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CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE

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THE EPIS OF THE CATEGORY OF ORDERED ALGEBRAS AND Z-CONTINUOUS HOMOMORPHISMS

by Ana PASZTOR

1. INTRODUCTION. THE MAIN RESULTS

A *subset system* Z is a class of posets containing the two-element chain and closed with respect to images of monotonic maps. If A is a poset, then $Z(A)$ is the set of all subposets of A which are in Z.

Let A and B be posets. Then a map $\phi: A \to B$ *is Z-continuous* if whenever $X \in Z(A)$ and $\sup_{\leq A} X$ exists, then $\sup_{\leq B} \phi(X)$ exists and equals $\phi(\sup_{\leq A} X)$.

Let Σ denote a *signature*, i. e. a set of function symbols. For any $f \in \Sigma$, $r(f)$ denotes the *arity of* f, which is an *arbitrary ordinal number.*

Ord denotes the class of all ordinal numbers.

A *partial* Σ*-algebra* $\underline{A}$ consists of a set A and of a family

$$< f_A : dom\, f_A \to A>_{f \in \Sigma}$$

of partial operations on A, i.e. $\bigwedge_{f \in \Sigma} dom\, f_A \subset A^{r(f)}$. Given two partial Σ-algebras $\underline{A}$ and $\underline{B}$, a homomorphism $\phi: \underline{A} \to \underline{B}$ is a map $\phi: A \to B$ which satisfies the following:

$$\bigwedge_{f \in \Sigma} \bigwedge_{a \in A^{r(f)}} (a \in dom\, f_A \to \phi \circ a \in dom\, f_B \wedge f_B(\phi \circ a) = \phi(f_A(a))).$$

A partial Σ-algebra $\underline{A}$ *is total* if $\bigwedge_{f \in \Sigma} dom\, f_A = A^{r(f)}$.

For details about subset system Z see Adámek-Nelson-Reiterman [1] or Nelson [6]. For more about the theory of partial Σ-algebras, see Andréka-Németi [2], Burmeister [3], Németi [7], Németi-Sain [8].

The frame category of the present paper will be ${}^{\perp}Z\,Palg_{\Sigma}$ defined as follows:

$\underline{A} \in Ob({}^{\perp}Z\,Palg_{\Sigma})$ and is called an *ordered partial* Σ*-algebra* if $\underline{A}$ is a partial Σ-algebra, A is a poset with a least element $\perp$ and all operations of $\underline{A}$ are monotonic with respect to $\leq_A$;

$\phi : \underline{A} \to \underline{B} \in Mor({}^{\perp}Z\,Palg_{\Sigma})$ if ϕ is a Z-continuous ${}^{\perp}$-preserving homorphism.

${}^{\perp}Z\,Alg_{\Sigma}$ denotes the full subcategory of ${}^{\perp}Z\,Palg_{\Sigma}$ defined by

$$Ob({}^{\perp}Z\,Alg_{\Sigma}) = \{\underline{A} \in Ob({}^{\perp}Z\,Palg_{\Sigma}) \mid \underline{A} \text{ is total}\}.$$

This paper provides a *characterization of the epis of* ${}^{\perp}Z\,Alg_{\Sigma}$ for any subset system Z and for any signature Σ .

Before giving the Main Result let us recall the characterization of epis of ${}^{\perp}Z\,Palg_{\Sigma}$ from Pasztor [9]. Throughout the paper, let a signature Σ and a subset system Z be arbitrary but fixed.

DEFINITION 1. Let $\underline{A}$ be an ordered partial Σ-algebra and let $X \subset A$. We define on A a relation $\xleftarrow{\alpha, X}$ for every $\alpha \in Ord$.

(A) $a \xleftarrow{0, X} b$ iff $a = b \in X$.

Suppose $\alpha > 0$. Then $a \xleftarrow{\alpha, X} b$ iff one of (B), (C) or (D) holds:

$$\text{(B)} \quad \bigvee_{c, d \in A} \; \bigvee_{\beta < \alpha} \; a \leq_A c \xleftarrow{\beta, X} d \leq_A b .$$

$$\text{(C)} \quad \bigvee_{f \in \Sigma} \; \bigwedge_{i < r(f)} \; \bigvee_{a_i, b_i \in A} \; \bigvee_{\alpha_i < \alpha} \; (a = f_A(a_i \mid i < r(f)) \wedge b = f_A(b_i \mid i < r(f)) \wedge a_i \xleftarrow{\alpha_i, X} b_i) .$$

$$\text{(D)} \quad \bigvee_{Y \in Z(A)} (a = \sup_{\leq_A} Y \wedge \bigwedge_{y \in Y} \; \bigvee_{\alpha_y < \alpha} \; y \xleftarrow{\alpha_y, X} b).$$

Then, let

$$\xleftarrow{X} := \bigcup_{\alpha \in Ord} \xleftarrow{\alpha, X} .$$

REMARKS. 1. This definition is equivalent to Definition 1 of Section 3 in Pasztor [9], but simplifies the proofs given there.

2. It is easy to see (by induction) that $\xleftarrow{X} \subset \leq_A$.

3. If t, s, q and r denote some term-functions of signature Σ and if $Y^i \in Z(A)$, then we could imagine $a \xleftarrow{X} b$ as drawn on Figure 1.

DEFINITION 2. Let $\underline{A}$ be an ordered partial Σ-algebra and $X \subset A$. Then

$$CL_{\Sigma}(X) := \{ a \in A \mid a \xleftarrow{X} a \}.$$

For the next result see Section 3, Theorem 2 in Pasztor [9].

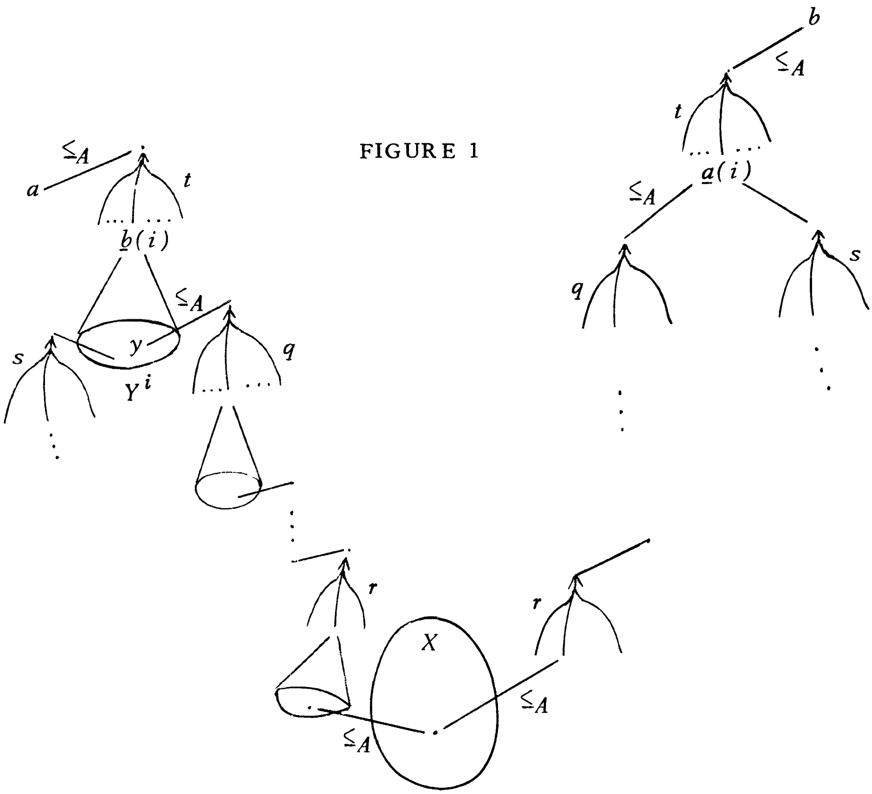


FIGURE 1

THEOREM 1. *$\phi : \underline{A} \to \underline{B} \in Mor({}^{\perp}Z\,Palg_{\Sigma})$ is an epi iff $CL_{\Sigma}(\phi(A)) = B$.*

Now we can state the *Main Result* of this paper. For the proof of Theorem 2, see Section 2.

THEOREM 2. *Every ordered partial Σ-algebra has a Z-continuous and $\perp$-preserving embedding into an ordered total Σ-algebra, i. e. for every object $\underline{A}$ of ${}^{\perp}Z\,Palg_{\Sigma}$ there exist*

$$\underline{B} \in Ob({}^{\perp}Z\,Alg_{\Sigma}) \quad \text{and} \quad u_A : \underline{A} \to \underline{B} \in Mor({}^{\perp}Z\,Palg_{\Sigma})$$

with u_A an embedding.

COROLLARY 1. *$\phi : \underline{A} \to \underline{B} \in Mor({}^{\perp}Z\,Alg_{\Sigma})$ is an epi iff $CL_{\Sigma}(\phi(A)) = B$.*

PROOF. It is clear that those morphisms of ${}^{\perp}Z\,Alg_{\Sigma}$ which are epis in ${}^{\perp}Z\,Palg_{\Sigma}$ are also epis in ${}^{\perp}Z\,Alg_{\Sigma}$. Hence if for $\phi : \underline{A} \to \underline{B} \in Mor({}^{\perp}Z\,Alg_{\Sigma})$, $CL_{\Sigma}(\phi(A)) = B$, then ϕ is epi in ${}^{\perp}Z\,Palg_{\Sigma}$ by Theorem 1 and hence epi in ${}^{\perp}Z\,Alg_{\Sigma}$, too. Now let us prove the other way round, i.e., suppose that $\phi: \underline{A} \to \underline{B}$ is an epi in ${}^{\perp}Z\,Alg_{\Sigma}$. We'll prove that ϕ is an epi in ${}^{\perp}Z\,Palg_{\Sigma}$, too and hence $CL_{\Sigma}(\phi(A)) = B$. Let therefore $r, \sigma: \underline{B} \to \underline{C}$ be arbitrary morphisms of ${}^{\perp}Z\,Palg_{\Sigma}$ such that $\phi \cdot \sigma = \phi \cdot r$. Let u_C be a Z-continuous ${}^{\perp}$-preserving embedding of $\underline{C}$ into the ordered total Σ-algebra $\underline{D}$. Then $\phi \cdot r \cdot u_C = \phi \cdot \sigma \cdot u_C$. Since ϕ is an epi in ${}^{\perp}Z\,Alg_{\Sigma}$, we have $r \cdot u_C = \sigma \cdot u_C$. But u_C is a mono, hence $r = \sigma$. □

COROLLARY 2. *Let* Φ *denote the subset system containing only the two element chain. Then* $\phi: \underline{A} \to \underline{B} \in Mor({}^{\perp}\Phi\,Alg_{\Sigma})$ *is an epi iff* $\phi(A) = B$, *i.e. iff* ϕ *is surjective.*

PROOF. In Pasztor [9], Corollary 9, we have proved that for $Z = \Phi$ - the class containing only the two-element chain - $CL_{\Sigma}(\phi(A)) = \phi(A)$. □

COROLLARY 3. *1. Let* Z *be bounded. Then for any signature* Σ, ${}^{\perp}Z\,Alg_{\Sigma}$ *is co(well-powered).*

2. Let Σ *be a signature with at least one* $f \in \Sigma$ *such that* $r(f) > 0$. *Then there is a subset system* $Z \subset \Delta$ *(i.e.* Y *is directed for any poset* A *and* $Y \in Z(A)$*) such that* ${}^{\perp}Z\,Alg_{\Sigma}$ *is not co(well-powered).*

PROOF. See Pasztor [9], Section 4, Corollary 29 and Proposition 30. □

2. PROOF OF THEOREM 2.

We want to prove that for any ordered partial Σ-algebra $\underline{A}$ *there is* a Z-continuous ${}^{\perp}$-preserving embedding into an ordered total Σ-algebra.

Before proving this let us recall from Pasztor [10] a construction of the *free* Σ*-completion* $\hat{\underline{A}}$ of a partial Σ-algebra $\underline{A}$. The free Σ-completion of $\underline{A}$ is just another name for the Alg_{Σ}*-reflection of* $\underline{A}$, where Alg_{Σ} is the category of total Σ-algebras and homomorphisms. Most of the denotations used here are adoptions of the denotations of Guessarian [5]. We denote by $\delta = \delta(\Sigma)$ the *ordinal dimension of* Σ, i.e. the least regular ordinal number δ such that $|\delta| > |r(f)|$ for each $f \in \Sigma$ (e.g. if for any $f \in \Sigma$, $r(f) \in \omega$, then $\delta(\Sigma) = \omega$). Then we denote by δ^* the set of all finite words over δ with λ as the empty word. A word m' *is a left* (resp. *right) factor* of a word m iff there is a word m'' such that $m = m'm''$ (resp.

$m = m'' m'$).

A *tree domain* D_t is a nonempty subset of δ^* satisfying the following two conditions :

(i) if $m = m_1 \dots m_q$ belongs to D_t, then every left factor $m_1 \dots m_p$, $p \leq q$ of m belongs also to D_t,

(ii) if $m = m_1 \dots m_{q-1} m_q$ belongs to D_t, then for every $m' < m_q$, $m_1 \dots m_{q-1} m'$ belongs also to D_t.

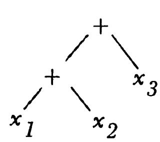
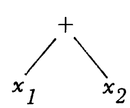
The elements of D_t are called the *nodes* of the tree domain and also of the trees we will associate with it. Let D_t be a tree domain, i an ordinal and $m, mi \in D_t$. Then m is the *father of* mi, which is in its turn the *son of* m. We call m' *an ancestor of* m iff m' is a left factor of m. Similarly, m' *is a descendant of* m iff m is a left factor of m'. The node λ is the *root of* D_t. A node having no descendant other than itself is called a *leaf*.

Let X be an arbitrary set. A *tree on* $X \cup \Sigma$ is a total mapping t from a tree domain D_t into $\Sigma \cup X$ with the property that for any $m \in D_t$, if $t(m) = s \in \Sigma \cup X$, then m has exactly $r(s)$ sons in D_t. The elements of X are by definition of arity 0. For any node m, $t(m)$ is its *label*.

We denote by $T(\Sigma, X)$ the set of all trees on $\Sigma \cup X$. For any tree t, $L(t)$ denotes the *set of leaves* in D_t. If t is a tree on $\Sigma \cup X$ and if $m \in D_t$, then $t' = t \backslash m$ is defined by

$$D_{t'} = \{ m' \mid mm' \in D_t \} \text{ and } t'(m') = t(mm') \text{ for any } m' \in D_{t'}.$$

E. g.

$t =$  , $\qquad t' = t \backslash 0 =$ 

A tree t on $\Sigma \cup X$ *is path-finite* if any countable sequence $m = n_0 n_1 n_2 \dots$ of nodes of D_t with n_i son of n_{i-1}, $i = 1, 2, \dots$, called a path is of finite length, i. e. there is a $q \in \omega$ such that $m = n_0 n_1 n_2 \dots n_{q-1}$. We denote the set of all path-finite trees on $\Sigma \cup X$ by $F(\Sigma, X)$.

Let $\underline{A}$ be a partial Σ-algebra and let $t \in F(\Sigma, A)$; we define t_A inductively as follows :

(i) if $D_t = \{ \lambda \}$ and $t(\lambda) \notin \Sigma$, then $t_A := t(\lambda)$;

(ii) If $t(\lambda) = f \in \Sigma$ then

$$t_A := f_A((t\backslash m)_A \mid m < r(f))$$

if this is defined and is undefined otherwise.

Note that if t_A is defined then for any $m \in D_t$, $(t\backslash m)_A$ is also defined.

We define recursively the *depth* $d(t)$ of a path-finite tree t on $\Sigma \cup X$:

(i) If $D_t = \{\lambda\}$ then $d(t) = 1$;

(ii) if $t(\lambda) = f \in \Sigma$ is not a constant, then $d(t)$ is the smallest ordinal greater then $d(t\backslash m)$ for each $m < r(f)$.

If $\underline{A}$ is a partial Σ-algebra, then we denote by $\hat{A}$ the set of all trees $t \in F(\Sigma, A)$ with the property that for any $m \in D_t$, if $(t\backslash m)_A$ is defined, then $m \in L(t)$ and $t(m)$ is not a constant symbol.

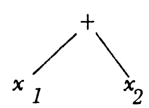
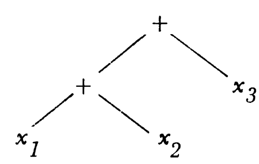
For any set X we make $T(\Sigma, X)$ into a total Σ-algebra as follows: let $f \in \Sigma$ and $t_i \in T(\Sigma, X)$ for $i < r(f)$. The tree

$$t := f_{T(\Sigma, X)}(t_i \mid i < r(f))$$

is defined by

$$D_t := \{\lambda\} \cup \{im \mid i < r(f),\ m \in D_{t_i}\},$$
$$t(\lambda) = f \text{ and } t(im) = t_i(m)$$

for all $i < r(f)$ and $m \in D_{t_i}$. E. g.:

$t_1 =$  , $t_2 = x_3$, $f = +$, $t =$ 

Of course $F(\Sigma, X)$ is closed under all these operations, so it is also a total Σ-algebra. Let $\underline{A}$ be a partial Σ-algebra. Note that if $f \in \Sigma$ and $t_i \in F(\Sigma, A)$, $i < r(f)$, such that for each $i < r(f)$, $(t_i)_A$ is not defined, then $(f_{T(\Sigma, A)}(t_i \mid i < r(f)))_A$ is not defined either.

For any partial Σ-algebra $\underline{A}$, $\hat{A}$ can be made into the total Σ-algebra $\hat{\underline{A}}$ as follows: Let $f \in \Sigma$ and $t_i \in \hat{A}$, $i < r(f)$. Then $f_{\hat{A}}(t_i \mid i < r(f))$ is: (i) t_a defined below if $a := f_A((t_i)_A \mid i < r(f))$ is defined, and

(ii) $f_{F(\Sigma, A)}(t_i \mid i < r(f))$ otherwise.

Let $u_A : A \to \hat{A}$ be defined as follows:

$$\bigwedge_{a \in A} u_A(a) = t_a\,, \quad \text{where } D_{t_a} := \{\lambda\} \text{ and } t_a(\lambda) = a.$$

Identifying A with $u_A(A)$ we get:

PROPOSITION 1. *For any partial* Σ*-algebra* $\underline{A}$, $\underline{\hat{A}}$ *is the free* Σ*-completion of* $\underline{A}$. *(Or:* $(u_A, \underline{\hat{A}})$ *is the* Alg_Σ*-reflection of* $\underline{A}$.*)*

PROOF. It is easy to show that $\underline{\hat{A}}$ satisfies the Axiom of Free Completion given in Theorem 6 of Burmeister-Schmidt [4]. □

In the following we will proceed like this: On $\hat{A}$ we will define a *quasi-order* $<$ with the following properties:

1. $<$ restricted to $A \times A$ is $\leq_A$.
2. The operations of $\hat{A}$ are monotonic with respect to $<$.
3. $<$ preserves suprema of sets in $Z(A)$.

Let

$$R_< := \{\langle a, b\rangle \in \hat{A} \times \hat{A} \mid a < b \text{ and } b < a\}.$$

Then $R_<$ is a congruence relation on $\hat{A}$ and

$$R_< \cap (A \times A) = Id_A .$$

Let $h_< : \underline{\hat{A}} \to \underline{\hat{A}}/R_<$ be the canonical homomorphism. Then

$$(h_< \circ <) =: \leq$$

is a partial order on $\hat{A}/R_<$ and

1. $\underline{\hat{A}}/R_< \in Ob(^\perp Z\, Alg_\Sigma)$.
2. $\underline{A}$ can be identically embedded into $\underline{\hat{A}}/R_<$ and this embedding is a Z-continuous, $\perp$-preserving homomorphism.

NOTATION. *id* denotes the identity function symbol, i.e. for any set A, id_A is the identity map on A. Of course $r(id) := 1$.

DEFINITION 1. For every ordinal α we define $<_\alpha \subset \hat{A} \times \hat{A}$ as follows:

(A) $a \leq_A b \iff a <_0 b$.

Let $\alpha > 0$. Then $a <_\alpha b$ iff either (B) or (C) holds:

(B) $$\bigvee_{f \in \Sigma \cup \{id\}} \ \bigwedge_{i < r(f)} \ \bigvee_{a_i, b_i \in \hat{A}} \ \bigvee_{\alpha_i < \alpha} (a = f_{\hat{A}}(a_i \mid i < r(f)) \wedge$$
$$b = f_{\hat{A}}(b_i \mid i < r(f)) \wedge \bigwedge_{i < r(f)} a_i <_{\alpha_i} b_i).$$

$$(C) \quad \bigvee_{X \in Z(A)} \bigvee_{\beta < \alpha} (a <_\beta \sup_{\leq_A} X \wedge \bigwedge_{x \in X} \bigvee_{\alpha_x < \alpha} x <_{\alpha_x} b).$$

Let

$$< \; := \bigcup_{\alpha \in Ord} <_\alpha \cup \{ (\perp, a) \mid a \in \hat{A} \}.$$

REMARKS. 1. Applying (B) for $f = id$, we get

$$\bigwedge_{\alpha \in Ord} ((\bigvee_{\beta < \alpha} a <_\beta b) \Rightarrow a <_\alpha b).$$

2. If Z is the trivial subset system Φ containing only the two-element chain, then (C) is equivalent to

$$(C') \quad \bigvee_{c \in A} \bigvee_{\beta, \gamma < \alpha} a <_\beta c <_\gamma b.$$

Since we assume that every subset system contains Φ, (C') implies $a <_\alpha b$ by applying (C) to $X = \{c\}$.

3. Notice that for $Z = \Phi$ this definition of $<$ is equivalent to Definition of $<$ in Pasztor [10], but here we do not use the special tree-construction of $\hat{A}$ and the proofs are much simpler, especially more transparent.

PROPOSITION 2. $\bigwedge_{a, b \in A} (a < b \Rightarrow a \leq_A b)$.

PROOF. Suppose $a, b \in A$ and $a < b$. Then $a <_\alpha b$ for some $\alpha \in Ord$.

a) If $\alpha = 0$, then by (A), $a \leq_A b$.

b) Suppose $\alpha > 0$ and that

$$\bigwedge_{\beta < \alpha} \bigwedge_{a, b \in A} (a <_\beta b \Rightarrow a \leq_A b).$$

Then one of (ba) and (bb) below holds:

ba) $a = f_{\hat{A}}(a_i \mid i < r(f))$, $b = f_{\hat{A}}(b_i \mid i < r(f))$
for some $f \in \Sigma \cup \{id\}$ and some $a_i, b_i \in \hat{A}$, $i < r(f)$ and

$$\bigwedge_{i < r(f)} \bigvee_{\alpha_i < \alpha} a_i <_{\alpha_i} b_i.$$

Since $a, b \in A$,

$$f_{\hat{A}}(a_i \mid i < r(f)) = f_A(a_i \mid i < r(f)), \quad \bigwedge_{i < r(f)} a_i \in A,$$

$$f_{\hat{A}}(b_i \mid i < r(f)) = f_A(b_i \mid i < r(f)) \text{ and } \bigwedge_{i < r(f)} b_i \in A.$$

By the induction hypothesis, $\bigwedge_{i < r(f)} a_i \leq_A b_i$ and then, by the monotonicity of f_A, $a \leq_A c$.

(bb) $a <_\beta \underset{\leq_A}{sup}\, X$ for some $X \in Z(A)$ and some $\beta < \alpha$, and

$$\bigwedge_{x \in X} \bigvee_{\alpha_x < \alpha} x <_{\alpha_x} b.$$

Then by the induction hypothesis $a \leq_A sup\, X$, $\bigwedge_{x \in X} x \leq_A b$, hence $a \leq_A b$.

PROPOSITION 3. $<$ *is a quasi-order.*

PROOF. 1. Reflexivity: Let $a \in \hat{A}$ be arbitrary. If $a \in A$, then $a \leq_A a$, hence by (A) $a <_0 a$. Suppose $a \notin A$. Then $a = t$ for a unique tree t. We prove $a < a$ by induction on the depth $d(t)$ of the tree t.

a) If $d(t) = 1$, then $D_t = \{\lambda\}$ and $t(\lambda) = c \in \Sigma$ a constant symbol. Since $a = t = c_A^\wedge$, $\bigwedge_{\alpha \in Ord} t <_\alpha t$ by (B) ($r(c) = 0$!). Hence $a < a$.

b) Let $d(t) = \sigma$ and $\sigma > 1$. Then $t(\lambda) = f \in \Sigma$ is not a constant symbol and $\bigwedge_{i < r(f)} d(t \setminus i) < \sigma$. By the induction hypothesis

$$\bigwedge_{i < r(f)} (t \setminus i) < (t \setminus i), \text{ i.e. } \bigwedge_{i < r(f)} \bigvee_{\alpha_i \in Ord} (t \setminus i) <_{\alpha_i} (t \setminus i).$$

Since

$$a = t = f_A^\wedge((t \setminus i) \mid i < r(f)),$$

by (B) we get $a <_\alpha a$ for an α greater than every α_i ($i < r(f)$). Hence $a < a$.

2. Transitivity: Suppose $a < b < c$, i.e. $a <_\alpha b <_\beta c$ for some α, β in Ord. If $a = \bot$, then $a < c$ per definitionem. Suppose $a \neq \bot$. We prove that $a < c$ *by induction on* α.

a) Let $\alpha = 0$. Then by (A) $b \in A$ and then by (C') $a <_{\beta+1} c$, hence $a < c$.

b) Let $\alpha > 0$ and suppose

$$\bigwedge_{\gamma < \alpha} \bigwedge_{a, b, c \in \hat{A}} (a <_\gamma b <_\beta c \Rightarrow a < c).$$

For $a <_\alpha b$ one of (ba) or (bb) below holds:

(ba) $$\bigvee_{X \in Z(A)} \bigvee_{\beta < \alpha} (a <_\beta \underset{\leq_A}{sup}\, X \wedge \bigwedge_{x \in X} \bigvee_{\alpha_x < \alpha} x <_{\alpha_x} b).$$

Then by the induction hypothesis $\bigwedge_{x \in X} \bigvee_{\beta_x \in Ord} x <_{\beta_x} c$ hence by (C) $a <_\tau c$ for some $\tau > \beta$, β_x ($x \in X$), i.e. $a < c$.

$$\text{(bb)} \quad \bigvee_{f \in \Sigma \cup \{id\}} \ \bigwedge_{i<r(f)} \ \bigvee_{a_i,\, b_i \in \hat{A}} (a = f_A^{\hat{}}(a_i \mid i < r(f)) \wedge b = f_A^{\hat{}}(b_i \mid i < r(f)) \wedge \bigwedge_{i<r(f)} \ \bigvee_{\alpha_i < \alpha} a_i <_{\alpha_i} b_i).$$

If $f = id$ then the induction hypothesis applies immediately and $a < c$. If $f \neq id$, then we prove $a < c$ *by induction on* β.

bba) If $\beta = 0$, then $a <_\alpha b <_0 c$ implies by (A) and by (C') (analogously to a) $a < c$ (without using the special form bb of $a <_\alpha b$).

bbb) Suppose

$$\beta > 0 \quad \text{and} \quad \bigwedge_{\gamma<\beta} \ \bigwedge_{c \in A} (a <_\alpha b <_\gamma c \implies a < c).$$

Then one of (bbba) or (bbbb) below holds:

$$\text{bbba)} \quad \bigvee_{X \in Z(A)} \ \bigvee_{\gamma<\beta} (b <_\gamma \sup_{\leq_A} X \wedge \bigwedge_{x \in X} \ \bigvee_{\beta_x<\beta} x <_{\beta_x} c).$$

Using the induction hypothesis for $a <_\alpha b$ and $b <_\gamma \sup_{\leq_A} X$, we get

$$a < \sup_{\leq_A} X, \quad \text{i.e.} \quad a <_\tau \sup_{\leq_A} X \ \text{for some} \ \tau \in Ord.$$

By (C) $a <_\sigma c$ for some $\sigma > \tau$, β_x $(x \in X)$, hence $a < c$.

$$\text{bbbb)} \quad \bigvee_{g \in \Sigma \cup \{id\}} \ \bigwedge_{i<r(g)} \ \bigvee_{b_i',\, c_i \in \hat{A}} (b = g_A^{\hat{}}(b_i' \mid i < r(g)) \wedge c = g_A^{\hat{}}(c_i \mid i < r(g)) \wedge \bigwedge_{i<r(g)} \ \bigvee_{\beta_i<\beta} b_i' <_{\beta_i} c_i).$$

If $g = id$, the induction hypothesis *on* β immediately applies and $a < c$ (without using the special form bb of $a <_\alpha b$). If $g \neq id$ but $b \in A$, then $a <_\alpha b <_\beta c$ and by (C') $a < c$. (We do not use bb). Suppose $g \neq id$ and $b \notin A$. Then by the definition of $\hat{A}$, $f = g$ and

$$\bigwedge_{i<r(f)} (b_i = b_i' \wedge a_i <_{\alpha_i} b_i <_{\beta_i} c_i).$$

By (B) $\bigwedge_{i<r(f)} b_i <_\beta c_i$ (see Remark 1 to Definition 1). By the induction hypothesis on α (see b) we get then $\bigwedge_{i<r(f)} a_i < c_i$. Then by (B) $a < c$. □

PROPOSITION 4. *The operations of* $\underline{A}$ *are monotonic with respect to* $<$.

PROOF. By (B).

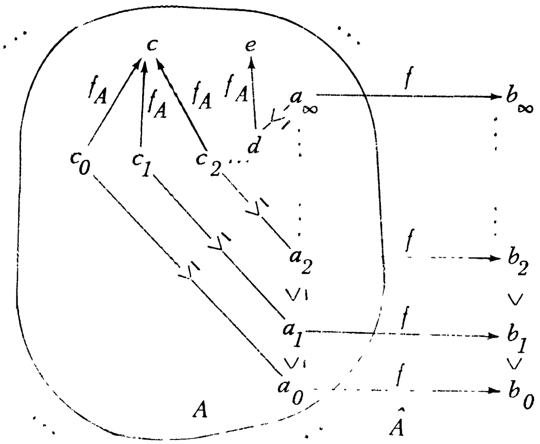
PROPOSITION 5. $<$ *preserves suprema of elements of* $Z(A)$.

PROOF. By (C).

REMARK. $\underline{A}/R_{<}$ is not Z-continuous. The following example demonstrates this. Let $\Sigma := \{f\}$ with $r(f) = 1$. Then $a_\infty = sup(a_n)_{n \in \omega}$ but

$$b_\infty := f_{\hat{A}}(a_\infty) \neq \underset{<}{sup}\,(f_{\hat{A}}(a_n))_{n \in \omega}$$

since $\bigwedge_{n \in \omega} b_n := f_{\hat{A}}(a_n) < c$, but $b_\infty \nless c$.



We cannot even force by definition $b_\infty < c$, because if we did it, by transitivity $e < b_\infty < c$ would imply $e < c$, but this would contradict $e \nleq_A c$.

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