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ON PRODUCT AND CHANGE OF BASE FOR TOPOSES

by Andrews PITTS

ABSTRACT. Le produit de deux S-topos bornés coïncide avec leur produit tensoriel comme catégories cocomplètes S-indexées. De plus, la catégorie indexée cocomplète sous-jacente au produit fibré d'un S-topos borné le long d'un morphisme géométrique est obtenue en considérant le topos comme une catégorie indexée cocomplète, puis en appliquant un certain foncteur adjoint associé au morphisme géométrique. Un corollaire montre la stabilité des morphismes essentiels et localement connexes par changement de base.

0. INTRODUCTION.

In this paper we prove some results about Grothendieck toposes and geometric morphisms that derive from regarding a Grothendieck topos (resp., the inverse image part of a geometric morphism) as a cocomplete category (resp., cocontinuous functor) with additional properties. Specifically, if S is an elementary topos with natural number object, then the product of two Grothendieck S-toposes coincides with their "tensor product" as cocomplete S-indexed categories. Moreover if $p: E \to S$ is a geometric morphism between elementary toposes (with natural number objects), then the pullback of a Grothendieck S-topos F along p coincides with the cocomplete S-indexed category obtained from F (regarded as a cocomplete S-indexed category) by "changing base along p". An immediate corollary of the latter fact is a result about essential geometric morphisms which generalises a theorem of Tierney on the pullback of locally connected geometric morphisms (unpublished, but see [12], V).

These results are seen as favourable evidence for the speculation that one may be able to describe Grothendieck toposes in terms of cocomplete categories in a way analogous to that in which Joyal and Tierney have recently described the theory of locales as part of the "commutative algebra" of complete lattices and arbitrary sup preserving maps : see [8]. Thus replacing locales by Grothendieck toposes and sups by colimits, Theorems 2.3 and 3.6 below are the analogues of the results in III.2 and VI.1 of [8]. However the proofs are not analogous : this is because in the one case they are immediate corollaries of the characterization of a locale as a certain kind of commutative monoid in the monoidal category of complete lattices and sup-preserving maps equipped with its tensor product (the monoid multplication giving binary meet in the locale and the unit giving the top element), whilst we do not know of an analogous characterization of a Grothendieck topos over the 2-category of cocomplete categories equipped with its tensor product (and possibly other, more complicated things). One of the fruitful results of Joyal and Tierney's characterization of locales is the ease with which one can deal with locale homomorphisms that have left adjoints (and in particular with open continuous maps), simply because these left adjoints necessarily preserve sups. Correspondingly here we obtain results about *essential* geometric morphisms, since the left adjoint of the inverse image part of such a geometric morphisms preserves colimits. A good test of the workability of any description of Grothendieck toposes over cocomplete categories would be whether it yielded more results along these lines. We have in mind for instance, the speculation that if

$$\begin{tikzcd} H \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\Rightarrow"] & G \arrow[d, "g"] \\ F \arrow[r, "f"] & E \end{tikzcd}$$

is a comma square of Grothendieck toposes and geometric morphisms, then f essential implies k essential (and $k_! h^* \simeq g^* f_!$). Such conjectures (and their application to proving results about coherent logic) were the original motivation for the considerations in this paper, but they were also influenced by Lawvere's lectures on "extensive and intensive quantities" at Aarhus in June 1983.

Evidently both the statement and the proofs of the results in this paper necessitate working in "category theory over an elementary topos S". The "small" part of this theory is contained within the *internal* mathematics of the topos (cf. [7]) and thus can be presented in an easily digestible form "as though S were the category of sets", or more precisely, using the language of type theory. The "large" part of category theory over S (and its interplay with the internal logic of S) requires the use of fibrations over S, or S-indexed categories; its study was initiated about a decade ago by Bénabou-Celeyrette and Paré-Schumacher (cf. [4, 5, 6, 13]) and is still under development. In particular an appropriate formal language and metatheorems that would allow automatic transfer of suitably constructive arguments in ordinary category theory to the S-indexed cas has not been worked out in full generality. (We have in mind here the analogues of the Mitchell-Bénabou language, Kripke-Joyal semantics, etc. in topos theory.) So we cannot refer the reader to a description of the formal language of category theory over an arbitrary base topos S. Nevertheless for ease of comprehension, the arguments of the first two sections of this paper are given in an informal version of such a language, i.e. given "as though S were the category of (constant) sets" (without of course using any of the non-constructive properties that it has). In the last section, where we consider category theory over *different* base toposes and the connections induced by a geometric morphism between the toposes, of necessity the use of indexed categories (or fibrations) is more overt.

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1. COCONTINUOUS FUNCTORS.

Let S be an elementary topos with natural number object. In this and the next section we are going to be working in "category theory over S". Within this context we need to recall some basic facts about categories with small colimits and functors preserving those colimits, which follow from the calculus of coends and left Kan extensions (cf. [9] and [10], Chapters IX § X). As noted in the Introduction, the development will be given as though S were the category of sets, and we make the following

Convention. In Sections 1 and 2, "category", "functor", "natural transformation", etc. will mean *S-indexed* category, functor, natural transformation, etc. Similarly "small" will mean *S-internal*. (See the references cited in the Introduction for an explanation of these concepts.)

Let COCTS_S denote the 2-category whose 0-cells are locally small categories with all small colimits, whose 1-cells are functors preserving these colimits and whose 2-cells are all natural transformations between such functors.

If $\mathbf{C}$ is a small category and $A \in \mathrm{COCTS}_S$, then the category of diagrams of type $\mathbf{C}$ in A, denoted $[\mathbf{C}, A]$, is again in COCTS_S and we have :

(1.1) Lemma. *For* $\mathbf{C}$ *a small category and* $A \in \mathrm{COCTS}_S$, $[\mathbf{C}^{op}, A]$ *(resp.* $[\mathbf{C}, A]$*) is the tensor (resp. the cotensor) of* A *by* $\mathbf{C}$ *in the 2-category* COCTS_S *, i.e. there is an equivalence*

$$\mathrm{COCTS}_S([\mathbf{C}^{op}, A], B) \simeq [\mathbf{C}, \mathrm{COCTS}_S(A, B)]$$

(resp. an isomorphism

$$\mathrm{COCTS}_S(B, [\mathbf{C}, A]) \simeq [\mathbf{C}, \mathrm{COCTS}_S(B, A)]$$

which is natural in B.

(To be precise, since the first of the above is an equivalence rather than an isomorphism, we mean "tensor" in the "up to isomorphism" sense appropriate to COCTS_S regarded as a bicategory ; cf [14].)

Proof. For $I \in S$ and $A \in A$, let $I.A \in A$ be the copower of A by I, i.e. the coproduct of I copies of A. Thus if $H : \mathbf{C} \hookrightarrow [\mathbf{C}^{op}, S]$ denotes the Yoneda embedding, then for $U \in \mathbf{C}$ and $A \in \mathcal{A}$ we have

$$H_U.A \in [\mathbf{C}^{op}, \mathcal{A}] \text{ sending } V \in \mathbf{C} \text{ to } H_U(V).A = \mathbf{C}(V, U).A.$$

Then given $F \in \mathrm{COCTS}_S([\mathbf{C}^{op}, \mathcal{A}], \mathcal{B})$, we get $\bar{F} \in [\mathbf{C}, \mathrm{COCTS}_S(\mathcal{A}, \mathcal{B})]$ by defining

$$\bar{F}U(A) = F(H_U.A) \qquad (U \in \mathbf{C},\ A \in \mathcal{A}).$$

Conversely given $G \in [\mathbf{C}, \mathrm{COCTS}_S(\mathcal{A}, \mathcal{B})]$, the coend

$$\bar{G}(X) = \int^U GU(XU) \qquad (X \in [\mathbf{C}^{op}, \mathcal{A}])$$

defines a functor $\bar{G} \in \mathrm{COCTS}_S([\mathbf{C}^{op}, \mathcal{A}], \mathcal{B})$.

The assignments $F \mapsto \bar{F}$ and $G \mapsto \bar{G}$ are functorial and define the required natural equivalence

$$\mathrm{COCTS}_S([\mathbf{C}^{op}, \mathcal{A}], \mathcal{B}) \simeq [\mathbf{C}, \mathrm{COCTS}_S(\mathcal{A}, \mathcal{B})].$$

The natural isomorphism

$$\mathrm{COCTS}_S(\mathcal{B}, [\mathbf{C}, \mathcal{A}]) \simeq [\mathbf{C}, \mathrm{COCTS}_S(\mathcal{B}, \mathcal{A})]$$

is simpy the restriction of that for the full functor categories

$$[\mathcal{B}, [\mathbf{C}, \mathcal{A}]] \simeq [\mathbf{C}, [\mathcal{B}, \mathcal{A}]] .$$ ◊

For $\mathcal{A} \in \mathrm{COCTS}_S$ and $A \in \mathcal{A}$, sending $I \in S$ to the copower $I.A$ of A by I gives a functor $(-).A : S \to \mathcal{A}$ which preserves colimits. Conversely any $F \in \mathrm{COCTS}_S(S, \mathcal{A})$ is determined by $F(1) \in \mathcal{A}$, since

$$F(I) \simeq F(I.1) \simeq I.F(1).$$

The assignments $A \mapsto (-).A$ and $F \mapsto F(1)$ are functorial and set up an equivalence of categories :

(1.2) Lemma. *There is an equivalence $\mathcal{A} \simeq \mathrm{COCTS}_S(S, \mathcal{A})$, which is natural in $\mathcal{A} \in \mathrm{COCTS}_S$.* ◊

Combining (1.1) and (1.2) we have :

(1.3) Corollary. *$[\mathbf{C}^{op}, S]$ is the free locally small cocomplete category on the small category $\mathbf{C}$ in the sense that restriction along the Yoneda embedding $H: \mathbf{C} \to [\mathbf{C}^{op}, S]$ induces an equivalence*

$$\mathrm{COCTS}_S([\mathbf{C}^{op}, S], \mathcal{A}) \simeq [\mathbf{C}, \mathcal{A}]$$

for any $\mathcal{A} \in \mathrm{COCTS}_S$. ◊

Given a functor $f : \mathbf{C} \to A$, the corresponding colimit-preserving functor $[\mathbf{C}^{op}, S] \to A$ is $\mathrm{colim}_H f$, the left extension of f along H :

$$(\mathrm{colim}_H f)(X) = \int^U XU.fU \qquad (X \in [\mathbf{C}^{op}, S]).$$

As usual, let $\Omega \in [\mathbf{C}^{op}, S]$ denote the subobject classifier of the topos of presheaves on the small category $\mathbf{C}$. Thus for $U \in \mathbf{C}$, $\Omega(U)$ is the set of *sieves* on U (i.e. the set of subobjects $R \rightarrowtail H_U$ in $[\mathbf{C}^{op}, S]$).

(1.4) Definition. Let P be any collection of sieves on $\mathbf{C}$. Say that a functor $f : \mathbf{C} \to A$ ($A \in \mathrm{COCTS}_S$) is *P-cocontinuous* if the corresponding colimit-preserving functor

$$\mathrm{colim}_H f : [\mathbf{C}^{op}, S] \longrightarrow A$$

sends each $R \rightarrowtail H$ in P to an isomorphism in A. Let P-cocts$(\mathbf{C}, A)$ denote the full subcategory of $[\mathbf{C}, A]$ whose objects are such functors.
Remark. It is not hard to see that $\mathrm{colim}_H f(R \rightarrowtail H)$ is an isomorphism in A iff, regarding R as a full subcategory of C/U, the diagram

$$R \hookrightarrow \mathbf{C}/U \longrightarrow \mathbf{C} \xrightarrow{f} A$$

(sending $\alpha : V \to U$ to $f(V)$) has

$$(f\alpha : fV \to fU \mid \alpha \in R)$$

as colimiting cone.

Now suppose that P is a subpresheaf of Ω in $[\mathbf{C}^{op}, S]$, i.e. that the collection P of sieves on $\mathbf{C}$ is closed under taking pullbacks. Let $\bar{P} \rightarrowtail \Omega$ denote the Grothendieck topology generated by $P \rightarrowtail \Omega$. So we have $\mathrm{Sh}_S(\mathbf{C}, P)$, the Grothendieck S-topos of sheaves on the site $(\mathbf{C}, P)$ and an associated sheaf functor

$$a : [\mathbf{C}^{op}, S] \longrightarrow \mathrm{Sh}_S(\mathbf{C}, \bar{P}),$$

which since it is left adjoint to the inclusion $\mathrm{Sh}_S(\mathbf{C}, P) \hookrightarrow [\mathbf{C}^{op}, S]$ is a morphism in COCTS_S for which the induced functor

$$a^* : \mathrm{COCTS}_S(\mathrm{Sh}_S(\mathbf{C}, \bar{P}), A) \longrightarrow \mathrm{COCTS}_S([\mathbf{C}^{op}, S], A)$$

is full and faithful (any $A \in \mathrm{COCTS}_S$).

We need the following mild generalization (from the case $\bar{P} = P$) of the kind of result to be found in [1], Exp. III.1 :

(1.5) Proposition. *The equivalence of Corollary (1.3) restricts along the*

the full and faithful functor a^* and along the inclusion

$$\text{P-cocts}(\mathbf{C}, A) \longrightarrow [\mathbf{C}, A]$$

to an equivalence

$$\mathrm{COCTS}_S(\mathrm{Sh}_S(\mathbf{C}, \bar{\mathrm{P}}), A) \simeq \text{P-cocts}(\mathbf{C}, A).$$

Proof. We must show that a functor $f : \mathbf{C} \to A$ is P-cocontinuous iff the corresponding morphism

$$F = \mathrm{colim}_H f : [\mathbf{C}^{op}, S] \longrightarrow A ,$$

in COCTS $_S$, factors up to isomorphism through a. Since $[\mathbf{C}^{op}, S]$ and $\mathrm{Sh}_S(\mathbf{C}, \mathrm{P})$ have small dense subcategories, colimit-preserving functors from them to $A \in \mathrm{COCTS}_S$ have right adjoints. In particular, the right adjoint of $F = \mathrm{colim}_H f : [\mathbf{C}^{op}, S] \to A$ is

$$\hat{F} : A \longrightarrow [\mathbf{C}^{op}, S]: \ A \mapsto \hat{F}A = A(f(-), A).$$

Consequently for any factorization of F through a in COCTS_S :

$$\begin{tikzcd}
{[\mathbf{C}^{op}, S]} \arrow[d, "a"'] \arrow[dr, "F"] \arrow[dr, phantom, "\simeq", shift right=3] & \\
\mathrm{Sh}_S(\mathbf{C}, \bar{\mathrm{P}}) \arrow[r, dashed] & A
\end{tikzcd}$$

there is a corresponding diagram of right adjoints :

$$\begin{tikzcd}
{[\mathbf{C}^{op}, S]} & \\
\mathrm{Sh}_S(\mathbf{C}, \bar{\mathrm{P}}) \arrow[u, hook] \arrow[ur, phantom, "\simeq", shift left=3] & A \arrow[l, dashed] \arrow[ul, "\hat{F}"']
\end{tikzcd}$$

It follows that F factors through a in COCTS_S iff for each $A \in A$, $\hat{F}A$ is a $\bar{\mathrm{P}}$-sheaf. But since P is a pullback stable collection of sieves, a presheaf is in $\mathrm{Sh}_S(\mathbf{C}, \bar{\mathrm{P}})$ just in case it satisfies the sheaf condition for sieves in P. (For a proof of this fact see for example [12], Theorem 24.) So F factors through a iff for all $A \in A$ and all $m : R \rightarrowtail H_U$ in P we have

$$[\mathbf{C}^{op}, S](H_U, \hat{F}A) \xrightarrow{m^*} [\mathbf{C}^{op}, S](R, \hat{F}A) \quad \text{is a bijection ;}$$

i.e. iff for all R, $(Fm)^* : A(FH_U, A) \to A(FR, A)$ is bijective, all $A \in A$; i.e. iff for all $R \rightarrowtail H_U$ in P, $F(R \rightarrowtail H_U)$ is an isomorphism in A ; i.e. iff F is P-cocontinuous. ◊

2. PRODUCT AND TENSOR PRODUCT.

For A, B, $C \in$ COCTS$_S$, call a functor $F : A \times B \to C$ a *bimorphism* if it preserves small colimits in each variable separately ; let BIM(A, B ; C) denote the category of such functors. Similarly for small categories **C**, D and collections P, Q of sieves in C and D respectively, call a functor $\mathbf{C} \times \mathrm{D} \to A$ P, Q-*cocontinuous* iff for all (U, V) $\in$ C $\times$ D

$$f(\mathrm{U}, -) \in \text{Q-cocts}(\mathbf{D}, A) \quad \text{and} \quad f(-, \mathrm{V}) \in \text{P-cocts}(\mathbf{C}, A) ;$$

let P,Q-cocts(**C**, D ; A) denote the full subcategory of $[\mathrm{C} \times \mathbf{D}, A]$ whose objects are such functors.

Note that since Q-cocts(**D**, A) is locally small and the inclusion Q-cocts(**D**, A) $\hookrightarrow$ [**D**, A] creates small colimits, we have that

$$\text{Q-cocts}(\mathbf{D}, A) \in \text{COCTS}_S$$

and the exponential adjointness

$$[\mathbf{C} \times \mathbf{D}, A] \simeq [\mathbf{C}, [\mathrm{D}, A]]$$

restricts to

$$\text{P,Q-cocts}(\mathbf{C}, \mathrm{D} ; A) \simeq \text{P-cocts}(\mathrm{C}, \text{Q-cocts}(\mathrm{D}, A)).$$

(2.1) Proposition. *Suppose that* P *and* Q *are pullback stable collections of sieves in* C *and* D *respectively ; let* $\bar{\mathrm{P}}$, $\bar{\mathrm{Q}}$ *denote the Grothendieck topologies generated by them and put* $E = \mathrm{Sh}_S(\mathrm{C}, \bar{\mathrm{P}})$, $F = \mathrm{Sh}_S(\mathrm{D}, \bar{\mathrm{Q}})$. *Then composition with*

$$\mathbf{C} \times \mathbf{D} \xrightarrow{\mathrm{H} \times \mathrm{H}} [\mathrm{C}, S] \times [\mathrm{D}, S] \xrightarrow{a \times a} \mathrm{Sh}_S(\mathbf{C}, \bar{\mathrm{P}}) \times \mathrm{Sh}_S(\mathrm{D}, \bar{\mathrm{Q}}) = E \times F$$

gives an equivalence

$$\mathrm{BIM}(E, F ; A) \simeq \text{P, Q-cocts}(\mathbf{C}, \mathbf{D} ; A)$$

(all $A \in$ COCTS$_S$).

Proof. Evidently the exponential *equivalence*

$$[E \times F ; A] \simeq [E, [F, A]]$$

(cf. [11], p. 35) restricts to one between BIM(E, F ; A) and the category of colimit-preserving functors from E to the category of colimit-preserving functors $F \to A$. So by repeatedly using Proposition (1.5) and by the Remark above, we have

$$\mathrm{BIM}(E, F ; A) \simeq \mathrm{COCTS}_S(E, \text{Q-cocts}(\mathbf{D}, A)) \simeq$$
$$\text{P-cocts}(\mathbf{C}, \text{Q-cocts}(\mathrm{D}, A)) \simeq \text{P,Q-cocts}(\mathbf{C}, \mathrm{D} ; A),$$

and this composite equivalence is induced by $(a \times a) \circ (H \times H)$. ◊

(2.2) Definition. The tensor product $A \otimes B$ of A and B in COCTS_S if it exists is the domain of the universal bimorphism from $A \times B$, i.e. there should be a bimorphism $\otimes : A \times B \to A \otimes_S B$, composition with which gives an equivalence

$$\mathrm{BIM}(A, B\ ;\ C) \simeq \mathrm{COCTS}_S(A \otimes_S B, C)$$

(all $C \in \mathrm{COCTS}_S$).

Without imposing further conditions on A and B, it is not clear that $A \otimes_S B$ always exists. (Recall that we require objects in COCTS_S not only to have small colimits, but also to be locally small.) However by (1.1) and (1.2) we have

$$\begin{gathered}\mathrm{BIM}(A, [\mathbf{C}, S]; C) \simeq \mathrm{COCTS}_S(A, [\mathbf{C}^{op}, C]) \simeq \\ [\mathbf{C}^{op}, \mathrm{COCTS}_S(A, C)] \simeq \mathrm{COCTS}_S([\mathbf{C}, A], C),\end{gathered}$$

so that $A \otimes_S [\mathbf{C}, S]$ is $[\mathbf{C}, A]$. In particular $[\mathbf{C}, S] \otimes_S [\mathbf{D}, S]$ is

$$[\mathbf{D}, [\mathbf{C}, S]] \simeq [\mathbf{C} \times \mathbf{D}, S].$$

The universal bimorphism

$$\otimes : [\mathbf{C}, S] \times [\mathbf{D}, S] \longrightarrow [\mathbf{C} \times \mathbf{D}, S]$$

sends $(X, Y) \in [\mathbf{C}, S] \times [\mathbf{D}, S]$ to $X \otimes Y \in [\mathbf{C} \times \mathbf{D}, S]$ where

$$(X \otimes Y)(U, V) = X(U) \times Y(V) \qquad (U \in \mathbf{C},\ V \in \mathbf{D}).$$

Note that

$$[\mathbf{C}, S] \otimes_S [\mathbf{D}, S] = [\mathbf{C} \times \mathbf{D}, S]$$

is also the *product* $[\mathbf{C}, S] \times_S [\mathbf{D}, S]$ of $[\mathbf{C}, S]$ and $[\mathbf{D}, S]$ in the 2-category GTOP_S of Grothendieck (= bounded) S-toposes and geometric morphisms (cf. [7], Cor. 4.36). More generally we have :

(2.3) Theorem. *For Grothendieck S-toposes E, F, the tensor product $E \otimes_S F$ of E and F regarded as objects in* COCTS_S *is given by their product $E \times_S F$ in* GTOP_S.

Proof. Suppose $E = \mathrm{Sh}_S(\mathbf{C}, J)$, $F = \mathrm{Sh}_S(\mathbf{D}, K)$ for small sites $(\mathbf{C}, J)$, $(\mathbf{D}, K)$. Then the product $E \times F$ can be constructed as the sheaf subtopos of the product $[(\mathbf{C} \times \mathbf{D})^{op}, S]$ of $[\mathbf{C}^{op}, S]$ and $[\mathbf{D}^{op}, S]$ given by the smallest Grothendieck topology on $\mathbf{C} \times \mathbf{D}$ that makes both

$$p_1^*(1 \xrightarrow{d} J) \qquad \text{and} \qquad p_2^*(1 \xrightarrow{d} K)$$

dense. Here

$$[\mathbf{C}^{op}, S] \xleftarrow{p_1} [(\mathbf{C} \times \mathbf{D})^{op}, S] \xrightarrow{p_2} [\mathbf{D}^{op}, S]$$

are the product projections (so that p_1^* is given by precomposition with the projection functor $(\mathbf{C} \times \mathbf{D})^{op} \to \mathbf{C}^{op}$ and similarly for p_2^*) and $d : 1 \rightarrowtail J$, $d : 1 \rightarrowtail K$ are the generic dense monomorphisms in $[\mathbf{C}^{op}, S]$ and $[\mathbf{D}^{op}, S]$.

The image of the map $p_1^* J \to \Omega$ classifying $p_1^*(d) : 1 \rightarrowtail p^*J$ in $[(\mathbf{C} \times \mathbf{D})^{op}, S]$ consists exactly of sieves of the form

$$R \otimes H = \{(\alpha, \beta) \mid \alpha \in R\} \quad (R \in J(U), \ (U, V) \in \mathbf{C} \times \mathbf{D}).$$

Similarly for $p_2^*(d) : 1 \rightarrowtail p^*K$. Thus with notation as in Section 1,

$$E \times_S F = \mathrm{Sh}_S(\mathbf{C} \times \mathbf{D}, \mathrm{P})$$

where P is the (pullback stable) collection

$$\mathrm{P} = \bigcup_{(U, V) \in \mathbf{C} \times \mathbf{D}} \{R \otimes H_V \mid R \in J(U)\} \cup \{H_U \otimes S \mid S \in K(V)\}$$

of sieves in $\mathbf{C} \times \mathbf{D}$. Hence by Proposition (1.5)

$$\mathrm{COCTS}_S(E \times_S F, A) \simeq \mathrm{P}\text{-cocts}(\mathbf{C} \times \mathbf{D}, A)$$

naturally in $A \in \mathrm{COCTS}_S$. The result will then follow by Proposition (2.1) if we can show that

$$\mathrm{P}\text{-cocts}(\mathbf{C} \times \mathbf{D}, A) = J, K\text{-cocts}(\mathbf{C}, \mathbf{D} ; A).$$

But given $f : \mathbf{C} \times \mathbf{D} \to A$, employing the "Fubini" property for iterated coends (cf. [10], IX.8) we have for any $V \in \mathbf{D}$ and $X \in [\mathbf{C}^{op}, S]$ that

$$(\mathrm{colim}_H f)(X \otimes H_V) = \int^{(U',V')} (X \otimes H_V)(U', V').\, f(U', V')$$

$$= \int^{(U',V')} (XU' \times H_V(V')).\, f(U', V') \simeq \int^{U'} XU'.\int^{V'} H_V(V').f(U', V')$$

$$\simeq \int^{U'} XU'.\, f(U', V) = (\mathrm{colim}_H f(-, V))(X),$$

naturally in X and V. Similarly for $U \in \mathbf{C}$

$$(\mathrm{colim}_H f)(H_U \otimes Y) \simeq (\mathrm{colim}_H f(U, -))(Y)$$

naturally in $Y \in [\mathbf{D}^{op}, S]$ and $U \in \mathbf{C}$. Thus $f \in \mathrm{P}\text{-cocts}(\mathbf{C} \times \mathbf{D}, S)$ iff for all $U \in \mathbf{C}$, $V \in \mathbf{D}$, $R \in J(U)$ and $S \in K(V)$

$$\mathrm{colim}_H f(R \otimes H_V \rightarrowtail H_U \otimes H_V = H_{(U,V)})$$

and

$$\mathrm{colim}_H f(H_U \otimes S \rightarrowtail H_U \otimes H_V = H_{(U,V)})$$

are isomorphisms in A, i.e.

$$\mathrm{colim}_H\, f(-, V)(R \rightarrowtail H_U) \quad \text{and} \quad \mathrm{colim}_H\, f(U, -)(S \rightarrowtail H_V)$$

are isomorphisms, i.e. iff $f \in J,K\text{-cocts}(\mathbf{C}, \mathbf{D}\,; A)$.

Remark. Letting

$$E \xleftarrow{\;p_1\;} E \times_S F \xrightarrow{\;p_2\;} F$$

be the product projections in GTOP_S, then the universal bimorphism

$$E \times F \longrightarrow E \otimes_S F = E \times_S F$$

simply sends $(\mathrm{E}, \mathrm{F}) \in E \times F$ to $(p_1^* \mathrm{E} \times p_2^* \mathrm{F}) \in E \times_S F$.

3. CHANGE OF BASE.

Suppose that $p : E \to S$ is a geometric morphism between elementary toposes (with natural number object). It induces a 2-functor

$$p_{\#} : \mathrm{COCTS}_E \longrightarrow \mathrm{COCTS}_S$$

where for $B \in \mathrm{COCTS}_E$, $p_{\#}B$ is the S-indexed category whose fibre over $\mathrm{I} \in S$ is

$$(p_{\#}B)^{\mathrm{I}} = B^{p^*\mathrm{I}}$$

and similarly for morphisms in S. By definition of diagram categories of the form $[\mathbf{C}, A]$, the above equality extends to

$$(3.1) \qquad [\mathbf{C}, p_{\#}B] = [p^*\mathbf{C}, B],$$

for any category $\mathbf{C}$ in S.

(3.2) Definition. For $A \in \mathrm{COCTS}_S$, $p^{\#}A \in \mathrm{COCTS}_E$ will denote the left adjoint of $p_{\#}$ at A (if it exists), in the sense that there is a morphism $\eta : A \to p_{\#}p^{\#}A$ in COCTS_S such that for each $B \in \mathrm{COCTS}_E$, the functor

$$(3.3) \qquad p_{\#}(-) \circ \eta : \mathrm{COCTS}_E(p^{\#}A, B) \longrightarrow \mathrm{COCTS}_S(A, p_{\#}B)$$

is an equivalence. As usual, the universal property ensures that $p^{\#}$ is a bicategory homomorphism where defined.

As in the case of the tensor product $\otimes_S$, whilst it is not clear that $p^{\#}$ is always defined, it is on the free objects of COCTS_S. Indeed by Corollary (1.3) and (3.1), we have equivalences

$$\mathrm{COCTS}_E([p^*\mathbf{C},\ E],B) \simeq [p^*\mathbf{C}^{op},B]=[\mathbf{C}^{op},\ p_\# B] \simeq \mathrm{COCTS}_S([\mathbf{C},\ S],p_\# B)$$

which are natural in $B \in \mathrm{COCTS}_S$. So we can take $p^\#[\mathbf{C},\ S]$ to be $[p^*\mathbf{C},\ E]$. Now recall ([7], Cor. 4.35) that

$$\begin{tikzcd} {[p^*\mathbf{C}^{op},\ E]} \arrow[r, "\bar{p}"] \arrow[d] \arrow[dr, phantom, "\simeq"] & {[\mathbf{C}^{op},\ S]} \arrow[d] \\ E \arrow[r, "p"] & S \end{tikzcd}$$

is a pullback square of toposes and geometric morphisms, where

$$\bar{p}^* : [\mathbf{C}^{op},\ S] \longrightarrow [p^*\mathbf{C}^{op},\ E]$$

is the functor p^* applied to presheaves (= discrete fibrations). Thus for the presheaf topos $[\mathbf{C}^{op},\ S]$, $p^\#[\mathbf{C}^{op},\ S]$ coincides with the pullback of $[\mathbf{C}^{op},\ S] \in \mathrm{GTOP}_S$ along p; indeed the unit

$$\eta : [\mathbf{C}^{op},\ S] \longrightarrow p_\# p^\#[\mathbf{C}^{op},\ S]$$

is given by $\bar{p}^*$ regarded as an S-indexed functor. We shall see that this holds for *all* Grothendieck S-toposes.

(3.4) Definition. Call a diagram of the form

$$\begin{tikzcd} A \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\varphi"] & B \arrow[r, "Q"] & C \end{tikzcd}$$

in COCTS_S a *coinverter diagram* if Q is universal with the property that $Q\varphi$ is an isomorphism, in the sense that for any $D \in \mathrm{COCTS}_S$,

$$Q^* : \mathrm{COCTS}_S(C,D) \longrightarrow \mathrm{COCTS}_S(B,\ D)$$

is full and faithful and has essential image those $H : B \to D$ for which $H\varphi$ is an isomorphism.

Note that for the above coinverter diagram, if $p^\# A$ and $p^\# B$ exist (then so do $p^\# F$, $p^\# G$, $p^\# \varphi$) and if

$$\begin{tikzcd} p^\# A \arrow[r, bend left, "p^\# F"] \arrow[r, bend right, "p^\# G"'] \arrow[r, phantom, "\Downarrow p^\#\varphi"] & p^\# B \arrow[r] & D \end{tikzcd}$$

is a coinverter diagram in COCTS_E, then we can take $p^\# C$ to be D.

To apply this when A and B are free on small categories, we need to examine more closely the morphisms between such free objects in COCTS_S. Up to equivalence these are just Bénabou's *profunctors* (or "distributeurs" or "modules" ; cf. [3] and [7], 2.4) ; for by (1.1) and (1.3) we have

$$\mathrm{COCTS}_S([\mathbf{C}^{op}, S], [\mathbf{D}^{op}, S]) \simeq [\mathbf{D}^{op}, \mathrm{COCTS}_S([\mathbf{C}^{op}, S], S)$$
$$[\mathbf{D}^{op}, [\mathbf{C}, S]] \simeq [\mathbf{D}^{op} \times \mathbf{C}, S].$$

We shall take a profunctor *from* $\mathbf{C}$ *to* $\mathbf{D}$ to be a diagram $f \in [\mathbf{D}^{op} \times \mathbf{C}, S]$ and denote it $f : \mathbf{C} \looparrowright \mathbf{D}$. Let Prof_S denote the bicategory of internal categories, profunctors and natural transformations in S: the composition of $f : \mathbf{C} \looparrowright \mathbf{D}$ and $g : \mathbf{D} \looparrowright \mathbf{E}$ in Prof_S is given by the usual coend formula :

$$(g \circ f)(U, W) = \int^V f(V, U) \times g(W, V). \qquad (U \in \mathbf{C}, \quad W \in \mathbf{E}).$$

Now denoting the presheaf category $[\mathbf{C}^{op}, S]$ by $\hat{\mathbf{C}}$, the assignment $\mathbf{C} \mapsto \hat{\mathbf{C}}$ extends to a homomorphism of bicategories

$$(\hat{\ }) : \mathrm{Prof}_S \longrightarrow \mathrm{COCTS}_S$$

which is full and faithful in the sense that

$$(\hat{\ }) : \mathrm{Prof}_S(\mathbf{C}, \mathbf{D}) \longrightarrow \mathrm{COCTS}_S(\hat{\mathbf{C}}, \hat{\mathbf{D}})$$

is an equivalence. Indeed this equivalence is given by (1.1) and (1.3) : for $f \in \mathrm{Prof}_S(\mathbf{C}, \mathbf{D}) = [\mathbf{D}^{op} \times \mathbf{C}, S]$, $\hat{f} : \hat{\mathbf{C}} \to \hat{\mathbf{D}}$ in COCTS_S is given by

$$\hat{f}X(V) = \int^U XU \times f(V, U) \qquad (X \in [\mathbf{C}^{op}, S]).$$

(Thus $(\hat{\ }) \simeq \mathrm{Prof}_S(\mathbf{1}, -)$ where $\mathbf{1}$ is the terminal category in S.)

Given a profunctor $f : \mathbf{C} \looparrowright \mathbf{D}$ in S, $F = \hat{f} : \hat{\mathbf{C}} \to \hat{\mathbf{D}}$ in COCTS_S induces

$$p^{\#} F : p^{\#}\hat{\mathbf{C}} = (p^*\mathbf{C})\hat{\ } \longrightarrow (p^*\mathbf{D})\hat{\ } = p^{\#}\hat{\mathbf{D}},$$

and this corresponds to a profunctor $p^*\mathbf{C} \looparrowright p^*\mathbf{D}$ in E. Tracing through the equivalence (3.3), one finds that this profunctor is just

$$\bar{p}^* : [\mathbf{D}^{op} \times \mathbf{C}, S] \longrightarrow [p^*(\mathbf{D}^{op} \times \mathbf{C}), E] \simeq [p^*\mathbf{D}^{op} \times p^*\mathbf{C}, E]$$

applied to f. Thus

$$\begin{tikzcd}
\mathrm{Prof}_S(\mathbf{C}, \mathbf{D}) \arrow[r, "(\hat{\ })", "\simeq"'] \arrow[d, "\bar{p}^*"'] \arrow[dr, phantom, "\simeq"] & \mathrm{COCTS}_S(\hat{\mathbf{C}}, \hat{\mathbf{D}}) \arrow[d, "p^{\#}"] \\
\mathrm{Prof}_E(p^*\mathbf{C}, p^*\mathbf{D}) \arrow[r, "(\hat{\ })", "\simeq"'] & \mathrm{COCTS}_S(p^{\#}\hat{\mathbf{C}}, p^{\#}\hat{\mathbf{D}})
\end{tikzcd}$$

commutes up to isomorphism.

(3.5) Proposition. *Suppose $X \in [\mathbf{C}^{op}, S]$: thus X is given by a discrete fibration over $\mathbf{C}$ in S, which we denote by $\pi : \mathbb{X} \to \mathbf{C}$. Let $\pi_\bullet$ denote the profunctor $\mathbb{X} \dashrightarrow \mathbf{C}$ induced by π viz*

$$\pi_\bullet(U, x) = \mathbf{C}(U, \pi x) \qquad (U \in \mathbf{C}, \ x \in \mathbb{X}).$$

(i) Subobjects $A \rightarrowtail X$ in $[\mathbf{C}^{op}, S]$ correspond to subobjects $\dot{D} \rightarrowtail \pi_\bullet$ in $\mathrm{Prof}_S(\mathbb{X}, \mathbf{C})$ which are strict in the sense that for each $f : x \to y$ in $\mathbb{X}$

$$\begin{tikzcd} D(-, x) \arrow[r, tail] \arrow[d, "D(-, f)"'] & \pi_\bullet(-, x) = H_{\pi x} \arrow[d, "H_{\pi f}"] \\ D(-, y) \arrow[r, tail] & \pi_\bullet(-, y) = H_{\pi y} \end{tikzcd}$$

is a pullback square in $[\mathbf{C}^{op}, S]$. Moreover this correspondence is preserved under change of base : if $m : A \rightarrowtail X$ corresponds to $i : D \rightarrowtail \pi_\bullet$ then $p^*m : p^*A \rightarrowtail p^*X$ corresponds to

$$\bar{p}^*(i) : \bar{p}^*D \rightarrowtail \bar{p}^*(\pi_\bullet) = (\bar{p}^*\pi)_\bullet.$$

(ii) If $D \rightarrowtail \pi_\bullet$ corresponds to $A \rightarrowtail X$ as in (i), then

$$\begin{tikzcd} {[\mathbb{X}^{op}, S]} \arrow[r, bend left, "\hat{D}"] \arrow[r, bend right, "\hat{\pi}_\bullet"'] \arrow[r, phantom, "\Downarrow\hat{i}"] & {[\mathbf{C}^{op}, S]} \arrow[r, "a"] & \mathrm{Sh}_S(\mathbf{C}, J) \end{tikzcd}$$

is a coinverter diagram in COCTS_S, where J is the least topology on $\mathbf{C}$ making $A \rightarrowtail X$ dense, and a is the associated sheaf functor. (So in particular every Grothendieck S-topos has a "free presentation" in COCTS_S by a coinverter of the above form.)

Proof. (i) Given $A \rightarrowtail X$, define $D \rightarrowtail \pi_\bullet$ by letting, for each $x \in \mathbb{X}$,

$$D(-, x) \rightarrowtail \pi_\bullet(-, x) = \mathbf{C}(-, \pi x) = H_{\pi x}$$

be given by the pullback

$$\begin{tikzcd} D(-, x) \arrow[r, tail] \arrow[d] & H_{\pi x} \arrow[d, "\ulcorner x \urcorner"] \\ A \arrow[r, tail] & X \end{tikzcd}$$

in $[\mathbf{C}^{op}, S]$ (where $\ulcorner x \urcorner$ corresponds to x under the Yoneda isomorphism

$$[\mathbf{C}^{op}, S](H_{\pi x}, X) \simeq X(\pi x)).$$

Evidently $D(-, x) \rightarrowtail H_{\pi x}$ is natural in x and gives a strict monomorphism $D \rightarrowtail \pi_\bullet$.

Conversely given a strict monomorphism $D \rightarrowtail \pi_\bullet$, for each $x \in X$, let $\chi_x : H_{\pi x} \rightarrowtail \Omega$ classify $D(-, x) \rightarrowtail \pi_\bullet(-, x) = H_{\pi x}$:

$$\begin{tikzcd} D(-, x) \arrow[r, tail] \arrow[d] & H_{\pi x} \arrow[d, "\chi_x"] \\ 1 \arrow[r, tail, "\top"] & \Omega . \end{tikzcd}$$

Since $D \rightarrowtail \pi_\bullet$ is strict, the maps $(\chi_x \mid x \in X)$ form a cone under

$$X \xrightarrow{\ \pi\ } \mathbf{C} \xhookrightarrow{\ H\ } [\mathbf{C}^{op}, S].$$

But the colimit of this diagram is just $X \in [\mathbf{C}^{op}, S]$; so on taking colimits over X, the above pullback squares yield another pullback square

$$\begin{tikzcd} \operatorname{colim}_X D(-, x) \arrow[r, tail] \arrow[d] & X \arrow[d, "\chi"] \\ 1 \arrow[r, tail, "\top"] & \Omega \end{tikzcd}$$

and hence $A = \operatorname{colim}_X D(-, x) \rightarrowtail X$ is a monomorphism and determines a subobject of X.

That the above two constructions are mutually inverse is immediate from their definitions and the usual properties of finite limits and (internal) colimits in a topos. Moreover p^* perserves such limits and colimits : so if $i : D \rightarrowtail \pi_\bullet$ is a strict monomorphism in $\mathrm{Prof}_S(X, \mathbf{C})$, then

$$\bar{p}^* i \ : \bar{p}^* D \rightarrowtail \bar{p}^* \pi_\bullet \simeq (\bar{p}^* \pi)_\bullet$$

is again one in $\mathrm{Prof}_E(p^*X, p^*\mathbf{C})$ and

$$\begin{tikzcd}[column sep=small] p^*(\operatorname{colim}_X D) \arrow[r, phantom, "\simeq"] \arrow[d] & \operatorname{colim}_{p^*X}(\bar{p}^* D) \arrow[d] \\ p^*(\operatorname{colim}_X \pi_\bullet) \arrow[r, phantom, "\simeq"] \arrow[dr, phantom, "\simeq"] & \operatorname{colim}_{p^*X}(\bar{p}^* \pi)_\bullet \arrow[d, phantom, "\simeq"] \\ & p^*X \end{tikzcd}$$

commutes. The last part of (i) follows from this.

(ii) Let $\chi : X \to \Omega$ classify $A \rightarrowtail X$ in $[\mathbf{C}^{op}, S]$, and let $P \rightarrowtail \Omega$ be the image of χ. Thus P consists precisely of the sieves $D(-, x) \rightarrowtail H_{\pi x}$ $(x \in X)$ where $D \rightarrowtail \pi_*$ is defined from $A \rightarrowtail X$ as in (i). So given

$$F : [\mathbf{C}^{op}, S] \longrightarrow A$$

in COCTS_S, $F\hat{\imath}$ is an isomorphism in

$$\mathrm{COCTS}_S([X^{op}, S], A) \simeq [X, A]$$

iff $F\hat{\imath}_H$ is an isomorphism in $[X, A]$, i.e. iff for all $x \in X$,

$$F(D(-, x) \rightarrowtail H_{\pi x})$$

is an isomorphism in A, i.e. iff $F \circ H : \mathbf{C} \to A$ is P-cocontinuous. The result now follows from proposition (1.5) and the fact ([7], 3.59 (ii)) that $\overline{P}$, the least Grothendieck topology containing P, is indeed the least topology making $A \rightarrowtail X$ dense. ◊

(3.6) Theorem. *Let $p : E \to S$ be a geometric morphism between elementary toposes with natural number objects. Then for any bounded S-topos $F \to S$, regarding F as an object of* COCTS_S, *$p^{\#}F$ exists and coincides with the pullback (in the bicategory of toposes and geometric morphisms) of $F \to S$ along p.*

Proof. Suppose $F = \mathrm{Sh}_S(\mathbf{C}, J)$, $(\mathbf{C}, J)$ a site in S. To the generic J-dense monomorphism $d : 1 \rightarrowtail J$ in $[\mathbf{C}^{op}, S]$ there corresponds by (3.5) (i) a strict monomorphism $i : D \rightarrowtail \pi_*$ in $\mathrm{Prof}_S(J, \mathbf{C})$; and by (3.5) (ii)

$$\begin{tikzcd}
{[J^{op}, S]} \arrow[r, bend left, "\hat{D}"] \arrow[r, bend right, "\hat{\pi}_*"'] \arrow[r, phantom, "\hat{\imath}\Downarrow"] & {[\mathbf{C}^{op}, S]} \arrow[r, "a"] & F
\end{tikzcd}$$

is a coinverter diagram in COCTS_S. Applying $p^{\#}$ to $\hat{\imath} : \hat{D} \rightarrowtail \hat{\pi}_*$ we get

$$\begin{tikzcd}
{[p^*J^{op}, E]} \arrow[rr, bend left, "(\bar{p}^* D)\hat{}"] \arrow[rr, bend right, "(\bar{p}^*\pi_*)\hat{}"'] \arrow[rr, phantom, "\Downarrow (\bar{p}^*i)\hat{}"] & & {[p^*\mathbf{C}, E]}
\end{tikzcd}$$

and since by (3.5) (i), $\bar{p}^*i : \bar{p}^*D \rightarrowtail \bar{p}^*\pi_*$ is the strict monomorphism corresponding to $p^*(d) : 1 \rightarrowtail p^*J$ in $[p^*\mathbf{C}^{op}, E]$, the above diagram has

$$[p^*\mathbf{C}, E] \xrightarrow{\ a\ } \mathrm{Sh}_E(p^*\mathbf{C}, K)$$

as its coinverter, where K is the least Grothendieck topology making $p^*(d) : 1 \to p^*J$ dense. By the remarks after definition (3.4), it follows that we can take $p^{\#}F$ to be $Sh_E(p^*\mathbf{C}, K)$. But K is precisely the "pull-back topology" (cf. [7], 3.59 (iii)), i.e.

$$\begin{tikzcd} Sh_E(p^*\mathbf{C}, K) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\simeq"] & Sh_S(\mathbf{C}, J) \arrow[d, hook] \\ {[p^*\mathbf{C}^{op}, E]} \arrow[r, "\bar{p}"] \arrow[d] \arrow[dr, phantom, "\simeq"] & {[\mathbf{C}^{op}, S]} \arrow[d] \\ E \arrow[r, "p"] & S \end{tikzcd}$$

are pullback squares of toposes and geometric morphisms. ◊

(3.7) Remark. If $F \in GTOP_S$ and

$$\begin{tikzcd} p^{\#}F \arrow[r, "q"] \arrow[d] \arrow[dr, phantom, "\simeq"] & F \arrow[d] \\ E \arrow[r, "p"] & S \end{tikzcd}$$

is a pullback square, then the unit $\eta : F \to p_{\#}p^{\#}F$ is given by q^* as an S-indexed functor.

By an *S-essential* morphism in $GTOP_S$ we mean one $f : G \to F$ for which $f^*: F \longrightarrow G$ has an S-indexed left adjoint, denoted $f_! : G \to F$. Thus for $F \in GTOP_S$, $F \longrightarrow S$ is S-essential precisely when F is a *molecular* Grothendieck S-topos (or : when $F \to S$ is bounded and *locally connected*) ; cf. [2]. Tierney (unpublished) proved that the property of being molecular is stable under pullback along a geometric morphism and that the resulting pullback square

$$\begin{tikzcd} G \arrow[r, "q"] \arrow[d, "g"'] \arrow[dr, phantom, "\simeq"] & F \arrow[d, "f"] \\ E \arrow[r, "p"] & S \end{tikzcd}$$

satisfies a *Beck-Chevalley condition* for objects, i.e. the canonical map $f^*p_* \to q_*g^*$ is an isomorphism, or equivalently, the canonical map $g_!q^* \to p^*f_!$ is an isomorphism. (See [12], V, for a proof of the symmetric case : p bounded, f locally connected.) More generally, we have :

(3.8) Corollary. *S-essential morphisms in* $GTOP_S$ *are stable under change* *of base, i.e. given $f : G \to F$ in GTOP_S, forming pullback squares :*

$$\begin{tikzcd}
p^{\#}G \arrow[r, "r"] \arrow[d, "g"'] \arrow[dr, phantom, "\simeq"] & G \arrow[d, "f"] \\
p^{\#}F \arrow[r, "q"] \arrow[d] \arrow[dr, phantom, "\simeq"] & F \arrow[d] \\
E \arrow[r, "p"] & S
\end{tikzcd}$$

we have : if f is S-essential, then g is E-essential. In this case the canonical map $g_! r^* \to q^* f_!$ is an isomorphism. Moreover if f is also connected (which happens iff f^* is full and faithful, iff the unit of $f_! \dashv f^*$ is an isomorphism), then so is g.

Proof. $p^{\#}$ is a homomorphism of bicategories where defined. So

$f_! \dashv f^*$ in COCTS_S (with counit an isomorphism if f connected)

gives

$p^{\#}(f_!) \dashv p^{\#}(f^*)$ in COCTS_E (" " " " " " " " " ")

In particular, $p^{\#}(f^*)$ is the inverse image of an E-essential geometric morphism $g : p^{\#}G \to p^{\#}F$ over E, which is connected if f is. Furthermore since the unit of $p^{\#} \dashv p_{\#}$ is natural, in the fibre over $1 \in S$ we have that

$$\begin{tikzcd}
p^{\#}G \arrow[r, "\eta"] \arrow[d, "p^{\#}(f^*)"'] \arrow[dr, phantom, "\simeq"] & G \arrow[d, "f^*"] \\
p^{\#}F \arrow[r, "\eta"] & F
\end{tikzcd}$$

commutes up to (canonical) isomorphism, i.e. by (3.7) that

$$\begin{tikzcd}
p^{\#}G \arrow[r, "r"] \arrow[d, "g"'] \arrow[dr, phantom, "\simeq"] & G \arrow[d, "f"] \\
p^{\#}F \arrow[r, "q"] & F
\end{tikzcd}$$

commutes up to (canonical) isomorphism. Thus g is indeed the (essentially unique) factorization of the pullback square

$$\begin{tikzcd}
p^{\#}G \arrow[r, "r"] \arrow[d] \arrow[dr, phantom, "\simeq"] & G \arrow[d] \\
E \arrow[r, "p"] & S
\end{tikzcd}$$

through the pullback square

$$\begin{tikzcd} p^{\#}F \arrow[r, "q"] \arrow[d] \arrow[dr, phantom, "\simeq"] & F \arrow[d] \\ E \arrow[r, "p"] & S \end{tikzcd}$$

Applying the naturality of η to $f_! : G \to F$ in COCTS_S, we have (in the fibre over $1 \in S$) that

$$\begin{tikzcd} p^{\#}G \arrow[d, "g_! = p^{\#}(f_!)"'] \arrow[dr, phantom, "\simeq"] & G \arrow[l, "\eta"'] \arrow[d, "f_!"] \\ p^{\#}G & F \arrow[l, "\eta"'] \end{tikzcd}$$

commutes up to (canonical) isomorphism, i.e. (by (3.7)) $g_! r^* \simeq q^* f_!$, as required. ◊

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