, A_2, A) = Q_2(-)k\, Q_1Q_2 \quad \text{if } \exists\, k \in U(Q),\ A_X = A_2(-)k\, A_1A_2.$$

Similarly, Y is defined considering A'_2 and A_Y .

It is clear that

$$<p_1, X, Y> : S_2 x P \to S_2 x L x L$$

is an isomorphism and then that, in E/S_2, $(S_2 x P \to S_2) \simeq R x R$.

Proposition 3.10. *The object $p_1 : (S_2 x S_2 \to S_2)$ of E/S_2 is isomorphic to*

$$\bar{S}_2 = [\![((x_1, y_1), (x_2, y_2)) \in R^2 x R^2 \mid (x_1 - x_2) \text{ inv } \vee\ (y_1 - y_2) \text{ inv}]\!].$$

Proof. By Axiom S and 1.11, we have : if $(A_1, A_2) \in S_2$, $(A, B) \in P x P$:

$$(A, B) \in S_2 \iff (A_X, B_X) \in S_2 \vee (A_Y, B_Y) \in S_2$$

and, if we denote for all $(A_1, A_2, A) \in S_2 x P$ (leaving out the notation A_1A_2)

$x_A = (A_1, A_2, X(A_1, A_2, A)) \in R$ and $y_A = (A_1, A_2, Y(A_1, A_2, A)) \in R$,

it is easy to prove that in $\hat{E}/S_2$ we have :

if $(A_1, A_2, A, B) \in S_2 x P x P$: $(A_1, A_2, A_X, B_X) \in S_2 x S_2 \iff x_A - x_B$ inv

and for (A_Y, B_Y) analogously.

Proposition 3.11. *With the notations above mentioned, in E/S_2 we have :*

If $(A_1, A_2, M, N, A) \in S_2 x S_2 x P$:

$A \in r(M, N, -) \iff \exists\ t \in R,\ \ x_A = x_M + t(x_M - x_N) \wedge y_A = y_M + t(y_M - y_N)$.

Proof. Let us define $C \in P$ by means of the isomorphism $<p_1, X, Y>$, by $x_C = x_M + Q_2$ and $y_C = y_M$ and $C' = (M + \overline{MC})^{\pi}_{MC}$; let us prove $\Rightarrow$; from 1.5, we have

$$\exists\, h \in U(Q),\ A = M(+)h MN \vee \exists\, k \in U(Q),\ A = N(-)k\, MN ;$$

in the first case let us denote $t = (A_1, A_2, Q_1(+)h\, Q_1Q_2)$ and in the second $t = (A_1, A_2, Q_2(-)k\, Q_1Q_2)$. We obtain $\Rightarrow$ by distinguishing the four cases

of the proposition 3.6, applied to (M, C, N) and (M, C', N) $\in \bar{S}$, and considering 1.7.
Let us prove $\Leftarrow$; by 1.11 we have (M, C, N) $\in T \vee$ (M, C', N) $\in T$; in the first case, by 1.8

$$\exists A' \in P, \ A' \in r(M, N, -) \wedge y_{A'} = y_A$$

and, by $\Rightarrow$

$$\exists t' \in R, \ x_{A'} = x_M + t'(x_M - x_N) \wedge y_{A'} = y_M + t'(y_M - y_N) \ ;$$

as $y_M - y_N$ is invertible, we have $t = t'$ and A = A' $\in r(M, N, -)$. In the second case, the proof is analogous.

Using some results of [16], we can prove :

Proposition 3.12. *If* $(A_1, A_2), (A, B) \in S_2$:

$$\bar{AB} = \|(x_B - x_A)^2 + (y_B - y_A)^2\|^{\frac{1}{2}} \ \overline{A_1A_2} \ .$$

Using also 3.11, we have :

Proposition 3.13. *Let* $<p_1, f> : S_2 x S_2 x P \to S_2 x P$ *be the map defined, under the canonical isomorphisms, by the map* $\bar{S}_2 x RxR \to RxR$ *given by*

$$(x_1, y_1, x_2, y_2, x, y) \mapsto (mx - ny + c, \ nx + my + c'),$$

with $m = x_2 - x_1$, $n = y_2 - y_1$, $c = x_1$, $c' = y_1$; *we have (roughly speaking) :*

i) $<p_1, p_2, f>$ *is isomorphic.*
ii) "f preserves the alignment".
iii) "f multiplies by $\|(m^2 + n^2)^{\frac{1}{2}}\|$ *the distance of two points which lie on a line".*
iv) "f preserves the perpendicularity".
v) "f multiplies by $\|m^2 + n^2\|$ *the areas".*

Proposition 3.14. *Let* $\bar{a} : \bar{S}_2 x RxR \to R^{\geq 0}$ *be the map given by*

$$(x_1, y_1, x_2, y_2, x, y) \mapsto |x_1y_2 + x_2y + xy_1 - xy \qquad y_1 - x_1y| \ ;$$

the following diagram commutes (in E/S_2*) :*

$$\begin{array}{ccc} S_2 x S_2 x P & \xrightarrow{S_2 x\, a} & S_2 x\, A \\ \downarrow \wr & & \downarrow \wr \\ \bar{S}_2 x RxR & \xrightarrow[\bar{a}]{} & R^{\geq 0} \end{array}$$

Proof. For every $(A_1, A_2, M, N, B) \in S_2 x S_2 x P$, let $A \in P$ be such that $f(A_1, A_2, M, N, A) = B$ (3.13, i) ; then

$$a\,(A_1, A_2, A) = \alpha\,(\overline{A_1 A_2}, h^{A}_{A_1 A_2}) = \| y_A \| \; \alpha\,(\overline{A_1 A_2}, \overline{A_1 A_2})$$

and, by 3.13 v,

$$a(M, N, B) = \| (x_N - x_M)^2 + (y_N - y_M)^2 \| \, a\,(A_1, A_2, A).$$

As R satisfies 2.7, we obtain

$$a(M, N, B) \;=\; \| x_M y_N + x_N y_B + x_B y_M - x_B y_N - x_N y_M - x_M y_B \| \alpha(\overline{A_1 A_2}, \overline{A_1 A_2})$$

and then the statement.

Proposition 3.15. *Let $\overline{\pi} : \overline{S}_2 x R x R \to R x R$ be the map given by :*

$$(x_1 y_1, x_2 y_2, t', t) \mapsto$$

$$(x_1 + dt'(x_2 - x_1) + d(t'-t)(y_2 - y_1),\ (y_1 + dt'(y_2 - y_1) + d(t - t')(x_2 - x_1))$$

with

$$d = ((x_2 - x_1)^2 + (y_2 - y_1)^2)^{-\frac{1}{2}} ;$$

let us define the set $\mathcal{J}$ of maps of E/S_2 as in Section 2 ; there is a map b from $\mathcal{J}$ such that $(S_2 x \pi)$ is given, under canonical isomorphisms, by $b\overline{\pi}$.

Proof. By canonical isomorphisms, $\overline{\pi}$ gives a map

$$\langle p_1, \pi' \rangle : S_2 x S_2 x L x L \to S_2 x P$$

and π' (with the obvious modifications) satisfies (in E) all the properties $\pi_1, \ldots, \pi_5$ and because of Proposition 3.12 we can prove that there are two maps

$$\overline{f} : S_2 x S_2 x L \to L \quad \text{and} \quad \overline{g} : S_2 x S_2 x L x L \to L,$$

an $(S_2 x S_2)$-isometry and an $(S_2 x S_2 x L)$-isometry respectively, such that

$$\langle p_1, p_2, p_{123}\overline{f}, \overline{g} \rangle \langle p_1, \pi' \rangle = (S_2 x \pi).$$

Under canonical isomorphisms, $\langle p_1, p_2, p_{123}\overline{f}, \overline{g} \rangle$ is a map of $\mathcal{J}$.

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