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A CONVENIENT SETTING FOR DIFFERENTIAL GEOMETRY AND GLOBAL ANALYSIS

by Peter MICHOR

ABSTRACT

A theory of smooth manifolds and vector bundles, where smooth curves take the place of charts and atlases, which is cartesian closed, is developped. In the finite dimensional case the manifolds turn out to be the usual ones.

CONTENTS

Sections 5-8 will be published in Volume XXV-2.

INTRODUCTION

This paper contains a theory of smooth manifolds and vector bundles, which coincides with the existing theories in the finite dimensional case. The whole theory aims at cartesian closedness from the beginning, so $S(M, N)$, the space of smooth mappings from a manifold M to a manifold N is again a manifold and the equation

$$S(M \times N, P) \approx S(M, S(N, P))$$

holds in general.

The general ideas are the following ones :

1. We forget about charts and atlases. There are at least two reasons for this : In Michor [1, 11.9] it is shown that the natural chart construction on spaces of smooth mappings does not allow cartesian closedness in general. The (topological) theory of manifolds modelled on Frechet spaces shows that these are open subsets of the modelling spaces in the most important cases, so they are rather simple objects.

2. We take the structure of smooth curves in a manifold as the basic notion, instead of charts. Another possible choice would be the structure of smooth real valued functions, which has been investigated via sheaf theory, schemes, etc, or a combination of both as Frölicher [2] proposes. The smooth curves alone are a «thin» structure, so we need a lot of other data as well : tangent spaces, differential operators for curves.

3. In view of 2, for vector bundles we do not require local triviality over open sets, but only triviality along smooth curves. The trivialisation we require to have some structure, they should be parallel transports along any smooth curve, depending smoothly on the curve too.

4. Lastly we require a geodesic structure on each manifold. This is a section for the differential operator for smooth curves in particular.

Our aim has been to construct a class of manifolds as small as possible such that we get cartesian closedness and get the usual theory in finite dimensions.

So a manifold M is a set of data (M1)-(M8) as follows :

(M1) Two sets M, TM, and a mapping $\pi_M : TM \to M$ such that each fibre is a locally convex space of a certain type (described in 1).

(M2) A set $S(\mathbb{R}, M)$ of curves in M, closed under C^∞-reparametrizations and containing all constants.

(M3) For each $t \in \mathbb{R}$ a mapping $\delta_t : S(\mathbb{R}, M) \to TM$ such that

$$\pi_M \circ \delta_t = ev_t \ , \ \delta_t(c \circ f) = f'(t).\delta_{f(t)}\, c,$$
$$c = constant \ \text{ if } \ \delta_t\, c = 0 \ \text{ for all } t.$$

(M4) A mapping $Pt^{TM} = Pt : S(\mathbb{R}, M) \times \mathbb{R} \to L(TM, TM)$ such that: $Pt(c, t) : T_{c(0)}M \to T_{c(t)}M$ is linear and continuous,

$$Pt(c, 0) = Id, \ \ Pt(c, f(t)) = Pt(c \circ f, t).Pt(c, f(0)).$$

(M5) $t \mapsto Pt(c, t)^{-1}.(\delta_t c)$ is a C^∞-curve in the l.c.s. $T_{c(0)}M$.

(M6) A mapping $Geo^M = Geo : TM \to S(\mathbb{R}, M)$ such that

$$Geo(t.v)(s) = Geo(v)(ts), \ \ \delta_t(Geo\, v) \ = Pt(Geo(v), t).v,$$
$$Geo(\delta_t Geo(v))(s) = Geo(v)(t+s).$$

A set of data like this is called a *premanifold*. We can show that TM is again a premanifold, so we have the whole tower of iterated tangent bundles and use them to define smooth mappings between premanifolds: they should map smooth curves to smooth curves and with a differentiation factor over to a tangent mapping, which should satisfy the same conditions, etc. We have to develop a lot of theory before we can formulate the next conditions:

(M7) $Pt : S(\mathbb{R}, M) \times \mathbb{R} \to L(TM, TM)$ is smooth.

(M8) $Geo : TM \to S(\mathbb{R}, M)$ is smooth.

The category of these objects (manifolds) and smooth mappings turns out to be cartesian closed (7.14). In 8.4 it is shown that the differentiable structure of a manifold does not change if we change the parallel transport to another one which is smooth and has a connection.

A manifold is called regular if the smooth real valued functions separate points (in a stronger sense, 8.7) on it. Regular manifolds with finite dimensional fibres for the tangent bundle turn out to be usual finite dimensional C^∞-manifolds (with charts), and conversely.

The theory developped here gives a cartesian closed (convenient) category of manifolds containing all finite dimensional ones and some of the usual infinite dimensional ones (e. g. Hilbert manifolds); and all manifolds in there have a lot of geometric structure (parallel transport, covariant derivative, geodesics, connections). By cartesian closedness it seems to be a good setting for variational calculus. Some of its drawbacks are: no chance for an Implicit Function Theorem. Not a good setting for infinite dimensional Lie groups (the general linear group of a locally convex space is not a smooth group in general). But the theory of principal fibre bundles might work, where the (smooth) monoid of all continuous endomorphisms takes the rôle of the group of isomorphisms. We do not go into this here. We also leave out the de Rham cohomology of manifolds and curvature.

In comparison with Synthetic Differential Geometry (see Kock, e. g.) there are no infinitesimal manifolds and we do not have a topos (no subobject classifier). On the other hand our manifolds are sets with structure mappings on them and not sheafs on categories of C^∞-algebras.

Let us now give a short description of the contents of all sections: 1 is an exposition of Kriegl [2, 3], of a convenient setting for differential calculus on locally convex spaces. The results later depend heavily on its special features. Most of the theory later on would remain valid if we take the only other cartesian closed setting for calculus in the literature, Seip [1]. The whole content of 1 is due to Kriegl.

2 defines premanifolds and pre-vector bundles and shows that the total space of a pre-vector bundle is a premanifold again. Using this, in 3 we can define smooth mappings between premanifolds and we show (3.5, 3.6) that the smooth mappings $\mathbb{R} \to M$ are exactly those in $S(\mathbb{R}, M)$ (with a surprisingly difficult proof).

In 4 we show that certain structure mappings (like π_M) are smooth and treat pullbacks of pre-vector bundles. In 5, the main result is that smooth sections of a pre-vector bundle form a convenient l.c.s. in the sense of 1, which is needed later to show that $S(M, N)$ is again a premanifold. In the course of the proof, we need the covariant derivative, so it is constructed and investigated before.

6 leaves the realm of premanifolds and gives a sort of differentiable

structure on $S(M,N)$ and the minimum of lemmas and concepts necessary for 7 where we introduce manifolds and vector bundles and prove cartesian closedness. The most difficult part of this is the construction of the flip mapping $\kappa_M : T^2 M \to T^2 M$ in 7.7.

8 completes the whole set up and shows the relations to the usual notions of manifolds.

Some remarks to the history of the ideas represented here : The use of smooth curves instead of charts is due to Seip [2] who treats subsets of sequentially complete l.c.s. and emplois a sort of weak geodesic structure to define manifolds and get cartesian closedness. In 1979-81 Kriegl and the author worked through Seip's paper and discussed the ideas of using parallel transports, geodesic structure, and the C^∞-curve final topology. In his dissertation Kriegl [1] improved Seip's setting with these ideas, treating subsets of locally convex spaces. A revised version of Kriegl [1] is to appear in Springer Lecture Notes. This paper contains the (one ?) embedding free approach which succeeded only after Kriegl [2, 3] developped the convenient setting for calculus as basis for it.

The main parts of this paper were presented in a lecture course in 1981/82 in Vienna. I want to thank the audience of this course, Mr. G. Kainz and A. Kriegl for the very stimulating cooperation and lots of discussion.

1. KRIEGL'S CONVENIENT SETTING FOR DIFFERENTIAL CALCULUS ON LOCALLY CONVEX SPACES.

In this chapter we give a somewhat streamlined account of the setting for differential calculus developped by Kriegl [2, 3]. We leave out all counterexamples and we only comment on the connections to existing settings like Keller. For the missing proofs, we refer to Kriegl.

1.1. *Bornological* locally convex vector spaces.

Let E be a real locally convex vector space (lcs). Let B be an absolutely convex bounded set in E. Then by E_B we mean the linear span of B in E, equipped with the Minkowski functional p_B of B as norm, i. e.

$$p_B(x) = \inf\{\lambda > 0 \mid x \in \lambda . B\}.$$

This is a normed space.

Recall that bE, the *bornologicalization of* E, is given as the locally convex limit of all the spaces E_B, where $E_B \to E_{B'}$ is a contraction if $B \subset B'$:

$$bE = \varinjlim\{E_B \mid B \subset E\}.$$

Clearly b is a functor from the category *lcs* of locally convex spaces and linear continuous maps into the full subcategory *blcs* of bornological lcs. (In fact *blcs* is monoreflexive in *lcs* in the sense of Herrlich - Strecker.)

1.2. LEMMA. *Let* (x_n) *be a sequence in a locally convex space* E. *Then the following properties are equivalent:*

1. There is some B *in* E *with* $x_n \to x$ *in* E_B *(i. e.* $p_B(x_n - x) \to 0$*).*

2. There is a sequence (μ_n) *in* $\mathbb{R}$, $\mu_n \to \infty$, *such that*

$$\{\mu_n(x_n - x) \mid n \in \mathbb{N}\}$$

is bounded in E.

3. There is a strictly increasing sequence (η_n) *in* $\mathbb{R}$, $\eta_n > 0$, $\eta_n \to \infty$, *such that* $\{\eta_n(x_n - x)\}$ *is bounded in* E.

DEFINITION. A sequence satisfying these equivalent conditions is called *Mackey convergent to* x. If we want to emphasize the particular sequence (η_n) in 3, we call (x_n) *η-falling to* x. If x is not relevant, we call (x_n) *a Mackey sequence, or η-falling.*

1.3. LEMMA. *Let* $x_n \to x$ *in* E, *let* (t_n) *be a sequence in* $\mathbb{R}$ *with* $t_n \downarrow 0$ *strictly such that*

$$\{(x_n - x_{n+1})/(t_n - t_{n+1}) \mid n \in \mathbb{N}\}$$

is bounded for all k. *Then there is a* C^∞*-curve* $c : \mathbb{R} \to E$ *with* $c(t_n) = x_n$, $c(0) = x$, *such that* c' *is* ∞*-flat at each* t_n *and at* 0.

c' is ∞-flat at r means: the infinite Taylor development of c' about r is the zero series. A mapping $f : \mathbb{R}^m \to E$ is called C^∞ iff all partial derivatives exist and are continuous - this is a concept without problems.

For the proof, let $\phi : \mathbb{R} \to \mathbb{R}$ be a C^∞-mapping, $\phi = 0$ locally about 0 and $\phi = 1$ locally about 1, $0 \leq \phi \leq 1$ elsewhere. Then put

$$c(t) = x \text{ for } t \leq 0, \quad c(t) = x_0 \text{ for } t_0 \leq t,$$
$$c(t) = \phi((t - t_{n+1})/(t_n - t_{n+1})).(x_n - x_{n+1}) + x_{n+1}$$

for $t_{n+1} \leq t \leq t_n$.

1.4. COROLLARY. *If $q > 1$ and (x_n) is q^{n^2}-falling to x, then there is a C^∞-curve c with $c(q^{-n}) = x_n$ and $c(0) = x$.*

1.5. DEFINITION. Let $c^\infty E$ denote the lcs E equipped with the final topology with respect to all C^∞-curves $\mathbb{R} \to E$.

1.6. A curve $c : \mathbb{R} \to E$ is said to be a *Lipschitz curve* if the set

$$\left\{ \frac{c(t) - c(s)}{t - s} \;\middle|\; t \neq s \right\}$$

is bounded in E. Let $\mathbb{N}_\infty$ denote the one-point compactification of $\mathbb{N}$. With these notions, we have:

LEMMA. *The final topologies with respect to the following sets of mappings into E coincide:*

$C^\infty(\mathbb{R}, E)$, Lipschitz curves, Mackey sequences (considered as mappings $\mathbb{N}_\infty \to E$), η-falling sequences (for any fixed η),

$$\{ E_B \hookrightarrow E,\ B \text{ bounded absolutely convex in } E \}.$$

So, in particular, $c^\infty E$ is the topological direct limit of all the spaces E_B.

The proof consists of showing that the adherence of a set A in E

$$\bigcup_f f(\text{closure of } f^{-1}(A))$$

is the same for all these mapping classes.

1.7. A circled set U (i.e. $x \in U$ implies $[-1, 1].x \subset U$) in E is called *bornivorous* if U absorbs each bounded set (i.e. each $B \subset \lambda.U$ for some λ).

LEMMA. *Let U in E be circled. Then the following properties are equivalent:*

1. U is bornivorous.

2. *For all* B *(as in 1.1)* $U \subset E_B$ *is a zero-neighbborhood in* E_B.
3. U *absorbs each compact set in* E.
4. U *absorbs Mackey sequences.*
5. U *absorbs* η*-falling sequences (any fixed* η*).*
6. U *absorbs* $c([-1, 1])$ *for all Lipschitz curves* c.
7. U *absorbs* $c([1\cdot, 1])$ *for all* C^∞*-curves* c.

1.8. COROLLARY. *Let* $f: E^k \to F$ *be a k-linear mapping between lcs. Then the following properties of* f *are equivalent:*

1. f *is bounded (i. e. maps bounded sets to bounded sets).*
2. *For all* B *in* E *the mapping* $E_B{}^k \to E^k \xrightarrow{f} F$ *is continuous.*
3. f *maps compact sets to bounded ones.*
4. f *maps Mackey sequences to bounded sets.*
5. f *maps* η*-falling sequences to bounded sets.*
6. f *maps compact pieces of Lipschitz curves to bounded sets.*
7. f *maps compact pieces of* C^∞*-curves to bounded sets.*
8. f *maps Mackey sequences to Mackey sequences.*
9. f *maps* η*-falling sequences to* η*-falling sequences.*
10. f *maps Lipschitz curves to local Lipschitz curves.*
11. f *maps* C^∞*-curves to* C^∞*-curves.*

1.9. COROLLARY. *The bornologicalization* bE *bears the finest locally convex topology with one (hence all) of the following equivalent properties:*

1. *It has the same bounded sets as* E.
2. *It has the same Mackey sequences as* E.
3. *It has the same* η*-falling sequences as* E.
4. *It has the same Lipschitz curves as* E.
5. *It has the same* C^∞*-curves as* E.
6. *It has the same bounded linear mappings into arbitrary lcs.*
7. *It has the same continuous linear mappings from normed spaces to* E.

1.10. THEOREM. *The category blcs of bornological lcs and continuous linear mappings is a symmetric monoidal closed category with unit* $\mathbb{R}$, *i. e.* $L(E, F)$ *with a bornological topology described below satisfies:*

$$L(E \otimes F, G) \approx L(E, L(F, G)),$$

$$E \otimes F = F \otimes E, \quad (E \otimes F) \otimes G = E \otimes (F \otimes G), \quad E \otimes \mathbb{R} = E,$$

where $E \otimes F$ *is the tensor product, suitably topologized.*

$L(E, F)$ is the space of all continuous (= bounded) linear mappings from E into F, equipped with the bornologicalization of the topology of uniform convergence on compact pieces of C^∞-curves. On $E \otimes F$ we put the following topology: consider C^∞-curves $c_1 : \mathbb{R} \to E$, $c_2 : \mathbb{R} \to F$; this gives a curve $\mathbb{R} \to E \otimes F$. Each absolutely convex set in $E \otimes F$ absorbing compact pieces of such curves is then a zero neighborhood. This gives a bornological space, and all the properties hold.

1.11. DEFINITION. A sequence (x_n) in E is called a *Mackey Cauchy sequence* if there is some bounded set $B \subset E$ such that (x_n) is a Cauchy sequence in the normed space E_B.

LEMMA. *Let* (x_n) *be a sequence in a lcs* E. *Then the following properties are equivalent:*

1. (x_n) *is a Mackey Cauchy sequence.*

2. *There is a double sequence* (t_{mn}) *in* $\mathbb{R}$, $t_{mn} \neq 0$, $t_{mn} \to 0$, *such that* $(x_m - x_n)/t_{mn}$ *is bounded.*

3. $(x_m - x_n)_{mn}$ *is Mackey convergent to* 0.

1.12. DEFINITION. A lcs E is called C^∞*-complete* if each Mackey Cauchy sequence has a limit in E.

1.13. THEOREM. *The following properties of a lcs* E *are equivalent:*

1. E *is* C^∞*-complete.*

2. *If* (x_n) *is bounded in* E *and* $\lambda = (\lambda_n) \in l^1$, *then the series* $\Sigma \lambda_n x_n$ *converges in* E.

3. *If* B *is bounded, closed, absolutely convex, then* E_B *is a Banach space.*

4. *For any* B *there is a* B' *such that* $B \subset B'$ *and* $E_{B'}$ *is a Banach space.*

5. *Any continuous linear mapping from a normed space* N *into* E *has a continuous extension to the completion* $\bar{N}$ *of* N.

6. *The closed absolutely convex hull of a Mackey sequence converg-*

ing to 0 is compact.

7. *Any Lipschitz curve in E is locally Riemann integrable.*

8. *For any $c \in C^\infty(\mathrm{R}, E)$ there is a $d \in C^\infty(\mathrm{R}, E)$ with $d' = c$. (Existence of antiderivatives)*

9. *If E is a topological linear subspace of F, then E is closed in $c^\infty F$ (cf. 1.5, 1.6).*

10. *E is a c^∞-closed linear subspace of a C^∞-complete lcs.*

1.14. REMARKS. 1. Any sequentially complete lcs is C^∞-complete (cf. 1.12), but not conversely.

2. E is C^∞-complete iff its bornologicalization bE is C^∞-complete, since this property depends only on the bounded sets.

3. If E is C^∞-complete, then bE is barreled (for it is a direct limit of Banach spaces then). Then even $(E, \sigma(E, E'))$, i.e. E with the weak topology, is C^∞-complete, since in barreled spaces weakly bounded sets are bounded and so $b(E, \sigma(E, E')) = bE$. Now use 2.

4. The full subcategory of C^∞-complete lcs is epireflexive in *lcs* and closed under formation of direct sums and strict inductive limits. The C^∞-completion of E is the closure of E in $c^\infty(E)$.

5. If E is bornological, then its C^∞-completion is bornological too.

1.15. THEOREM. *Let E be a lcs. Then the following properties are equivalent:*

1. *E is C^∞-complete.*
2. *If $f: \mathrm{R}^n \to E$ is scalarwise $C^{k\cdot}$, then f is $C^{k\cdot}$ for $k > 1$.*
3. *If $c: \mathrm{R} \to E$ is scalarwise C^∞ then c is differentiable at 0.*

Here a mapping $f: \mathrm{R}^n \to E$ is called $C^{k\cdot}$ if all partial derivatives up to order $k-1$ exist and are locally Lipschitz. f scalarwise C^α means that $\lambda \circ f$ is a C^α-function $\mathrm{R}^n \to \mathrm{R}$ for all $\lambda \in E'$.

1.16. DEFINITION. Let E, F be lcs. A mapping $f: E \to F$ is called C^∞ if $f \circ c \in C^\infty(\mathrm{R}, F)$ for each $c \in C^\infty(\mathrm{R}, E)$, i.e. if

$$f_*: C^\infty(\mathrm{R}, E) \to C^\infty(\mathrm{R}, F)$$

makes sense.

Let $C^\infty(E, F)$ denote the space of all C^∞-mappings from E to F. Then we have

$$C^\infty(E, F) = C^\infty(bE, bF),$$

since the C^∞-curves depend only on the bounded sets (cf. 1.9.5). Constant maps are C^∞; multilinear mappings are C^∞ iff they are bounded by 1.8. Clearly composition of C^∞-mappings gives again a C^∞-mapping. For $E = \mathbb{R}^n$ we get the usual C^∞-mappings as is shown by the following lemma. Later on, we will see that the differential operator

$$d: C^\infty(E, F) \to C^\infty(E, L(E, F))$$

exists and is linear and bounded. But C^∞-mappings need not be continuous (they are continuous in the c^∞-topologies).

1.17. LEMMA. *Let $f: \mathbb{R}^n \to F$, where F is C^∞-complete. f is C^∞ iff all partial derivatives* $\dfrac{\partial^{|\alpha|} f}{\partial x^\alpha}: \mathbb{R}^n \to F$ *exist and are continuous.*

This is true if F is not C^∞-complete, with a more intricate proof.

PROOF. If $f: \mathbb{R}^n \to F$ maps smooth curves to smooth curves, then for all $\lambda \in F'$ the function $\lambda \circ f: \mathbb{R}^n \to \mathbb{R}$ maps C^∞-curves to C^∞-curves. By the beautiful theorem of Boman this suffices to see that $\lambda \circ f$ is a C^∞-mapping in the usual sense. So $f: \mathbb{R}^n \to F$ is scalarwise C^∞, hence C^∞ in the usual sense by 1.15.2.

1.18. *Topology on $C^\infty(E, F)$.*

We equip the space $C^\infty(\mathbb{R}, F)$ with the bornologicalization of the topology of uniform convergence on compact sets, in all derivatives separately. Then we equip the space $C^\infty(E, F)$ with the bornologicalization of the initial topology with respect to all mappings

$$c^*: C^\infty(E, F) \to C^\infty(\mathbb{R}, F), \quad c^*(f) = f \circ c \text{ for all } c \in C^\infty(\mathbb{R}, E).$$

1.19. LEMMA. *If F is C^∞-complete, then $C^\infty(E, F)$ is C^∞-complete too.*

The proof is decomposed in the following steps:

1. Let X be a set, let $B(X, F)$ be the linear space of all bounded mappings $X \to F$ (i. e. $f(X)$ is bounded), equipped with the topology of

uniform convergence on X. Then $B(X, F)$ is a C^∞-complete lcs.

2. Any product of C^∞-complete spaces is C^∞-complete.

3. $C(\mathbb{R}, F)$, the space of all continuous mappings from $\mathbb{R}$ to F, is a closed linear subspace of the product $\prod_n B([-n, n], F)$.

4. $C^\infty(\mathbb{R}, F)$ is a closed linear subspace in $\prod_n C(\mathbb{R}, F)$, via

$$c \mapsto (c^{(n)}).$$

5. $C^\infty(E, F)$ is a closed linear subspace of $\prod_{c \in C^\infty(\mathbb{R}, E)} C^\infty(\mathbb{R}, F)$.

1.20. LEMMA. *Let E, F be bornological spaces. Then we have:*

1. $L(E, F)$, with the topology defined in the proof of 1.10, is a closed linear subspace of $C^\infty(E, F)$, bornologicalized.

2. If F is C^∞-complete, then $L(E, F)$ is C^∞-complete.

3. If E is C^∞-complete, then a curve $c: \mathbb{R} \to L(E, F)$ is C^∞ iff $t \mapsto c(t)(x)$ is a C^∞-curve in F for all $x \in E$.

1.21. THEOREM. *The category of all C^∞-complete bornological lcs and C^∞-mappings is cartesian closed, i. e. we have a natural bijection:*

$$C^\infty(E \times F, G) \simeq C^\infty(E, C^\infty(F, G)).$$

PROOF. The natural bijection is defined as follows:

$$C^\infty(E \times F, G) \underset{\wedge}{\overset{\vee}{\rightleftarrows}} C^\infty(E, C^\infty(F, G))$$

where

$$f^\vee(x)(y) = f(x, y) \quad \text{and} \quad \hat{g}(x, y) = g(x)(y).$$

This is clearly natural and we have to show that it makes sense. It is first proved in the case $E = \mathbb{R} = F$. Using this result, the theorem is proved as follows:

Let $f \in C^\infty(E, C^\infty(F, G))$. Then for all $c_E \in C^\infty(\mathbb{R}, E)$ we have $f \circ c_E = C^\infty(\mathbb{R}, C^\infty(F, G))$. For all $c_F \in C^\infty(\mathbb{R}, F)$, the mapping

$$c_F^* : C^\infty(F, G) \to C^\infty(\mathbb{R}, G)$$

is linear and continuous by the construction of the topology on $C^\infty(F, G)$. so $c_F^* \circ f \circ c_E : \mathbb{R} \to C^\infty(\mathbb{R}, G)$ is C^∞. Using the above result, we see that the mapping

$$(c_F^* \circ f \circ c_E)\hat{} = \hat{f} \circ (c_E \times c_F): R^2 \to G$$

is C^∞, so

$$\hat{f} \circ (c_E \times c_F) \circ diag: \mathrm{R} \to \mathrm{R}^2 \to G$$

is C^∞. Each $c \in C^\infty(\mathrm{R}, E \times F)$ is of the form

$$(c_E \times c_F) \circ diag = (c_E, c_F),$$

so we conclude that $f: E \times F \to G$ is C^∞.

On the other hand let $g \in C^\infty(E \times F, G)$. Then for any $c_E \in C^\infty(\mathrm{R}, E)$ and any $c_F \in C^\infty(\mathrm{R}, F)$ we have $g \circ (c_E \times c_F) \in C^\infty(\mathrm{R}^2, G)$, so by the above result:

$$(g \circ (c_E \times c_F)) = c_F^* \circ g \circ c_E \in C^\infty(\mathrm{R}, C^\infty(\mathrm{R}, G)).$$

So the mapping

$$g \circ c_E : \mathrm{R} \to \prod_{C^\infty(\mathrm{R}, F)} C^\infty(\mathrm{R}, G)$$

is C^∞ and has values in the closed linear subspace $C^\infty(F, G)$ (see 1.19). So $g \circ c_E: \mathrm{R} \to C^\infty(F, G)$ is C^∞, hence $g \in C^\infty(E, C^\infty(F, G))$.

1.22. COROLLARY. *Let all spaces be C^∞-complete bornological lcs. Then the following natural mappings are C^∞:*

$$ev: C^\infty(E, F) \times E \to F, \quad ev(f, x) = f(x),$$
$$ins: E \to C^\infty(F, E \times F), \quad ins(x)(y) = (x, y),$$
$$\hat{}: C^\infty(E, C^\infty(F, G)) \to C^\infty(E \times F, G),$$
$$\check{}: C^\infty(E \times F, G) \to C^\infty(E, C^\infty(F, G)),$$
$$comp: C^\infty(F, G) \times C^\infty(E, F) \to C^\infty(E, G),$$
$$C^\infty(-,-): C^\infty(F, F') \times C^\infty(E', E) \to C^\infty(C^\infty(E, F), C^\infty(E', F')$$
$$\Pi: \Pi\, C^\infty(E_i, F_i) \to C^\infty(\Pi\, E_i, \Pi\, F_i).$$

1.23. COROLLARY.

$$\hat{}: C^\infty(E, C^\infty(F, G)) \to C^\infty(E \times F, G)$$

is a linear isomorphism of topological vector spaces.

1.24. REMARK. The (bornologicalized) topology on $C^\infty(E, F)$ is uniquely determined if cartesian closedness is asked for: Let $C^\infty_\tau(E, F)$ be equipped with any locally convex topology such that

$$C^\infty(\mathrm{R}, C^\infty_\tau(E, F)) = C^\infty(\mathrm{R} \times E, F)$$

as sets, then $bC^\infty_\tau(E,F) = C^\infty(E,F)$. For we have

$$C^\infty(\mathrm{R}, C^\infty_\tau(E,F)) \simeq C^\infty(\mathrm{R}\times E, F) \simeq C^\infty(\mathrm{R}, C^\infty(E,F)),$$

so $C^\infty_\tau(E,F)$ and $C^\infty(E,F)$ have the same C^∞-curves and thus the same bornologicalizations by 1.9.

1.25. THEOREM. *Let E, F be C^∞-complete bornological lcs. Then the differential operator $d: C^\infty(E,F) \to C^\infty(E, L(E,F))$ exists and is linear and bounded (so continuous), where*

$$df(x).v = \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t}.$$

PROOF. Consider $d^{\wedge\wedge}: C^\infty(E,F)\times E\times E \to F$, given by

$$d^{\wedge\wedge}(f,x,y) = \lim_{t\to 0}\frac{f(x+ty)-f(x)}{t} = \frac{d}{dt}\Big|_0 f(x+ty),$$

which is well defined.

1. It is first proved that $d^{\wedge\wedge}$ is C^∞. Hence, by cartesian closedness:

$$d^\wedge: C^\infty(E,F)\times E \to C^\infty(E,F)$$

is C^∞.

2. $\hat{d}(f,x): E \to F$ is linear for all $f \in C^\infty(E,F)$, $x \in E$. To prove this, for $v, w \in E$ consider the C^∞-mapping:

$$\mathrm{R}^2 \to F: (s,t) \mapsto f(x+sv+tw)$$

and use 1.17 to compute

$$d(f,x)(v+w) = \frac{d}{dt}\Big|_0 f(x+tv+tw) =$$

$$= \frac{\partial}{\partial s}\Big|_0 f(x+sv+0w) + \frac{\partial}{\partial t}\Big|_0 f(x+0v+tw) =$$

$$= d(f,x)(v) + d(f,x)(w),$$

$$d(f,x)(rv) = \frac{d}{dt}\Big|_0 f(x+trv) = r.\frac{d}{dt}\Big|_0 f(x+tv) = r.d(f,x)(v).$$

So $d(f,x) \in L(E,F)$ since it is continuous by 1.8.11.

3. $L(E,F)$ is a closed subspace in $C^\infty(E,F)$ by 1.20.1.

$$\hat{d}: C^\infty(E,F)\times E \to L(E,F) \to C^\infty(E,F)$$

is C^∞, so $\hat{d}: C^\infty(E,F)\times E \to L(E,F)$ is C^∞ since the topology on

$L(E,F)$ is the bornologized subspace topology from $C^\infty(E,F)$. Then by cartesian closedness again $d: C^\infty(E,F) \to C^\infty(E, L(E,F))$ is C^∞.

1.26. PROPOSITION *(Chain rule). Let $f: E \to F$, $g: F \to G$ be C^∞-mappings between C^∞-complete bornological lcs. Then $g \circ f$ is C^∞ and*

$$d(g \circ f)(x) = dg(f(x)) \circ df(x).$$

The proof twice uses the following

SUBLEMMA. *If $c \in C^\infty(\mathbb{R}, E)$, then for each $f \in C^\infty(E,F)$ we have*

$$\frac{d}{dt}\Big|_0 (f \circ c) = df(c(0))(c'(0)).$$

PROOF. In general we have

$$\frac{c(t) - c(0)}{t} = \int_0^1 c'(ts)\,ds,$$

which is C^∞ as a function of t. So the curve

$$t \mapsto df(c(0))\Big(\frac{c(t) - c(0)}{t}\Big)$$

is C^∞ by 1.8.10.

$$df(c(0))(c'(0)) = df(c(0))\Big(\lim_{t\to 0} \frac{c(t) - c(0)}{t}\Big) =$$

$$= \lim_{t \to 0} df(c(0))\Big(\frac{c(t) - c(0)}{t}\Big)$$

$$= \lim_{t\to 0} \lim_{s \to 0} \frac{1}{s}\Big(f\Big(c(0) + s\,\frac{c(t) - c(0)}{t}\Big) - f(c(0))\Big).$$

Note that the last expression is in $C^\infty(\mathbb{R}^2, F)$ as a function of (s,t), for it may be written as

$$\int_0^1 \frac{\partial \tilde{f}}{\partial x_2}(t, s.v)\,dv, \quad \text{where } \tilde{f}(t,v) := f\Big(c(0) + v\int_0^1 c'(tu)\,du\Big),$$

and clearly $\tilde{f} \in C^\infty(\mathbb{R}^2, F)$. So the double limit of the expression above can be computed along any curve in $\mathbb{R}$ going to 0. We compute it along (t,t) for $t \to 0$, and we find that it is equal to

$$\lim_{t \to 0} \frac{1}{t}\Big(f\Big(c(0) + t\,\frac{c(t) - c(0)}{t}\Big) - f(c(0))\Big) =$$

$$= \frac{d}{dt}\Big|_0 (f \circ c)(t).$$

1.27. REMARKS. 1. In general a C^∞-mapping $f: E \to F$ *is not* continuous. This cannot be avoided if one wants cartesian closedness. But clearly $f: c^\infty E \to c^\infty F$ is continuous, so $f: E \to F$ is continuous if $c^\infty E = E$ (e.g. if E is a Fréchet space, or has the property that any sequentially closed set is closed (sequentially determined)).

2. The notion of differentiability C^∞ of Kriegl is weaker than the notion C^∞_c of Keller. Since C^∞_c is the weakest notion with a chain rule, among all notions that can be described with the use of limit structures, the notion of Kriegl cannot be described with the use of convergence structures. But again if $c^\infty F = E$, then $f: E \to F$ is C^∞ iff C^∞_σ iff C^∞_b in the sense of Keller.

3. The exposition of Kriegl's theory given here follows Kriegl [2, 3] closely, with a special emphasis on the results needed later, leaves out all counterexamples and gives some results only in specialized settings (we have assumed C^∞-complete bornological whenever it simplified proofs).

2. PREMANIFOLDS AND PRE-VECTOR BUNDLES.

2.1. DEFINITION. *A premanifold* M is a set of data as follows:

(M 1) Two sets M, TM and a mapping $\pi_M: TM \to M$ such that

$$\pi_M^{-1}(x) := T_x M$$

is a C^∞-complete bornological lcs for each $x \in M$. It follows that π_M is surjective since $0_x \in T_x M$ for each x in M.

(M 2) A subset $S(\mathrm{R}, M)$ of $M^{\mathrm{R}} = Set(\mathrm{R}, M)$ such that $c \circ f \in S(\mathrm{R}, M)$ for each $c \in S(\mathrm{R}, M)$ and $f \in C^\infty(\mathrm{R}, \mathrm{R})$, containing all constant mappings $\mathrm{R} \to M$. Elements of $S(\mathrm{R}, M)$ are called *smooth curves in* M.

(M 3) For each $t \in \mathrm{R}$, a mapping $\delta_t: S(\mathrm{R}, M) \to TM$ such that:

$$\pi_M \circ \delta_t(c) = c(t), \quad c \in S(\mathrm{R}, M),$$

$$\delta_t(c \circ f) = \frac{d}{dt} f(t) \cdot \delta_{f(t)}(c), \quad c \in S(\mathrm{R}, M),\ f \in C^\infty(\mathrm{R}, \mathrm{R}),$$

$$\delta_t(c) = 0_{c(t)} \text{ for all } t \text{ implies that } c \text{ is constant.}$$

$\delta_t(c)$ is called the *differential at* t of the smooth curve c.

(M 4) A mapping

$$Pt = Pt^{TM} : S(\mathrm{R}, M) \times \mathrm{R} \to L(TM, TM) := \bigcup_{x,y \in M} L(T_xM, T_yM)$$

such that

$Pt(c,t) \in L(T_{c(0)}M, T_{c(t)}M)$ for all smooth curves c and all t in R,

$Pt(c,0) = Id_{T_{c(0)}M}$ for all smooth curves c,

$Pt(c, f(t)) = Pt(c \circ f, t) \circ Pt(c, f(0))$ for $f \in C^\infty(\mathrm{R}, \mathrm{R})$,

Here $L(T_xM, T_yM)$ denotes the space of all continuous linear mappings $T_xM \to T_yM$. The mapping Pt is called *parallel transport*. It follows that $Pt(c,t) : T_{c(0)}M \to T_{c(t)}M$ is a topological linear isomorphism with inverse $Pt(c(.+t), -t)$.

(M5) *(Soldering)* For each $c \in S(\mathrm{R}, M)$ the mapping

$$t \mapsto Pt(c,t)^{-1}(\delta_t c) = Pt(c(.+t), -t)(\delta_t c) : \mathrm{R} \to T_{c(0)}M$$

is a C^∞-curve in the bornological lcs $T_{c(0)}M$.

(M6) A mapping $Geo^M : TM \to S(\mathrm{R}, M)$ such that:

$Geo^M(t.v_x)(s) = Geo^M(v_x)(t.s)$, $\delta_t Geo^M(v_x) = Pt(Geo^M(v_x), t)$,

$\delta_t Geo^M(v_x) = Pt(Geo^M(v_x), t)(v_x)$,

$Geo^M(\delta_t Geo^M(v_x))(s) = Geo^M(v_x)(s+t)$.

REMARK. (M6) implies that $\delta_t : S(\mathrm{R}, M) \to TM$ is surjective, since

$$\delta_0(Geo(v_x)) = Pt(Geo(v_x), 0)(v_x) = v_x,$$

$$\delta_t(Geo(v_x)(.-t)) = \delta_0 Geo(v_x) = v_x.$$

2.2. Let M be a premanifold. The natural topology on M is the final topology with respect to all smooth curves

$$c : \mathrm{R} \to M, \quad c \in S(\mathrm{R}, M),$$

i.e. the finest topology such that all c are continuous. In general, this topology is not Hausdorff.

2.3. EXAMPLES. Any paracompact smooth finite dimensional manifold in the usual sense is a premanifold. For let $\pi_M : TM \to M$ be the tangent bundle, let $S(\mathrm{R}, M)$ be the space of all smooth curves, let

$$\delta_t(c) := \frac{d}{dt} c(t) \in T_{c(t)} M ;$$

then choose a complete Riemannian metric g on M (which exists by the result of Nomizu-Ozeki or Morrow), let ∇ denote its Levi-Civita covariant derivative, let Pt be the induced parallel transport,

$$Geo(v_x)(t) = exp(t.v_x).$$

Then (M1)-(M6) are satisfied.

2.4. REMARK. Instead of (M2) consider the following condition:

(M2') There is a subset $S(\mathrm{R}, M)$ of M^{R} such that $c \circ f \in S(\mathrm{R}, M)$ for all $c \in S(\mathrm{R}, M)$ and $f: \mathrm{R} \to \mathrm{R}$ any affine mapping (polynomial of degree ≤ 1).

Adapt (M3) similarly. This is something to be called a geometric space. Any complete Riemannian manifold would then be a geometric space, with $S(\mathrm{R}, M)$ the set of all geodesics.

2.5. DEFINITION. Let M be a premanifold. By a *pre-vector bundle* (E, p, M) we mean a set of data as follows:

(VB1) E is a set, $p: E \to M$ is a mapping such that $p^{-1}(x) =: E_x$ is a C^∞-complete bornological lcs for each x in M.

It follows that p is surjective, since $0_x \in p^{-1}(x)$.

(VB2) There is a mapping

$$Pt^E: S(\mathrm{R}, M) \times \mathrm{R} \to L(E, E) =: \bigcup_{x, y \in M} L(E_x, E_y)$$

such that:

$$Pt^E(c, t) \in L(E_{c(0)}, E_{c(t)}), \quad Pt^E(c, 0) = Id_{E_{c(0)}},$$
$$Pt^E(c, f(t)) = Pt^E(c \circ f, t) \circ Pt^E(c, f(0)).$$

Clearly $Pt^E(c, t): E_{c(0)} \to E_{c(t)}$ is a topological linear isomorphism with inverse

$$Pt^E(c, t)^{-1} = Pt^E(c(.+t), -t).$$

Note that (TM, π_M, M) is a pre-vector bundle for each manifold M.

2.6. THEOREM. *If (E, p, M) is a pre-vector bundle over a premanifold M then the total space E is itself a premanifold in a natural way.*

PROOF. (2.7) Define

$$S_{const}(\mathbb{R},E) =: \bigcup_{x \in M} C^{\infty}(\mathbb{R},E_x),$$

where the union is disjoint; this is the set of all «vertical» smooth curves in E. Then consider the following pullback in the category *Set* of sets and mappings:

$$\begin{tikzcd}
S(\mathbb{R},M) \times_M S_{const}(\mathbb{R},E) \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & S_{const}(\mathbb{R},E) \arrow[d, "p"] \\
S(\mathbb{R},M) \arrow[r, "ev_0"] & M
\end{tikzcd}$$

Use the parallel transport Pt^E of the pre-vector bundle E to define

(2.8) $$Cart^E : S(\mathbb{R},M) \times_M S_{const}(\mathbb{R},E) \to Set(\mathbb{R},E) = E^{\mathbb{R}},$$

$$Cart^E(c_1,c_2)(t) := Pt^E(c_1,t).c_2(t).$$

Then the following diagram commutes:

$$\begin{tikzcd}
S(\mathbb{R},M) \times_M S_{const}(\mathbb{R},E) \arrow[r, "Cart^E"'] \arrow[d, "pr_1"'] & Set(\mathbb{R},E) \arrow[d, "Set(\mathbb{R},p)"] \\
S(\mathbb{R},M) \arrow[r, "incl"] & Set(\mathbb{R},M).
\end{tikzcd}$$

Claim: The mapping $Cart = Cart^E$ is injective. Suppose

$$Cart(c_1,c_2) = Cart(d_1,d_2),$$

then $c_1 = d_1$ by the diagram above, so

$$Pt^E(c_1,t).c_2(t) = Pt^E(c_1,t).d_2(t),$$

hence $c_2(t) = d_2(t)$ for all t, since $Pt^E(c_1,t)$ is an isomorphism.

(2.9) We define $S(\mathbb{R},E) =:$ image of $Cart^E$ in $Set(\mathbb{R},E)$. (The name Cart was chosen in order to indicate that it is a sort of «cartesian» decomposition of the smooth curves in $S(\mathbb{R},E)$).

Claim: $Cart(c_1,c_2) \circ f = Cart(c_1 \circ f, Pt^E(c_1,f(0)) \circ c_2 \circ f)$.

So (M2) holds. (We write Pt instead of Pt^E when no confusion arises.)

$$\begin{aligned} Cart(c_1, c_2)(f(t)) &= Pt(c_1, f(t)).\, c_2(f(t)) = \\ &= Pt(c_1 \circ f, t) \circ Pt(c_1, f(0)).\, c_2(f(t)) = \\ &= Cart(c_1 \circ f,\ Pt(c_1, f(0)) \circ c_2 \circ f)(t). \end{aligned}$$

Consider the following equivalence relation on $S(\mathrm{R}, E)$:

$$Cart(c_1, c_2) \sim Cart(d_1, d_2) \text{ iff } \delta_0 c_1 = \delta_0 d_1,\ c_2(0) = d_2(0), \\ \frac{d}{dt} c_2(0) = \frac{d}{dt} d_2(0).$$

(2.10) We define $TE := S(\mathrm{R}, E)/\sim$. Then we have mappings

$$\begin{tikzcd} S(\mathrm{R}, E) \arrow[rr, "\delta_0"] \arrow[dr, "ev_0"'] & & S(\mathrm{R}, E)/\sim\ = TE \arrow[dl, "\pi_E"] \\ & E & \end{tikzcd}$$

Put

$$\delta_t : S(\mathrm{R}, E) \to TE, \quad \delta_t c = \delta_0(c(\,.\,+t)).$$

Then clearly $\pi_E \circ \delta_t = ev_t$.

(2.11) *Claim*: There is a canonically given bijective mapping

$$Dec^E : TE \to TM \times_M E \times_M E,$$

called *decomposition*, fitting commutatively into the following diagram:

$$\begin{tikzcd} S(\mathrm{R}, M) \times_M S_{const}(\mathrm{R}, E) \arrow[rr, "Cart"] \arrow[d, "\delta_0 \times \tilde{\delta}_0"'] & & S(\mathrm{R}, E) \arrow[d, "\delta_0"] \\ TM \times_M E \times_M E \arrow[dr, "pr_2"'] & & TE \arrow[ll, "Dec^E"'] \arrow[dl, "\pi_E"] \\ & E & \end{tikzcd}$$

Here $\tilde{\delta}_t : S_{const}(\mathrm{R}, E) \to E \times_M E$ is given by

$$\tilde{\delta}_t(c) = (c(t), \frac{d}{dt} c(t)).$$

This is seen as follows: By the definition of the equivalence relation in (2.9) we see that the mapping $(\delta_0 \times \tilde{\delta}_0) \circ (Cart)^{-1}$ factors over δ_0: $S(\mathrm{R}, E) \to TE$ to an injective mapping $Dec = Dec^E$ which is surjective too. As an immediate application we see that

$$T_{u_x}E = \pi_E^{-1}(u_x) = T_xM \times \{u_x\} \times E_x$$

(via *Dec*) is a C^∞-complete bornological lcs, as required by (M 1).

(2.12) *Claim:* $Dec \circ \delta_t \circ Cart(c_1, c_2) =$

$$= (\delta_t c_1, Pt(c_1, t).c_1(t), Pt(c_2, t).\tfrac{d}{dt}c_2(t))$$

$$= (Id_{TM} \times Pt(c_1, t) \times Pt(c_1, t)) \circ (\delta_t \times \tilde{\delta}_t)(c_1, c_2).$$

$$Dec \circ \delta_t \circ Cart(c_1, c_2) = Dec \circ \delta_0 (Cart(c_1, c_2)(.+t)) =$$

$$= Dec \circ \delta_0(Pt(c_1, t+.).c_2(t+.))$$

$$= Dec \circ \delta_0 (Pt(c_1(t+.),.) \circ Pt(c_1, t)c_2(t+.)) \quad \text{by (VB2)}$$

$$= Dec \circ \delta_0 \circ Cart(c_1(t+.), Pt(c_1, t)c_2(t+.))$$

$$= (\delta_0 \times \tilde{\delta}_0)(c_1(t+.), Pt(c_1, t)c_2(t+.)) \quad \text{by (2.11)}$$

$$= (\delta_t c_1, Pt(c_1, t)c_2(t), \tfrac{d}{ds}\big|_0 Pt(c_1, t)c_2(s+t))$$

$$= (\delta_t c_1, Pt(c_1, t)c_2(t), Pt(c_1, t).\tfrac{d}{dt}c_2(t)).$$

By (2.11) the fibre scalar multiplication in the bundle (TE, π_E, E) is given by

$$t.(Dec)^{-1}(u_x, v_x, w_x) = Dec^{-1}(t.u_x, v_x, t.w_x).$$

(2.13) *Claim:* For $f \in C^\infty(\mathbb{R},\mathbb{R})$ and $c \in S(\mathbb{R}, E)$ we have

$$\delta_t(c \circ f) = \tfrac{d}{dt}f(t).\delta_{f(t)}(c).$$

If $c = Cart(c_1, c_2) = Pt(c_1, .) \circ c_2(.)$, then

$$c \circ f = Cart(c_1, c_2) \circ f = Cart(c_1 \circ f, Pt(c_1, f(0)) \circ c_2 \circ f)$$

by (2.9),

$$Dec \circ \delta_t(c \circ f) = Dec \circ \delta_t \circ Cart(c_1 \circ f, Pt(c_1, f(0)) \circ c_2 \circ f)$$

$$= (\delta_t(c_1 \circ f), Pt(c_1 \circ f, t) \circ Pt(c_1, f(0)) \circ c_2 \circ f(t),$$

$$Pt(c_1 \circ f, t).\tfrac{d}{dt}\{Pt(c_1, f(0)).c_2(f(t))\})$$

$$= (f'(t).\delta_{f(t)}(c_1), Pt(c_1, f(t)).c_2(f(t)), f'(t).Pt(c_1, f(t)).c_2'(f(t)))$$

$$= Dec(f'(t).(\delta_{f(t)} Cart(c_1, c_2))) = Dec(f'(t).\delta_{f(t)}(c)).$$

(2.14) *Claim*: Let $c \in S(\mathrm{R}, E)$ be such that $\delta_t c = 0_{c(t)}$ for all t in R. Then $c = constant$. Let $c = Cart(c_1, c_2)$. Then

$$Dec \circ \delta_t \circ Cart(c_1, c_2) = (\delta_t c_1, Pt(c_1, t)c_2(t), Pt(c_1, t)\frac{d}{dt} c_2(t))$$

$$= 0_{c(t)} = (0_{c_1(t)}, Pt(c_1, t).c_2(t), 0_{c_1(t)}).$$

So $\delta_t c_1 = 0_{c_1(t)}$ for all t, so $c_1 = $ const. by (M3) for M, and

$$\frac{d}{dt} c_2(t) = 0_{c_1(0)} \quad \text{for all } t,$$

so $c_2 = $ const. in $E_{c_1(0)}$, so finally $c = $ const.

All requirements of (M3) are satisfied now.

(2.15) Define

$$Pt^{TE} = Pt(TE, \pi_E, E) : S(\mathrm{R}, E) \times \mathrm{R} \to \bigcup_{u, v \in E} L(T_u E, T_v E) = L(TE, TE)$$

by:

$$Dec(Pt^{TE}(Cart(c_1, c_2), t).(Dec)^{-1}(u_x, v_x, w_x)) :=$$

$$= (Pt^{TM}(c_1, t).u_x, Pt(c_1, t).c_2(t), Pt(c_1, t).w_x)$$

$$= (Pt^{TM}(c_1, t).u_x, Cart(c_1, c_2)(t), Pt(c_1, t).w_x).$$

Claim: Pt^{TE}, so defined, satisfies all requirements of (M4).

$Pt^{TE}(c, t) : T_{c(0)} E \to T_{c(t)} E$ is linear and continuous by construction.

$$Pt^{TE}(c, 0).Dec^{-1}(u_x, v_x, w_x) =$$

$$= Dec^{-1}(Pt^{TM}(c_1, 0).u_x, c(0) = v_x, Pt(c_1, 0).w_x)$$

$$= Dec^{-1}(u_x, v_x, w_x).$$

$$Pt^{TE}(c, f(t)).Dec^{-1}(u_x, v_x, w_x) =$$

$$= Dec^{-1}(Pt^{TM}(c_1, f(t)).u_x, c(f(t)), Pt(c_1, f(t)).w_x)$$

$$= Dec^{-1}(Pt^{TM}(c_1 \circ f, t).Pt^{TM}(c_1, f(0)).u_x, c(f(t)),$$

$$Pt(c_1 \circ f, t).Pt(c_1, f(0)).w_x)$$

$$= Pt^{TE}(c_1 \circ f, t).Dec^{-1}(Pt^{TM}(c_1, f(0)).u_x, c(f(0)), Pt(c_1, f(0)).w_x)$$

$$= Pt^{TE}(c_1 \circ f, t).Pt^{TE}(c_1, f(0)).Dec^{-1}(u_x, v_x, w_x).$$

Claim: Pt^{TE} satisfies (M5).

$$Dec \circ \delta_t (c) = Dec \circ \delta_t \circ Cart(c_1, c_2) =$$
$$= (\delta_t c_1, Pt(c_1, t).c_2(t), Pt(c_1, t).\frac{d}{dt} c_2(t))$$

by (2.12).

$$Dec \circ Pt^{TE}(c(.+t), -t).\delta_t c =$$
$$= Dec \circ Pt^{TE}(Cart(c_1(.+t), Pt(c_1, t).c_2(.+t)), -t).$$
$$.(\delta_t c_1, Pt(c_1, t).c_2(t), Pt(c_1, t).\frac{d}{dt} c_2(t))$$
$$= (Pt^{TM}(c_1(.+t), -t).\delta_t c_1, Pt(c_1(.+t), -t).Pt(c_1, t).c_2(-t+t),$$
$$Pt(c_1(.+t), -t).Pt(c_1, t).\frac{d}{dt} c_2(t))$$
$$= (Pt^{TM}(c_1, t)^{-1}(\delta_t c_1), c_2(0) = c(0), \frac{d}{dt} c_2(t)).$$

This is a C^∞-curve in the bornological lcs

$$T_{c(0)}E = T_{c_1(0)} M \times \{c(0)\} \times E$$

by (M5) for M.

(2.16) Define $Geo = Geo^E : TE \to S(\mathrm{R}, E)$ by the formula

$$Geo\, Dec^{-1}(u_x, v_x, w_x)(t) = Cart(Geo^M(u_x), v_x + .\, w_x)(t) =$$
$$= Pt\ (Geo^M(u_x), t).(v_x + t w_x).$$

Claim: Geo, so defined, satisfies all requirements of (M6).

$$Geo(t.\, Dec^{-1}(u_x, v_x, w_x))(s) = Geo(Dec^{-1}(t.u_x, v_x, t.w_x))(s) =$$
$$= Pt(Geo^M(t.u_x), s).(v_x + s t w_x) = Pt(Geo^M(u_x), st).(v_x + s t w_x)$$
$$= Geo(Dec^{-1}(u_x, v_x, w_x))(st).$$

$$Dec \circ \delta_t \circ Geo \circ (Dec)^{-1}(u_x, v_x, w_x) = Dec \circ \delta_t \circ Cart(Geo^M(u_x), v_x + .\, w_x)$$
$$= (\delta_t Geo^M(u_x), Pt(Geo^M(u_x), t)(v_x + t w_x), Pt(Geo^M(u_x), t).w_x)$$

by (2.12)

$$= Dec \circ Pt^{TE}(Cart(Geo^M(u_x), v_x + .\, w_x), t).Dec^{-1}(u_x, v_x, w_x)$$
$$= Dec \circ Pt^{TE}(Geo(Dec^{-1}(u_x, v_x, w_x)), t).Dec^{-1}(u_x, v_x, w_x).$$

$$Geo(\delta_t Geo(Dec^{-1}(u_x, v_x, w_x)))(s) =$$

$$= Geo(Dec^{-1} \circ Dec \circ \delta_t \circ Geo \circ Dec^{-1}(u_x, v_x, w_x))(s)$$

$$= Geo(Dec^{-1}(\delta_t\, Geo^M(u_x), Pt(Geo^M(u_x), t)(v_x + tw_x), Pt(Geo^M(u_x), t).w_x)(s)$$

where we used the computation above,

$$= Pt(Geo^M(\delta_t Geo^M(u_x), s).(Pt(Geo^M(u_x), t)(v_x + tw_x)$$
$$.(Pt(Geo^M(u_x), t)(v_x + tw_x) + s.Pt(Geo^M(u_x), t).w_x)$$
$$= Pt(Geo^M(u_x)(\cdot + t), s).Pt(Geo^M(u_x), t).(v_x + tw_x + sw_x)$$
$$= Geo(Dec^{-1}(u_x, v_x, w_x))(s+t).$$

QED

2.17. COROLLARY. *For any premanifold M the tangent bundle is a pre-vector bundle (TM, π_M, M), so TM is itself a premanifold. In turn we get the whole sequence of iterated tangent bundles:*

$$\dots \to T^{n+1}M \xrightarrow[\pi_{T^nM}]{} T^nM \to \dots \to T^2M \xrightarrow[\pi_{TM}]{} TM \xrightarrow[\pi_M]{} M.$$

3. SMOOTH MAPPINGS.

3.1. DEFINITION. Let M, N be premanifolds. A mapping $f: M \to N$ is called *smooth* if there is a sequence of mappings

$$(T^n f)_{n \geq 0} \quad \text{with} \quad T^0 f = f \quad \text{and} \quad T^n f: T^n M \to T^n N$$

such that for each n the following diagram makes sense and commutes:

$$\text{(a)} \qquad \begin{tikzcd} S(\mathbb{R}, T^n M) \arrow[r, "(T^n f)_*"] \arrow[d, "\delta_0"'] & S(\mathbb{R}, T^n N) \arrow[d, "\delta_0"] \\ T^{n+1}M \arrow[r, "T^{n+1}f"] & T^{n+1}N \end{tikzcd} \quad .$$

Note that

$$\pi_{T^nM} \circ T^{n+1} f = T^n f \circ \pi_{T^nM}$$

by the following commutative diagram (b), in which the two triangles commute by (M3), so that the bottom rectangle commutes since δ_0 surjective.

(b)
$$\begin{tikzcd}
 & S(\mathbb{R}, T^nM) \arrow[rr, "(T^nf)_*"] \arrow[d, "\delta_0"] \arrow[ddl, "ev_0"'] & & S(\mathbb{R}, T^nN) \arrow[d, "\delta_0"'] \arrow[ddr, "ev_0"] & \\
 & T^{n+1}M \arrow[rr, "T^{n+1}f"] \arrow[dl, "\pi_{T^nM}"] & & T^{n+1}N \arrow[dr, "\pi_{T^nN}"'] & \\
T^nM \arrow[rrrr, "T^nf"] & & & & T^nN
\end{tikzcd}$$

Note too that for any smooth mapping $f: M \to N$ and any $t \in \mathbb{R}$ the following diagram commutes:

(c)
$$\begin{tikzcd}
S(\mathbb{R}, T^nM) \arrow[r, "(T^nf)_*"] \arrow[d, "\delta_t"'] & S(\mathbb{R}, T^nM) \arrow[d, "\delta_t"] \\
T^{n+1}M \arrow[r, "T^{n+1}f"] & T^{n+1}N
\end{tikzcd}$$

since

$$\delta_t(T^n f \circ c) = \delta_0(T^n f \circ c(\,.\,+t)) = T^{n+1}f \circ \delta_0(c(\,.\,+t)) = T^{n+1}f \circ \delta_t\, c.$$

Note finally that each $T^n f: T^n M \to T^n N$ is uniquely determined by f (since all δ_0 are surjective) and are again smooth with

$$T^k(T^n f) = T^{k+n} f.$$

3.2. LEMMA. *Any composition of smooth mappings between premanifolds is again smooth, each identity mapping is smooth. So we have a category whose objects are premanifolds and whose morphisms are smooth mappings.*

This category of premanifolds will be denoted by pMf.

3.3. LEMMA. *If $f: M \to N$ is smooth, then $T_x f: T_x M \to T_{f(x)} N$ is continuous and linear as a mapping between two C^∞-complete bornological lcs.*

PROOF. Note first that $T_x f$ is homogeneous of degree 1:

$$\begin{aligned}
(T_x f)(t.u_x) &= (T_x f)(t.\delta_0 c) \quad \text{for some } c \in S(\mathbb{R}, M) \text{ with } \delta_0 c = u_x, \\
&= (T_x f).\delta_0(c(t.)) \qquad \text{by (M3)} \\
&= \delta_0 f_*(c(t.)) \qquad \text{by 3.1} \\
&= \delta_0(f \circ c(t.)) = t.\delta_0(f \circ c) = t.(\delta_0 \circ f_*(c)) =
\end{aligned}$$

$$= t.(T_x f)(\delta_0 c) = t.(T_x f)(u_x).$$

Since Tf is again smooth the mapping $T_x f: T_x M \to T_{f(x)} N$ is C^∞, by the results cited in Section 1. Now the Taylor expansion at 0 of $T_x f$ reduces to the linear term since the mapping is homogeneous, so $T_x f$ is linear. Since it maps C^∞-curves to C^∞-curves, it is bounded, so continuous, since the spaces are bornological. QED

3.4. DEFINITION. Let us denote by $S(M,N)$ the space of all smooth mappings from M to N, where M, N are premanifolds.

We already introduced the notation $S(\mathrm{R}, M)$ in (M2). That we now defined the same space is shown by the next lemma.

3.5. LEMMA. *Let M be a premanifold. Then the set $S(\mathrm{R}, M)$ of (M2) is exactly the space of all smooth mappings in the sense of Definition 3.1 from the manifold* R *(cf Example 2.3) into M.*

PROOF. Let $c: \mathrm{R} \to M$ be a smooth mapping in the sense of 3.1. Then

$$c_*: S(\mathrm{R},\mathrm{R}) = C^\infty(\mathrm{R},\mathrm{R}) \to S(\mathrm{R}, M)$$

makes sense, so $c = c \circ Id_{\mathrm{R}} = c_*(Id_{\mathrm{R}})$ is an element of $S(\mathrm{R}, M)$.

Now suppose conversely that $c \in S(\mathrm{R}, M)$. We have to construct a sequence of mappings $c = T^0 c,\ T^1 c,\ T^2 c, \dots$ satisfying 3.1. Let f be in $S(\mathrm{R},\mathrm{R}) = C^\infty(\mathrm{R},\mathrm{R})$, then $c_*(f) = c \circ f \in S(\mathrm{R}, M)$ by (M2) and

$$\delta_0 \circ c_*(f) = \delta_0 (c \circ f) = f'(0).(\delta_{f(0)} c).$$

So if we define $Tc = T_1 c: T\mathrm{R} - \mathrm{R}^2 \to TM$ by

$$(Tc)(x_1, x_2) = x_2 . (\delta_{x_1} c) = \delta_0 (c(x_1 + . x_2) =: \frac{\partial}{\partial y_1}\Big|_0 c(x_1 + y_1 x_2),$$

then the following diagram commutes :

$$\text{(a)} \qquad \begin{array}{ccc} S(\mathrm{R},\mathrm{R}) & \xrightarrow{\ c_*\ } & S(\mathrm{R}, M) \\ \delta_0 \downarrow & & \downarrow \delta_0 \\ T\mathrm{R} & \xrightarrow{\ Tc\ } & TM \end{array} .$$

For the next step we need results and notation from Lemma 3.6 below. Suppose that

$$f = (f_1, f_2) \in S(\mathrm{R}, T\mathrm{R}) = C^\infty(\mathrm{R}, \mathrm{R}^2).$$

Then the mapping

$$t \mapsto Tc(f_1(t), f_2(t)) = \frac{\partial}{\partial y_1}\Big|_0 c(f_1(t) + y_1 f_2(t))$$

is in $S(\mathrm{R}, TM)$ by Lemma 3.6 below, and by the same lemma we have

$$\delta_0 \circ (Tc)_*(f) = \frac{\partial}{\partial t}\Big|_0 \frac{\partial}{\partial y_1}\Big|_0 c(f_1(t) + y_1 f_2(t))$$

$$= \frac{\partial}{\partial y_2}\Big|_0 \frac{\partial}{\partial y_1}\Big|_0 c(f_1(0) + y_2 f_1'(0) + y_1 f_2(0) + y_1 y_2 f_2'(0))$$

So if we define $T^2 f : T^2\mathrm{R} = \mathrm{R}^4 \to T^2 M$ by

$$(T^2 c)(x_1, x_2, x_3, x_4) = \frac{\partial}{\partial y_2}\Big|_0 \frac{\partial}{\partial y_1}\Big|_0 c(x_1 + y_1 x_2 + y_2 x_3 + y_1 y_2 x_4)$$

then the following diagram commutes :

(b)
$$\begin{array}{ccc} S(\mathrm{R}, T\mathrm{R}) & \xrightarrow{(Tc)_*} & S(\mathrm{R}, TM) \\ \downarrow{\scriptstyle \delta_0} & & \downarrow{\scriptstyle \delta_0} \\ T^2\mathrm{R} & \xrightarrow{T^2 c} & T^2 M \end{array}$$

If $f = (f_1, f_2, f_3, f_4) \in S(\mathrm{R}, T^2\mathrm{R}) = C^\infty(\mathrm{R}, \mathrm{R}^4)$, then

$$t \mapsto (T^2 c)(f(t)) = \frac{\partial}{\partial y_2}\Big|_0 \frac{\partial}{\partial y_1}\Big|_0 c(f_1(t) + y_1 f_2(t) + y_2 f_3(t) + y_1 y_2 f_4(t))$$

is in $S(\mathrm{R}, T^2 M)$ by Lemma 3.6 below, and

$$\delta_0 \circ (T^2 c)_*(f) =$$
$$= \frac{\partial}{\partial t}\Big|_0 \frac{\partial}{\partial y_2}\Big|_0 \frac{\partial}{\partial y_1}\Big|_0 c(f_1(t) + y_1 f_2(t) + y_2 f_3(t) + y_1 y_2 f_4(t))$$
$$= \frac{\partial}{\partial y_3}\Big|_0 \frac{\partial}{\partial y_2}\Big|_0 \frac{\partial}{\partial y_1}\Big|_0 c(f_1(0) + y_3 f_1'(0) + y_1 f_2(0) + y_1 y_3 f_2'(0)$$
$$+ y_2 f_3(0) + y_2 y_3 f_3'(0) + y_1 y_2 f_4(0) + y_1 y_2 y_3 f_4'(0)) .$$

So we may define $T^3 c : T^3\mathrm{R} = \mathrm{R}^8 \to T^3 M$ by

$$(T^3 c)((x_i)_{i=1}^8) = \frac{\partial}{\partial y_3}\Big|_0 \frac{\partial}{\partial y_2}\Big|_0 \frac{\partial}{\partial y_1}\Big|_0 c(x_1 + y_1 x_2 + y_2 x_3 + y_1 y_2 x_4$$
$$+ y_3 x_5 + y_1 y_3 x_6 + y_2 y_3 x_7 + y_1 y_2 y_3 x_8)$$

and continue as above. QED

3.6. LEMMA. *Let* M *be a premanifold. If* $c \in S(\mathrm{R}, M)$ *and* $f \in C^\infty(\mathrm{R}^k, \mathrm{R})$ *then the following hold:*

$$x_2 \mapsto \delta_0 (c \circ f(\ , x_2, \dots, x_k)) =: \frac{\partial}{\partial x_1} c \circ f(0, x_2, \dots, x_k)$$

is in $S(\mathrm{R}, TM)$ *and depends only on*

$$c,\ f(0, x_2, \dots, x_k),\ \frac{\partial f}{\partial x_1}(0, x_2, \dots, x_k).$$

$$x_3 \mapsto \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_1} c \circ f(0, 0, x_3, \dots, x_k)$$

is in $S(\mathrm{R}, T^2 M)$ *and depends only on*

$$c,\ f(0, 0, x_3, \dots, x_k),\ \frac{\partial f}{\partial x_1}(0, 0, x_2, x_3, \dots, x_k),$$

$$\frac{\partial f}{\partial x_2}(0, 0, x_3, \dots, x_k),\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(0, 0, x_3, \dots, x_k).$$

......

$\frac{\partial}{\partial x_k} \dots \frac{\partial}{\partial x_1} c \circ f(0, \dots, 0) \in T^k M$ *depends only on* c, $f(0)$,

$$\frac{\partial^l f}{\partial x_{i_1} \dots \partial x_{i_l}}(0, \dots, 0) \text{ for } 1 \le l \le k,\ 1 \le i_1 \le \dots \le i_l \le k.$$

This lemma means the following: If $f, g \in C^\infty(\mathrm{R}^k, \mathrm{R})$ and

$$\frac{\partial^l f}{\partial x_{i_1} \dots \partial x_{i_l}}(0) = \frac{\partial^l g}{\partial x_{i_1} \dots \partial x_{i_l}}(0)$$

for $0 \le l \le k$, $1 \le i_1 \le \dots \le i_l \le k$, then

$$\frac{\partial}{\partial x_k} \dots \frac{\partial}{\partial x_1} c \circ f(0) = \frac{\partial}{\partial x_k} \dots \frac{\partial}{\partial x_1} c \circ g(0).$$

PROOF. (1) First we put

$$\bar{c} = Pt(c(. + t), -t)\delta_t c = Pt(c, t)^{-1} \delta_t c.$$

Then $\delta c = Cart(c, \bar{c})$. Here and below Pt means always Pt^{TM} and $Cart$ is $Cart^{TM}$, and a running variable is indicated by an empty place like in $c(\ + t)$ instead of $c(. + t)$ (to avoid confusion with ...).

(2) $$\frac{\partial}{\partial x_1} c \circ f(0, x_2, \dots, x_k) = \delta_0 (c \circ f(\ , x_2, \dots, x_k))$$

$$= \frac{d}{dt}\Big|_0 f(t, x_2, \ldots, x_k).\, \delta_{f(0,x_2,\ldots,x_k)}(c) \qquad \text{by (M3)}$$

$$= \frac{\partial f}{\partial x_1}(0, x_2, \ldots, x_k).\, (\delta c) \circ f(0, x_2, \ldots, x_k) =$$

$$= \frac{\partial f}{\partial x}(0, x_2, \ldots, x_k).\, Cart(c, \bar{c})(f(0, x_2, \ldots, x_k)) \qquad \text{by (1)}$$

$$= \frac{\partial f}{\partial x}(0, x_2, \ldots, x_k).\, Pt(c, f(0, x_2, \ldots, x_k)).\, \bar{c}(f(0, x_2, \ldots, x_k))$$

$$= Pt(c, f(0, x_2, \ldots, x_k)).\{\frac{\partial f}{\partial x}(0, x_2, \ldots, x_k).\, \bar{c}(f(0, x_2, \ldots, x_k))\}$$

$$= Pt(c \circ f(0, \ , x_3, \ldots, x_k), x_2).\, Pt(c, f(0, 0, x_3, \ldots, x_k)).\{\ldots\}$$

$$= Pt(c \circ f(0, \ , x_3, \ldots, x_k), x_2).\, Pt(c \circ f(0, 0, \ , x_4, \ldots, x_k), x_3).$$
$$.\, Pt(c, f(0, 0, 0, x_4, \ldots, x_k)).\{\ldots\}$$

$$= Pt(c \circ f(0, \ , x_3, \ldots), x_2).\, Pt(c \circ f(0, 0, \ , x_4, \ldots), x_3), \ldots$$
$$..\, Pt(c \circ f(0, \ldots, 0, \ ), x_k).\, Pt(c, f(0)).\{\frac{\partial f}{\partial x}(0, x_2, \ldots).\, \bar{c}(f(0, x_2, \ldots))\}.$$

(3) For short we put

$$Pt^{j,k}(c \circ f, x_j, x_{j+1}, \ldots, x_k) :=$$
$$:= Pt(c \circ f(0, \ldots, 0, \ , x_{j+1}, \ldots, x_k), x_j) \ldots Pt(c \circ f(0, \ldots, 0, \ ), x_k).$$

(4) $c_1(x_2, \ldots, x_k) := Pt(c, f(0)).\{\frac{\partial f}{\partial x_1}(0, x_2, \ldots, x_k).\, \bar{c}(f(0, x_2, \ldots, x_k))\}.$

Then $c_1 : \mathbb{R}^{k-1} \to T_{c(0)}M$ is a C^∞-mapping. By (2) we have

(5) $$\frac{\partial}{\partial x_1} c \circ f(0, x_2, \ldots, x_k) = Pt^{2,k}(c \circ f, x_2, \ldots, x_k).\, c_1(x_2, \ldots, x_k)$$

$$= Pt(c \circ f(0, \ , x_3, \ldots, x_k), x_2).\, Pt^{3,k}(c \circ f, x_3, \ldots, x_k).\, c_1(x_2, \ldots, x_k)$$

$$= Cart(c \circ f(0, \ , x_3, \ldots, x_k), Pt^{3,k}(c \circ f, x_3, \ldots, x_k).\, c_1(\ , x_3, \ldots, x_k))(x_2).$$

So we have proved the first claim of the lemma. We continue:

(6) $$\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} c \circ f(0, 0, x_3, \ldots, x_k) = \delta_0(\frac{\partial}{\partial x_1} c \circ f(0, \ , x_3, \ldots, x_k))$$

$$= \delta_0 \circ Cart(c \circ f(0, \ , x_3, \ldots, x_k), Pt^{3,k}(c \circ f, x_3, \ldots, x_k).\, c_1(\ , x_3, \ldots, x_k))$$

$$= Dec^{-1} \circ (\delta_0 \times \tilde{\delta}_0)(c \circ f(0, \ , x_3, \ldots), Pt^{3,k}(c \circ f, x_3, \ldots).\, c_1(\ , x_3, \ldots))$$

$$= Dec^{-1}(\delta_0(c \circ f(0, \,, x \;, \dots)), Pt^{3,k}(c \circ f, x_3, \dots).c_1(0, x_3, \dots), \\ , Pt^{3,k}(c \circ f, x_3, \dots).\frac{\partial}{\partial x_2} c_1(0, x_3, \dots)).$$

Looking at (2) we see that

$$(7)\ \delta_0(c \circ f(0, \,, x_3, \dots, x_k)) = \frac{\partial}{\partial x_2} c \circ f(0, 0, x_3, \dots, x_k)$$

$$= Pt^{3,k}(c \circ f, x_3, \dots).Pt(c, f(0)).\frac{\partial f}{\partial x_2}(0, 0, x_3, \dots).\bar{c}(f(0, 0, x_3, \dots)).$$

(8) Put

$$c_1^2(x_3, \dots, x_k) = Pt(c, f(0)).\frac{\partial f}{\partial x_2}(0, 0, x_3, \dots).\bar{c}(f(0, 0, x_3, \dots)).$$

Then $c_1^2 : \mathbb{R}^{k-2} \to T_{c(0)}M$ is a C^∞-mapping and we have by (6):

$$(9)\ Dec(\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} c \circ f(0, 0, x_3, \dots, x_k)) =$$

$$= (Pt^{3,k}(c \circ f, x_3, \dots).c_1^2(x_3, \dots), Pt^{3,k}(c \circ f, x_3, \dots).c_1(0, x_3, \dots), \\ Pt^{3,k}(c \circ f, x_3, \dots).\frac{\partial}{\partial x_2} c_1(0, x_3, \dots))$$

$$= (Pt(c \circ f(0, 0, \,, x_4, \dots), x_3).Pt^{4,k}(c \circ f, x_4, \dots).c_1^2(x_3, \dots), \\ , Pt(c \circ f(0, 0, \,, x_4, \dots), x_3).Pt^{4,k}(c \circ f, x_4, \dots).c_1(0, x_3, \dots), \\ , Pt(c \circ f(0, 0, \,, x_4, \dots), x_3).Pt^{4,k}(c \circ f, x_4, \dots).\frac{\partial}{\partial x_2} c_1(0, x_3, \dots))$$

$$= Dec^{-1} Pt_2(Cart(c \circ f(0, 0, \,, x_4, \dots), Pt^{4,k}(c \circ f, x_4, \dots).c_1(0, \,, x_4, \dots)), x_3) \\ .Dec^{-1}(Pt^{4,k}(c \circ f, x_4, \dots).c_1^2(x_3, \dots), Pt^{4,k}(c \circ f, x_4, \dots).c_1(0, 0, x_4, \dots). \\ , Pt^{4,k}(c \circ f, x_4, \dots).\frac{\partial}{\partial x_2} c_1(0, x_3, \dots)),$$

where $Pt_2 = Pt^{T^2M}$ and where we used Definition (2.15) for Pt^{T^2M}.

But note that

$$(10)\ Cart(c \circ f(0, 0, \,, x_4, \dots), Pt^{4,k}(c \circ f, x_4, \dots), c_1(0, \,, x_4, \dots))(x_3)$$

$$= Pt(c \circ f(0, 0, \,, x_4, \dots), x_3).Pt^{4,k}(c \circ f, x_4, \dots).c_1(0, x_3, \dots)$$

$$= Pt^{3,k}(c \circ f, x_3, \dots).c_1(0, x_3, \dots) = Pt^{2,k}(c \circ f, 0, x_3, \dots).c_1(0, x_3, \dots)$$

$$= \frac{\partial}{\partial x_1} c \circ f(0, 0, x_3, \dots, x_k) \qquad \text{by (5).}$$

Putting (10) into (9) we get:

$$(11)\ \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}\ c\circ f(0,0,x_3,\dots,x_k)$$

$$= Pt_2(\frac{\partial}{\partial x_1}\ c\circ f(0,0,\ ,x_4,\dots),x_3).Dec^{-1}(Pt^{4,k}(c\circ f,x_4,\dots).c_1^2(x_3,\dots),$$

$$,Pt^{4,k}(c\circ f,x_4,\dots).c_1(0,0,x_4,\dots),Pt^{4,k}(c\circ f,x_4,\dots).\frac{\partial}{\partial x_2}c_1(0,x_3,\dots))$$

$$= Pt_2(\frac{\partial}{\partial x_1}\ c\circ f(0,0,\ ,x_4,\dots),x_3).$$

$$.Pt_2(Cart(c\circ f(0,0,0,\ ,x_5,\dots),Pt^{5,k}(c\circ f,x_5,\dots).c_1(0,0,\ ,x_5,\dots)),x_4).$$

$$.Dec^{-1}(Pt^{5,k}(c\circ f,x_5,\dots).c_1^2(x_3,\dots),$$

$$Pt^{5,k}(c\circ f,x_5,\dots).c_1(0,0,0,x_5,\dots),$$

$$Pt^{5,k}(c\circ f,x_5,\dots).\frac{\partial}{\partial x_2}c_1(0,x_3,\dots))\qquad \text{by (2.15)}$$

$$= Pt_2(\frac{\partial}{\partial x_1}\ c\circ f(0,0,\ ,x_4,\dots),x_3).Pt_2(\frac{\partial}{\partial x_1}\ c\circ f(0,0,0,\ ,x_5,\dots),x_4).$$

$$.Dec^{-1}(Pt^{5,k}(c\circ f,x_5,\dots).c_1^2(x_3,\dots),$$

$$Pt^{5,k}(c\circ f,x_5,\dots).c_1(0,0,0,x_5,\dots),$$

$$Pt^{5,k}(c\circ f,x_5,\dots).\frac{\partial}{\partial x_2}c_1(0,x_3,\dots))\qquad \text{by (10)}$$

$$= Pt_2(\frac{\partial}{\partial x_1}\ c\circ f(0,0,\ ,x_4,\dots),x_3).Pt_2(\frac{\partial}{\partial x_1}\ c\circ f(0,0,0,\ ,x_5,\dots),x_4)\dots$$

$$\dots Pt_2(\frac{\partial}{\partial x_1}\ c\circ f(0,\dots,0,\ ),x_k).Dec^{-1}(c_1^2(x_3,\dots),c_1(0),\frac{\partial}{\partial x_2}c_1(0,x_3,\dots))$$

by recursion

$$= Pt_2^{3,k}(\frac{\partial}{\partial x_1}c\circ f,x_3,\dots,x_k).Dec^{-1}(c_1^2(x_3,\dots),c_1(0),\frac{\partial}{\partial x_2}c_1(0,x_3,\dots)).$$

Here we have used convention (3) for $Pt_2 = Pt^{T^2M}$ to define $Pt_2^{j,k}$.

(12) Put

$$c_2(x_3,\dots,x_k) = Dec^{-1}(c_1^2(x_3,\dots),c_1(0),\frac{\partial}{\partial x_2}c_1(0,x_3,\dots)),$$

then $c_2 : \mathrm{R}^{k-2} \to T_{c_1(0)}TM$ is a C^∞-mapping. Using this in (11) we get:

$$(13)\ \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}\ c\circ f(0,0,x_3,\dots,x_k)$$

$$= Pt_2^{3,k}(\frac{\partial}{\partial x_1}\ c\circ f(x_3,\dots,x_k).c_2(x_3,\dots,x_k))$$

$$= Pt_2(\frac{\partial}{\partial x_1} c \circ f(0, 0, , x_4, \dots), x_3). Pt^{4,k}(\frac{\partial}{\partial x_1} c \circ f, x_4, \dots). c_2(x_3, \dots)$$

$$= Cart_2(\frac{\partial}{\partial x_1} c \circ f(0, 0, , x_4, \dots), Pt_2^{4,k}(\frac{\partial}{\partial x_1} c \circ f, x_4, \dots). c_2(, x_4, \dots))(x_3),$$

which is a smooth curve in the parameter x_3, depending only on c,

$$f(0, 0, x_3, \dots), \ \frac{\partial f}{\partial x_1}(0, 0, x_3, \dots), \ \frac{\partial f}{\partial x_2}(0, 0, x_3, \dots). \ \frac{\partial^2 f}{\partial x_1 \partial x_2}(0, 0, x_3, \dots).$$

So we have proved the second claim of the formula. In the formula above

$$Cart_2 = Cart^{T^2M}.$$

(14) Now we put down the general recursion formulas:

$$c_1(x_2, \dots, x_k) = Pt(c, f(0)). \frac{\partial f}{\partial x_1}(0, x_2, \dots, x_k). \bar{c}(f(0, x_2, \dots, x_k)),$$

$$c_1^l(x_{l+1}, \dots, x_k) = Pt(c, f(0)). \frac{\partial f}{\partial x_l}(0, \dots, 0, x_{l+1}, \dots, x_k).$$

$$. \bar{c}(f(0, \dots, 0, x_{l+1}, \dots, x_k)),$$

for $l = 2, 3, \dots, k-1$, and

$$c_1^k = Pt(c, f(0)). \frac{\partial f}{\partial x_k}(0). \bar{c}(f(0)) \in T_{c(f(0))}M.$$

Define c_j, c_j^l for $j = 2, \dots, k$, $l = j+1, \dots, k$ by

$$c_j(x_{j+1}, \dots, x_k) =$$

$$= Dec_{j-1}^{-1}(c_{j-1}^j(x_{j+1}, \dots, x_k), c_{j-1}(0), \frac{\partial}{\partial x_j} c_{j-1}(0, x_{j+1}, \dots, x_k)),$$

$$c_j^l(x_{j+1}, \dots, x_k) =$$

$$= Dec_{j-1}^{-1}(c_{j-1}^l(x_{l+1}, \dots, x_k), c_{j-1}(0), \frac{\partial}{\partial x_l} c_{j-1}(0, \dots, 0, x_{l+1}, \dots)),$$

....

$$c_k = Dec_{k-1}^{-1}(c_{k-1}^k, c_{k-1}(0), \frac{\partial}{\partial x_k} c_{k-1}(0)).$$

Then c_j, $c_j^l : \mathbb{R}^{k-l} \to T_{c_{j-1}(0)} T^{j-1}M$ are all C^∞-mappings.

(15) Claim: With the formulas of (14) we have

$$\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_{j-1}} \dots \frac{\partial}{\partial x_1} c \circ f(0, \dots, 0, x_{j+1}, \dots, x_k) =$$

$$= Pt_j^{j+1,k}\left(\frac{\partial}{\partial x_{j-1}} \dots \frac{\partial}{\partial x_1} c \circ f, x_{j+1}, \dots, x_k\right) . c_j(x_{j+1}, \dots, x_k)$$

$$= Cart_j\left(\frac{\partial}{\partial x_{j-1}} \dots \frac{\partial}{\partial x_1} c \circ f(0, \dots, 0, \ , x_{j+1}, \dots, x_k),\right.$$

$$\left., Pt_j^{j+2,k}\left(\frac{\partial}{\partial x_{j-1}} \dots \frac{\partial}{\partial x_1} c \circ f, x_{j+2}, \dots, x_k\right) . c_j(\ , x_{j+2}, \dots, x_k)(x_{j+1})\right).$$

This claim proves inductively the lemma (by (3) the expression involving $Pt_j^{j+2,k}$ depends only on the terms indicated in the lemma). The claim itself may be proved by induction. The proof of the induction step is essentially the same as the proof of the second step ((6)-(13)). QED

4. SMOOTHNESS OF CERTAIN STRUCTURE MAPPINGS.

4.1. THEOREM. *Let* $(M_\alpha)_{\alpha \in A}$ *be a family of premanifolds, then* $\prod_{\alpha\in A} M_\alpha$ *is a premanifold in a natural way,*

$$T\left(\prod_{\alpha\in A} M_\alpha\right) = \prod_{\alpha\in A} TM_\alpha,$$

and each projection $pr_\beta : \prod_{\alpha\in A} M_\alpha \to M_\beta$ *is smooth. Furthermore the couple* $\left(\prod_{\alpha\in A} M_\alpha, pr_\alpha\right)$ *is a product in the category* pMf.

PROOF. (M1) Define

$$\begin{tikzcd} T\left(\prod_{\alpha\in A} M_\alpha\right) \arrow[dr, "\pi_{\prod M_\alpha}"'] & = & \prod_{\alpha\in A} TM_\alpha \arrow[dl, "\prod \pi_{M_\alpha}"] \\ & \prod_{\alpha\in A} M_\alpha & \end{tikzcd}$$

Then

$$\pi_{\prod M_\alpha}^{-1}((x_\alpha)) = \prod_{\alpha\in A} T_{x_\alpha} M_\alpha$$

is a C^∞-complete bornological lcs (at least if $card(A)$ is smaller than the least inaccessible cardinal number by Section 1 and the theorem of Mackey-Ulam; if not, one has to take first the bornological locally convex topology on the product).

(M2) Define $S(\mathrm{R}, \prod_\alpha M_\alpha)$ as the set $\prod_\alpha S(\mathrm{R}, M_\alpha)$; i.e. $c = (c_\alpha)$:

$\mathbb{R} \to \prod_\alpha M_\alpha$ is in $S(\mathbb{R}, \prod M_\alpha)$ iff each coordinate c_α is in $S(\mathbb{R}, M_\alpha)$.
If f is in $C^\infty(\mathbb{R}, \mathbb{R})$, then

$$c \circ f = (c_\alpha) \circ f = (c_\alpha \circ f)$$

is in $S(\mathbb{R}, \prod M_\alpha)$ again.

(M3) Define

$$\begin{tikzcd}
S(\mathbb{R}, \prod M_\alpha) \arrow[rr, "\cong"] \arrow[dr, "ev_t"] \arrow[dd, dashed, "\delta_t"'] & & \prod S(\mathbb{R}, M_\alpha) \arrow[dl, "\prod ev_t"'] \arrow[dd, "\prod \delta_t"] \\
& \prod M_\alpha & \\
T(\prod M_\alpha) \arrow[rr] \arrow[ur, "\pi_{\prod M_\alpha}"] & & \prod T M_\alpha \arrow[ul, "\prod \pi_{M_\alpha}"']
\end{tikzcd}$$

Then we have for c and f as above:

$$\delta_t(c \circ f) = \delta_t(c_\alpha \circ f) = (\delta_t(c_\alpha \circ f)) = (f'(t).\delta_{f(t)} c_\alpha)$$
$$= f'(t).(\delta_{f(t)} c_\alpha) = f'(t).\delta_{f(t)}(c_\alpha) = f'(t).\delta_{f(t)} c.$$

If $\delta_t c = 0_{c(t)}$ for all t, then $\delta_t c_\alpha = 0$ for all t, α, so each $c_\alpha =$ const hence $c =$ constant.

(M4) Define

$$Pt^{\prod M_\alpha}(c, t) = Pt^{\prod M_\alpha}((c_\alpha), t) = \prod Pt^{M_\alpha}(c_\alpha, t) :=$$
$$\prod T_{c_\alpha(0)} M_\alpha = T_{c(0)}(\prod M_\alpha) \to T_{c(t)}(\prod M_\alpha) = \prod T_{c_\alpha(t)} M_\alpha .$$

This mapping is continuous and linear. The functional equations of (M4) are easily seen to be satisfied.

(M5) This can be checked component-wise.

(M6) Put

$$Geo^{\prod M_\alpha}((u_\alpha))(t) = (Geo^{M_\alpha}(u_\alpha)(t)) \in \prod M_\alpha .$$

then

$$Geo^{\prod M_\alpha}((u_\alpha)) = (Geo^{M_\alpha}(u_\alpha)) \in \prod S(\mathbb{R}, M_\alpha) = S(\mathbb{R}, \prod M_\alpha).$$

The functional equations for $Geo^{\prod M_\alpha}$ can be checked component-wise.

So $\prod M_\alpha$ is a premanifold in a natural way.

Claim: $pr_\beta : \Pi M_\alpha \to M_\beta$ is smooth.

$$\begin{tikzcd}
S(\mathbb{R}, \Pi M_\alpha) \arrow[rr, "(pr_\beta)_*"] \arrow[dr, "="'] \arrow[dd, "\delta_0"'] & & S(\mathbb{R}, M_\beta) \arrow[dd, "\delta_0"] \\
& \Pi S(\mathbb{R}, M_\alpha) \arrow[ur, "pr_\beta"'] \arrow[dd, "\Pi\delta_0"] & \\
T(\Pi M_\alpha) \arrow[rr, "T(pr_\beta)"] \arrow[dr, "="'] & & TM_\beta \\
& \Pi TM_\alpha \arrow[ur, "pr_\beta"'] &
\end{tikzcd}$$

So $T(pr_\beta) \ll = \gg pr_\beta$ and we may iterate.

Claim: $(\Pi M_\alpha, pr_\alpha)$ is a product in pMf.

Consider smooth mappings $f_\alpha : P \to M_\alpha$, where P is a premanifold. Since $(\Pi M_\alpha, pr_\alpha)$ is a product in Set, there is a mapping $(f_\alpha) : P \to \Pi M_\alpha$, such that $pr_\beta \circ (f_\alpha) = f_\beta$. We have to check whether (f_α) is smooth. To see this we use the following diagram:

$$\begin{tikzcd}
S(\mathbb{R}, P) \arrow[rrr, "\delta_0"] \arrow[dr, "(f_\alpha)_*"] \arrow[ddr, "((f_\alpha)_*)"'] \arrow[ddd, "(f_\beta)_*"'] & & & TP \arrow[dl, "T((f_\alpha))"'] \arrow[ddl, "(Tf_\beta)"] \arrow[ddd, "Tf_\beta"] \\
& S(\mathbb{R}, \Pi M_\alpha) \arrow[r, "\delta_0"] \arrow[d, equal] \arrow[ddl, "(pr_\beta)_*"'] & T(\Pi M_\alpha) \arrow[d, equal] \arrow[ddr, "T(pr_\beta)"] & \\
& \Pi S(\mathbb{R}, M_\alpha) \arrow[r, "\Pi\delta_0"] \arrow[dl, "pr_\beta"] & \Pi TM_\alpha \arrow[dr, "pr_\beta"'] & \\
S(\mathbb{R}, M_\beta) \arrow[rrr, "\delta_0"] & & & TM_\beta
\end{tikzcd}$$

So $T((f_\alpha)) \ll = \gg (Tf_\alpha)$ and we may iterate to get the whole sequence $T^n((f_\alpha))$. QED

4.2. PROPOSITION. *Let M be a premanifold, let (E_i, p_i, M) be pre-vector bundles over M for $i = 1, 2$. Then the fibre product $E_1 \underset{M}{\times} E_2$ is a pre-vector bundle over M in a canonical way.*

REMARK. We are not yet in a position to show that $pr_i : E_1 \underset{M}{\times} E_2 \to E_i$ is smooth.

PROOF. (VB1)

$$(E_1 \underset{M}{\times} E_2)_x = (E_1)_x \times (E_2)_x$$

is a C^∞-complete bornological lcs.

(VB2) Put

$$Pt_{E_1 \underset{M}{\times} E_2}(c,t) = Pt^{E_1}(c,t) \times Pt^{E_2}(c,t):$$

$$(E_1)_{c(0)} \times (E_2)_{c(0)} = (E_1 \underset{M}{\times} E_2)_{c(0)} \to (E_1 \underset{M}{\times} E_2)_{c(t)} = (E_1)_{c(t)} \times (E_2)_{c(t)}$$

which is continuous and linear. The functional equations are easily checked. QED

4.3. PROPOSITION. *Let (E, p, M) be a pre-vector bundle and let N be another premanifold, let $f: M \to N$ be a smooth mapping. Then the pullback (f^*E, f^*p, N) is a pre-vector bundle over N in a canonical way.*

$$\begin{array}{ccc} f^*E & \xrightarrow{p^*f} & E \\ {\scriptstyle f^*p}\downarrow & & \downarrow{\scriptstyle p} \\ N & \xrightarrow{f} & M \end{array}$$

PROOF. (VB1)

$$(f^*E)_n = (f^*p)^{-1}(n) = E_{f(n)}$$

is a C^∞-complete bornological lcs.

(VB2) Put

$$Pt^{f^*E}(c,t) = Pt^E(f \circ c, t): E_{f(c(0))} = (f^*E)_{c(0)} \to (f^*E)_{c(t)} = E_{f(c(t))}.$$

This is linear and continuous, and the functional equations are easily checked. QED

4.4. Note that yet we do not know whether certain canonical mappings like the projection $p: E \to M$ of a pre-vector bundle or Dec are smooth - a scandal! It is not so easy to show that these are smooth without a circle conclusion, since they are interwoven into the differentiable structure themselves. In order to treat this rigourously we give the following definition:

DEFINITION. Let M, N be premanifolds, let $f: M \to N$ be a mapping. We say that *f is of class S^1* if $f_*: S(\mathrm{R}, M) \to S(\mathrm{R}, N)$ makes sense and if there is a mapping $Tf: TM \to TN$ such that $\delta_0 \circ f_* = Tf \circ \delta_0$.

Note that any S^1-mapping is continuous in the natural topologies of the premanifolds and that Tf is uniquely determined by f and is homogeneous on each fibre (to conclude that it is linear as in 3.1 we need more). Furthermore for any S^1-mapping f and any t in R we have

$$\delta_t \circ f_* = Tf \circ \delta_t .$$

This can be proved as the same assertions in 3.1.

Let us say inductively that *$f: M \to N$ is of class S^2* if f is of class S^1 and Tf is of class S^1 too, and that *f is S^k* if f is S^1 and Tf is S^{k-1}, for each finite k. Let $S^k(M, N)$ denote the set of all S^k-mappings of M into N. Clearly composites of S^k-mappings are again S^k, so we have a category pMf^k of premanifolds and S^k-mappings.

Note that S^1 is not an analogue of the usual notion C^1: an S^1 mapping has to map smooth curves on smooth curves; on a C^∞-complete bornological lcs a S^1-mapping is already C^∞; it might well be that in general S^1 equals smooth.

4.5. LEMMA. *If (E^i, p_i, M) are pre-vector bundles over the premanifold M, then $pr_i : E^1 \times_M E^2 \to E_i$ is of class S^1 for $i = 1, 2$.*

PROOF.

$$\begin{tikzcd}[column sep=large]
S(\mathrm{R},M) \times_M S_{const}(\mathrm{R},E^1) \times_M S_{const}(\mathrm{R},E^2) \arrow[r, "\delta_0 \times \tilde\delta_0 \times \tilde\delta_0"'] \arrow[d, equal] \arrow[dddd, bend right=60, "pr_{1,2}"'] & TM \times_M E^1 \times_M E^1 \times_M E^2 \times_M E^2 \arrow[d, "flip_{3,4}"] \arrow[dddd, bend left=60, "pr_{1,2,3}"] \\
S(\mathrm{R},M) \times_M S_{const}(\mathrm{R}, E^1 \times_M E^2) \arrow[r, "\delta_0 \times \tilde\delta_0"] \arrow[d, "Cart^{E^1 \times_M E^2}"] & TM \times_M (E^1 \times_M E^2) \times_M (E^1 \times_M E^2) \\
S(\mathrm{R}, E^1 \times_M E^2) \arrow[r, "\delta_0"] \arrow[d, "(Tpr_1)_*"] & T(E^1 \times_M E^2) \arrow[u, "Dec^{E^1 \times_M E^2}"'] \arrow[d, "T(pr_1)"] \\
S(\mathrm{R}, E^1) \arrow[r, "\delta_0"] & TE^1 \\
S(\mathrm{R}, M) \times_M S_{const}(\mathrm{R}, E^1) \arrow[r, "\delta_0 \times \tilde\delta_0"] \arrow[u, "Cart^{E^1}"] & TM \times_M E^1 \times_M E^1 \arrow[u, "Dec^{E^1}"']
\end{tikzcd}$$

This diagram commutes:

$$\begin{aligned}
((pr_1)_* \circ Cart^{E^1 \times_M E^2}(c_1, c_2, c_3))(t) &= \\
&= pr_1(Pt^{E^1 \times_M E^2}(c_1, t).(c_2(t), c_3(t)))) \\
&= pr_1(Pt^{E^1}(c_1, t).c_2(t)),\ Pt^{E^2}(c_1, t).c_3(t)) \\
&= Pt^{E^1}(c_1, t).c_2(t) = (Cart^{E^1} \circ pr_{1,2}(c_1, c_2, c_3))(t).
\end{aligned}$$

The rest is clear. So pr_1 is S^1. The same for pr_2. QED

4.6. LEMMA. *If (E, p, M) is a pre-vector bundle, then $p : E \to M$ is S^1.*

PROOF.

$$\begin{tikzcd}
S(\mathbb{R}, M) \times_M S_{const}(\mathbb{R}, E) \arrow[rr, "\delta_0 \times \tilde{\delta}_0"] \arrow[d, "Cart^E"] \arrow[dd, bend right=60, "pr_1"'] & & TM \times_M E \times_M E \arrow[dd, bend left=60, "pr_1"] \\
S(\mathbb{R}, E) \arrow[rr, "\delta_0"] \arrow[d, "p_*"] & & TE \arrow[u, "Dec^E"'] \arrow[d, "Tp"'] \\
S(\mathbb{R}, M) \arrow[rr, "\delta_0"] & & TM
\end{tikzcd}$$

QED

4.7. LEMMA. *Let (E^i, p_i, M) be pre-vector bundles over a premanifold M, then we have a canonical bijection*

$$T(E^1 \times_M E^2) \xrightarrow{\sim} TE^1 \times_{(Tp_1, TM, Tp_2)} TE^2,$$

given by the following diagram:

$$\begin{tikzcd}
T(E^1 \times_{(p_1, M, p_2)} E^2) \arrow[d, "Dec", "\wr"'] & & TE^1 \times_{(Tp_1, TM, Tp_2)} TE^2 \arrow[d, "Dec^{E^1} \times Dec^{E^2}", "\wr"'] \\
TM \times_M (E^1 \times_M E^2) \times_M (E^1 \times_M E^2) \arrow[dr, "flip_{3,4}"', "\sim"] & & (TM \times_M E^1 \times_M E^1) \times_{TM} (TM \times_M E^2 \times_M E^2) \arrow[dl, equal] \\
& TM \times_M E^1 \times_M E^1 \times_M E^2 \times_M E^2 &
\end{tikzcd}$$

Look at the diagram in 4.5 to see that this diagram makes sense.

4.8. LEMMA. *If (E^i, p_i, M) are pre-vector bundles over a premanifold M,*

$$(a)\qquad \begin{tikzcd} E^1 \underset{M}{\times} E^2 \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & E^1 \arrow[d, "p_1"] \\ E^2 \arrow[r, "p_2"] & M \end{tikzcd}$$

is a pullback in the category pMf^1 of premanifolds and S^1-mappings.

PROOF. Note first that (a) is a diagram in the category pMf^1 by Lemmas 4.4 and 4.6. Now let N be a premanifold and consider a diagram of the following form in pMf^1:

$$(b)\qquad \begin{tikzcd} N \arrow[dr, dashed, "{(f,g)}"] \arrow[drr, bend left, "f"] \arrow[ddr, "g"'] & & \\ & E^1 \underset{M}{\times} E^2 \arrow[r, "pr_1"] \arrow[d, "pr_2"] & E^1 \arrow[d, "p_1"] \\ & E^2 \arrow[r, "p_2"] & M \end{tikzcd}$$

Since diagram (a) is a pullback in *Set* there is a mapping (f, g) fitting commutatively into the diagram. It remains to show that (f, g) is S^1.

$$(c)\qquad \begin{tikzcd}[column sep=small] S(\mathbb{R},N) \arrow[dr, dashed] \arrow[ddd, "\delta_0"'] & & & & \\ & S(\mathbb{R},M) \underset{M}{\times} S_{const}(\mathbb{R},E^1) \underset{M}{\times} S_{const}(\mathbb{R},E^2) \arrow[rrr, "pr"] \arrow[dr, "Cart"] \arrow[drrr, "f_*"] \arrow[dddd, "g_*"'] \arrow[ddddd, bend right, "pr"'] & & & S(\mathbb{R},M) \underset{M}{\times} S_{const}(\mathbb{R},E^1) \arrow[d, "Cart"] \arrow[dddd, bend left, "pr"] \\ & & S(\mathbb{R}, E^1 \underset{M}{\times} E^2) \arrow[d] & & S(\mathbb{R},E^1) \arrow[d, "\delta_0"] \\ TN \arrow[rr, dashed] \arrow[rrrr, bend left, "Tf"] \arrow[drr, "Tg"'] & & T(E^1 \underset{M}{\times} E^2) \arrow[rr, "T(pr_1)"] \arrow[d, "T(pr_2)"] & & TE^1 \arrow[d, "T(p_1)"] \\ & & TE^2 \arrow[rr, "T(p_2)"] & & TM \\ & S(\mathbb{R},E^2) \arrow[ur, "\delta_0"] & & & S(\mathbb{R},M) \arrow[u, "\delta_0"] \\ & S(\mathbb{R},M) \times S_{const}(\mathbb{R},E^2) \arrow[u, "Cart"] \arrow[urrr, "pr"'] & & & \end{tikzcd}$$

Here we used the fact that the innermost square is a pullback by 4.7. Note that this diagram shows that

$$S(\mathrm{R}, E^1 \underset{M}{\times} E^2) = S(\mathrm{R}, E^1) \underset{S(\mathrm{R}, M)}{\times} S(\mathrm{R}, E^2)$$

holds. QED

4.9. LEMMA. *Let* (E^i, p_i, M) *be pre-vector bundles over a premanifold* M, *let* (F^i, q_i, N) *be pre-vector bundles over* N. *Let* $f: M \to N$, $g_i: E^i \to F^i$ *be* S^1*-mappings such that*

$$\begin{array}{ccc} E^i & \xrightarrow{g_i} & F^i \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle q_i} \\ M & \xrightarrow{f} & N \end{array}$$

commutes for $i = 1, 2$. *Then the mapping*

$$g_1 \times g_2 : E^1 \underset{M}{\times} E^2 \to F^1 \underset{N}{\times} F^2$$

is S^1.

Use Lemmas 4.5 and 4.8 for the mappings

$$g_i \circ pr_i : E^1 \underset{M}{\times} E^2 \to E^i \to F^i$$

to prove this result.

4.10. LEMMA. *Let* (E, p, M) *be a pre-vector bundle, let* N *be a premanifold and let* $f: N \to M$ *be a* S^1*-mapping (only). Then* (f^*E, f^*p, N) *is a pre-vector bundle in a canonical way, and the diagram*

$$(a) \qquad \begin{array}{ccc} f^*E & \xrightarrow{p^*f} & E \\ {\scriptstyle f^*p}\downarrow & & \downarrow{\scriptstyle p} \\ N & \xrightarrow{f} & M \end{array}$$

is a pullback in the category pMf^1.

PROOF. First we show that (f^*E, f^*p, N) is a pre-vector bundle.

(VB 1) $(f^*E)_n = E_{f(n)}$ is a C^∞-complete bornological lcs.

(VB 2) For $c \in S(\mathrm{R}, N)$ define

$$Pt^{f^*E}(c, t) = Pt^E(f \circ c, t) : (f^*E)_{c(0)} = E_{f(c(0))} \to E_{f(c(t))} = (f^*E)_{c(t)}$$

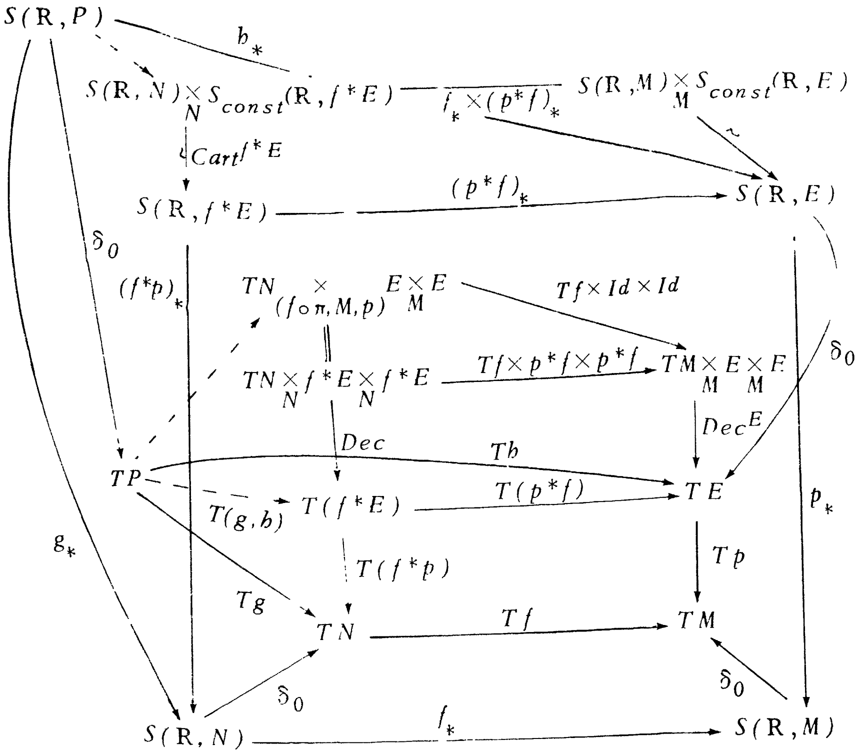
as in 4.3. The functional equations are easily checked.
So by Theorem 2.6, f^*E is a premanifold and by 4.6 the projections p and f^*p are S^1-mappings. It remains to check that p^*f is S^1. To see this look at the following diagram (b):

$$\begin{tikzcd}[column sep=small]
S(\mathbb{R},N) \underset{(f\circ ev_0,\, M,\, p)}{\times} S_{const}(\mathbb{R},E) \arrow[rrrrd, "f_*\times Id"] \arrow[rd, equal] \arrow[ddddd, "\delta_0\times\tilde{\delta}_0"'] & & & & \\
& S(\mathbb{R},N)\underset{N}{\times} S_{const}(\mathbb{R},f^*E) \arrow[rrr, "(f_*)\times(p^*f)_*"] \arrow[rd, "Cart^{f^*E}"] \arrow[ddd, "\delta_0\times\tilde{\delta}_0"'] & & & S(\mathbb{R},M)\underset{M}{\times} S_{const}(\mathbb{R},E) \arrow[ld, "Cart^E"'] \arrow[ddd, "\delta_0\times\tilde{\delta}_0"] \\
& & S(\mathbb{R},f^*E) \arrow[r, "(p^*f)_*"] \arrow[d, "\delta_0"] & S(\mathbb{R},E) \arrow[d, "\delta_0"'] & \\
& & T(f^*E) \arrow[r, dashed, "T(p^*f)"] \arrow[ld, "Dec^{f^*E}"] & TE \arrow[rd, "Dec^E"'] & \\
& TN\underset{N}{\times} f^*E \underset{N}{\times} f^*E \arrow[rrr, "Tf\times(p^*f)\times(p^*f)"] & & & TM\underset{M}{\times} E\underset{M}{\times} E \\
TN \underset{(f\circ\pi,\, M,\, p)}{\times} E\underset{M}{\times} E \arrow[rrrru, "Tf\times Id\times Id"'] \arrow[ru, equal] & & & &
\end{tikzcd}$$

Now we know that diagram (a) is in pMf^1. We show that it is a pullback in this category. So let P be another premanifold and consider a diagram of the following form in pMf^1:

(c)
$$\begin{tikzcd}
P \arrow[rrd, "h"] \arrow[rd, dashed, "(g,h)"] \arrow[rdd, "g"'] & & \\
& f^*E \arrow[r, "p^*f"] \arrow[d, "f^*p"] & E \arrow[d, "p"] \\
& N \arrow[r, "f"] & M
\end{tikzcd}$$

Here g, h are S^1-mappings. Since diagram (a) is a pullback in *Set* by construction, there is a mapping $(g,h): P \to f^*E$ fitting commutatively into diagram (c). We claim that (g,h) is S^1. We use the following diagram (d), in which we employ twice the universal property of pullbacks and we indicate in the diagram why the squares are pullbacks.



For further reference, we note that

$$T(f^*E) = TN \underset{(Tf,\, TM,\, Tp)}{\times} TE, \qquad S(\mathbb{R}, f^*E) = S(\mathbb{R},N) \underset{(f_*,\, S(\mathbb{R},M),\, p_*)}{\times} S(\mathbb{R},E)$$

QED

4.11. REMARK. If (E,p,M) is a pre-vector bundle, then the mapping $Dec^E : TE \to TM \underset{M}{\times} E \underset{M}{\times} E$ is an isomorphism between the following two pre-vector bundles:

$$Dec^E : (TE, \pi_E, E) \to (TM \underset{M}{\times} E \underset{M}{\times} E, pr_2, E).$$

In fact we used it to define the vector bundle structure on (TE, π_E, E) in the proof of Theorem 2.6. Clearly the following two pre-vector bundles coincide

$$(TM \underset{M}{\times} E \underset{M}{\times} E, pr_2, E) = (p^*TM \underset{E}{\times} p^*E, p^*p = p^*\pi_M, E),$$

the second pre-vector bundle being given in 4.2. But $TM \times_M E \times_M E$ is a pre-vector bundle over M too, applying 4.2 twice. We now want to show that Dec^E is actually a diffeomorphism between the two premanifolds.

4.12. LEMMA. *Consider* $TM \times_M E \times_M E$ *as a pre-vector bundle over* M *by* 4.2, *so it has a canonical premanifold structure by* 2.6. *Consider on* TE *the premanifold structure induced from the pre-vector bundle* (TE, π_E, E). *Then* $Dec^E : TE \to TM \times_M E \times_M E$ *is a* S^1*-diffeomorphism (isomorphism in the category* pMf^1*).*

PROOF. Look at the diagram on page 44. Most of it is trivially seen to comute (all squares involving δ_0 or $\tilde{\delta}_0$). It remains to check that the polygon on the left hand side commutes. So let

$$(c_1, c_2, c_3, c_4) \in S(\mathbb{R}, M) \times_M S_{const}(\mathbb{R}, TM) \times_M S_{const}(\mathbb{R}, E) \times_M S_{const}(\mathbb{R}, E)$$

Then writing

$$Dec = Dec^E, \quad Cart = Cart^E, \quad Pt = Pt^E,$$

we have

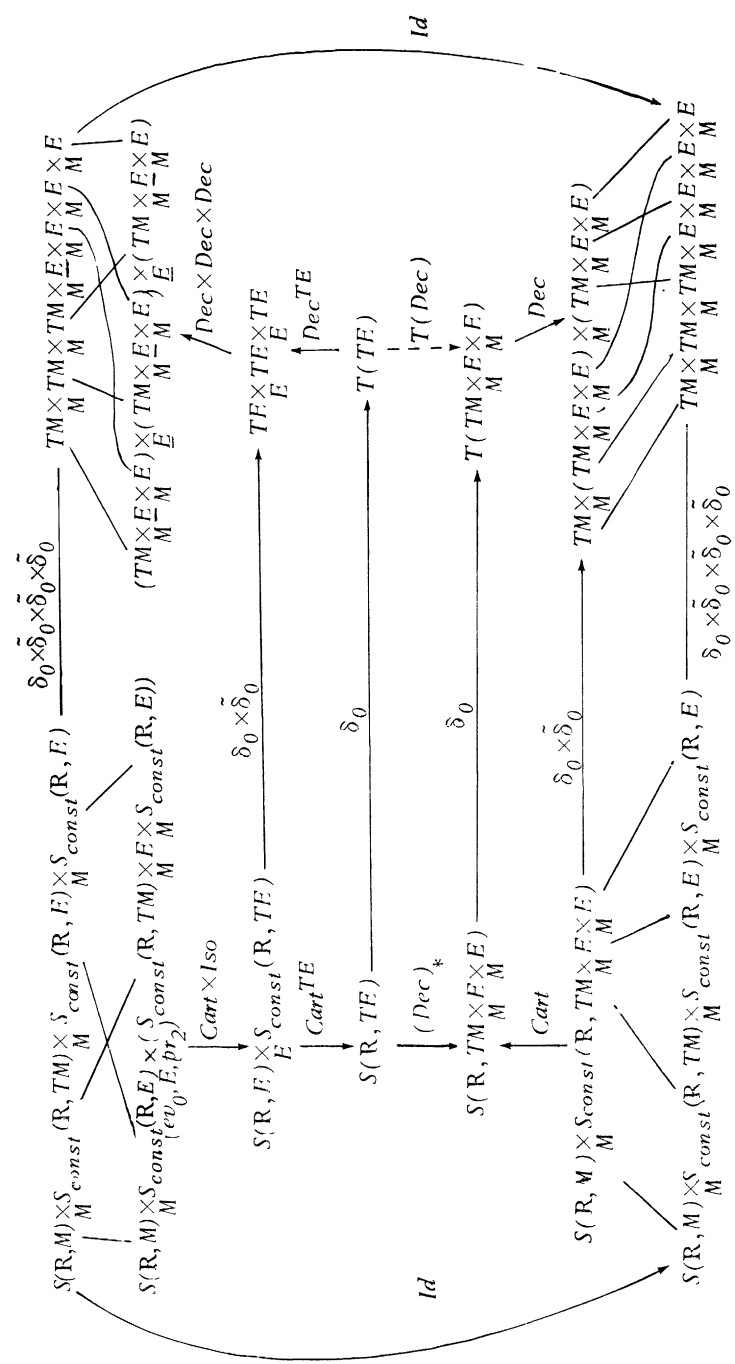
$$\begin{aligned}
&((Dec)_* \circ Cart^{TE} \circ (Cart \times Iso) \circ Iso(c_1, c_2, c_3, c_4))(t)\\
&= ((Dec)_* \circ Cart^{TE} \circ (Cart \times Iso)((c_1, c_3), (c_2, c_3(0), c_4)))(t)\\
&= ((Dec)_* \circ Cart^{TE}(Cart(c_1, c_3), Dec^{-1} \circ (c_2, c_3(0), c_4)))(t)\\
&= Dec(Pt^{TE}(Cart(c_1, c_3), t). Dec^{-1}(c_2(t), c_3(0), c_4(t)))\\
&= (Pt^{TM}(c_1, t). c_2(t), Pt(c_1, t). c_3(t), Pt(c_1, t). c_4(t)).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&Cart^{TM \times_M E \times_M E} \circ Iso(c_1, c_2, c_3, c_4))(t)\\
&= Cart(c_1, (c_2, c_3, c_4))(t) = Pt^{TM \times_M E \times_M E}(c_1, t).(c_2(t), c_3(t), c_4(t))\\
&= (Pt^{TM}(c_1, t). c_2(t), Pt(c_1, t). c_3(t), Pt(c_1, t). c_4(t)).
\end{aligned}$$

QED

4.13. THEOREM. *If* (E, p, M) *is a pre-vector bundle over a premanifold* M, *then* $p : E \to M$ *is smooth and*



Id
$S(\mathbb{R},M)\times_M S_{const}(\mathbb{R},TM)\times_M S_{const}(\mathbb{R},E)\times_M S_{const}(\mathbb{R},E)$
$\delta_0\times\tilde\delta_0\times\tilde\delta_0\times\tilde\delta_0$
$TM\times_M TM\times_M TM\times_M E\times_M E\times_M E$
$S(\mathbb{R},M)\times_M S_{const}(\mathbb{R},E)\times_M (S_{const}(\mathbb{R},TM)\times_M E\times_M S_{const}(\mathbb{R},E))$
(ev_0, E, pr_2)
$(TM\times_M E\times_M E)\times_E (TM\times_M E\times_M E)\times_E (TM\times_M E\times_M E)$
$Cart\times Iso$
$Dec\times Dec\times Dec$
$S(\mathbb{R},E)\times_E S_{const}(\mathbb{R},TE)$
$\delta_0\times\tilde\delta_0$
$TE\times_E TE\times_E TE$
$Cart^{TE}$
Dec^{TE}
$S(\mathbb{R},TE)$
δ_0
$T(TE)$
$(Dec)_*$
$T(Dec)$
$S(\mathbb{R},TM\times_M E\times_M E)$
δ_0
$T(TM\times_M E\times_M E)$
Dec
$Cart$
$S(\mathbb{R},M)\times_M S_{const}(\mathbb{R},TM\times_M E\times_M E)$
$\delta_0\times\tilde\delta_0$
$TM\times_M (TM\times_M E\times_M E)\times_M (TM\times_M E\times_M E)$
$S(\mathbb{R},M)\times_M S_{const}(\mathbb{R},TM)\times_M S_{const}(\mathbb{R},E)\times_M S_{const}(\mathbb{R},E)$
$\delta_0\times\tilde\delta_0\times\tilde\delta_0\times\tilde\delta_0$
$TM\times_M TM\times_M TM\times_M E\times_M E\times_M E$
Id

$$Dec = Dec^E : TE = p^*TM \underset{E}{\times} p^*E \to TM \underset{M}{\times} E \underset{M}{\times} E$$

is smooth with smooth inverse.

PROOF. By Lemma 4.12, Dec is S^1 and Dec^{-1} is S^1. Looking at the diagram in 4.12 we see that

$$T(Dec) = Dec^{TM\underset{M}{\times}E\underset{M}{\times}E} \circ Iso \circ Iso \circ Iso \circ (Dec \times Dec \times Dec) \circ Dec^{TE}$$

By the Lemmas 4.5, 4.8, 4.9, all the mappings called Iso are S^1 and all Dec's are S^1 too, so $T(Dec)$ is S^1, so Dec is S^2. The same argument applies for $(Dec)^{-1}$. By Lemma 4.6, $p : E \to M$ is S^1 and $Tp = pr_1 \circ Dec$, so Tp is S^1 and p is S^2.

By Lemma 4.5, $pr_1 : E^1 \underset{M}{\times} E^2 \to E$ is S^1 and

$$T(pr_1) = Dec^{E^1} \circ pr_{1,2,3} \circ Iso \circ Dec^{E^1\underset{M}{\times}E^2}$$

which is again S^1 ($pr_{1,2,3}$ is S^1 by 4.5 and 4.8 or 4.9), so pr_1 is S^2. Now consider the situation of Lemma 4.8: if f, g are S^2, then (f, g) is S^1 and $T(f, g) = (Tf, Tg)$ via some identifications along Dec and pr in 4.7; since all these identifications are S^2 already we see that $T(f, g)$ is S^2. So by Lemma 4.8 itself, (f, g) is S^2. So Lemma 4.8 remains true for S^2, also its Corollary 4.9. But then all components in $T(Dec)$ in 4.12 are S^2, so Dec is S^3.

But then $p : E \to M$ is S^3 and we can repeat the argument ad infinitum.

QED

4.14. THEOREM. *Let (E, p, M), (F, q, M) be pre-vector bundles and let $f : N \to M$ be smooth. Then the following two diagrams are pullbacks in the category pMf of premanifolds and smooth mappings.*

$$\begin{tikzcd} E \underset{M}{\times} F \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & F \arrow[d, "q"] \\ E \arrow[r, "p"] & M \end{tikzcd} \qquad \begin{tikzcd} f^*E \arrow[r, "p^*f"] \arrow[d, "f^*p"] & E \arrow[d, "p"] \\ N \arrow[r, "f"] & M \end{tikzcd} .$$

PROOF. This was established in the course of the proof of Theorem 4.13 and can directly be read of the diagrams in the proofs of Lemmas 4.10 and 4.8. QED

4.15. THEOREM. *Let (E, p, M) be a pre-vector bundle. Then (TE, Tp, TM) is a pre-vector bundle too and is isomorphic (via Dec^E) to the pre-vector bundle $(TM \underset{M}{\times} E \underset{M}{\times} E, pr_1, TM)$.*

PROOF. $Pt^{(TE, Tp, TM)}$ is given, for

$$c \in S(\mathbb{R}, TM) \quad \text{and} \quad (u_x, v_x, w_x) \in TM \underset{M}{\times} E \underset{M}{\times} E$$

with $c(0) = u_x$ by the formula:

$$Dec^E \circ Pt^{(TE, Tp, TM)}(c, t) \circ (Dec^E) \quad (u_x = c(0), v_x, w_x)$$

$$= (c(t), Pt^E(\pi_M \circ c, t).v_x, Pt^E(\pi_M \circ c, t).w_x)).$$

This satisfies all requirements. QED

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