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PULLBACK AND PUSHOUT SQUARES IN A SPECIAL DOUBLE CATEGORY WITH CONNECTION

by C. B. SPENCER and Y. L. WONG

INTRODUCTION.

This paper continues the work of [12] in an attempt to rephrase classical homotopy theory in the setting of a special double category with connection via the concepts of pullback and pushout squares. In the present paper we adopt the more convenient notation of thin squares employed initially by R. Brown and P. J. Higgins in their discussion of double groupoids and other higher dimensional objects [1, 3, 4 and 5]. Most of the results obtained arose from an effort to generalise the «cube theorems» of [11] to the present setting. While it was found that they are not true in general, the second author has obtained such results in the category of chain complexes [13]. The first two sections are devoted to basic ideas and definitions and in Section 3 we discuss homotopy equivalences and homotopy commutative cubes. An equivalence relation on squares, similar to that given in [11], is established. Next, pullback and pushout squares are recalled and some of their further properties established. We obtain versions of the well-known glueing and coglueing theorems in homotopy theory [2]. We include examples of pullback and pushout in the special double category with connection of chain complexes CC. This will be discussed further in [13].

1. DOUBLE CATEGORIES.

DEFINITION 1.1. A *double category* D consists of sets D_0, D_1 and D_2 (of points, edges and squares, resp.) together with boundary maps

$$\partial_j^\alpha : D_{i+1} \to D_i \,,\quad \alpha = 0, 1,\quad j = 0 \text{ for } i = 0,\quad j = 1, 2 \text{ for } i = 1,$$

and degeneracy maps

$$1 : D_0 \to D_1 \,,\quad \overline{} : D_1 \to D_2 \text{ and } |\;| : D_1 \to D_2$$

satisfying the rules for a two-dimensional cubical complex such that $(D_1, D_0, \partial_0^0, \partial_0^1, 1)$ is a category (composition in which is denoted by juxtaposition). In addition we must have composition laws $\underset{1}{+}$ and $\underset{2}{+}$, called resp. horizontal and vertical composition, defined on D_2 satisfying the following axioms :

(i) for each $a, b \in D_2$, $a \underset{i}{+} b$ is defined whenever

$$\partial_i^1 a = \partial_i^0 b, \quad i = 1, 2.$$

(ii)

$$\partial_j^\alpha (a \underset{i}{+} b) = \begin{cases} \partial_i^0 a & \text{if } \alpha = 0,\ i = j \\ \partial_i^1 b & \text{if } \alpha = 1,\ i = j \\ \partial_j^\alpha b\, \partial_j^\alpha a & \text{if } i \neq j. \end{cases}$$

(iii) for each $f \in D_1$, $=_f$ (resp. $\| _f$) is the identity for the composition $\underset{1}{+}$ (resp. $\underset{2}{+}$). $=_f$ and $\|_f$ are called resp. the horizontal and vertical identity.

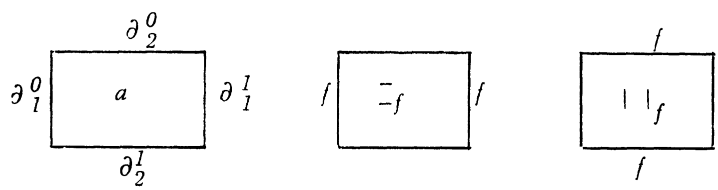
(iv) for each $x \in D_0$, $=_{1_x} = \|_{1_x}$. This square is called the double identity and is denoted by $\square_x$.

(v) $=_{fg} = =_g \underset{2}{+} =_f$ and $\|_{fg} = \|_g \underset{1}{+} \|_f$.

(vi) (the interchange law)

$$(a \underset{1}{+} b) \underset{2}{+} (c \underset{1}{+} d) = (a \underset{2}{+} c) \underset{1}{+} (b \underset{2}{+} d).$$

To make the picture more clear



are sometimes used to represent a square, $=_f$ and $\|_f$ in D_2. It is also convenient to use a matrix notation for compositions of squares. Thus, if a, b satisfy $\partial_1^1 a = \partial_1^0 b$, we write $[a\ b]$ for $a \underset{1}{+} b$, and if $\partial_2^1 a = \partial_2^0 b$, we write $\begin{bmatrix} a \\ b \end{bmatrix}$ for $a \underset{2}{+} b$. More generally, for squares

$$a_{ij} \quad (1 \leq i \leq m,\ 1 \leq j \leq n)$$

satisfying

$$\partial_1^1 a_{i(j-1)} = \partial_1^0 a_{ij} \quad (1 \leq i \leq m, \quad 2 \leq j \leq n),$$
$$\partial_2^1 a_{(i-1)j} = \partial_2^0 a_{ij} \quad (2 \leq i \leq m, \quad 1 \leq j \leq n)$$

we write

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

for

$$(a_{11} \underset{2}{+} \cdots \underset{2}{+} a_{m1}) \underset{1}{+} \cdots \underset{1}{+} (a_{1n} \underset{2}{+} \cdots \underset{2}{+} a_{mn}).$$

The notion of double category was first introduced by C. Ehresmann in [7].

In our definition, the horizontal and vertical edges form the same category D_1 whereas in the general case they may be two different categories. Let D_1 be a category, then the double category with squares consisting of all those commutative squares in D_1 is denoted by $\square D_1$. Consider squares in D of the form

$$\begin{array}{ccc} & 1 & \\ f & \square & g \\ & 1 & \end{array} \quad ;$$

they form a substructure of D which is refered to as the *horizontal sub-double category* $h(D)$ *of* D. Similarly we have the concept of *vertical sub-double category* $v(D)$ *of* D. They clearly constitue a 2-category structure under $\underset{i}{+}$. For more detailed discussions of double categories, confer [6, 7, 8 and 10].

DEFINITION 1.2. A double category D is said to be *with connection* if a double functor $\Delta : \square D_1 \to D$ is defined such that $\Delta|_{(\square D_1)_1}$ is the identity functor.

Functions $\lrcorner, \ulcorner : D_1 \to D_2$ for which $\lrcorner_f, \ulcorner_f$ have edges given by

$$\begin{array}{ccc} & f & \\ f & \boxed{\;\lrcorner_f\;} & 1 \\ & 1 & \end{array} \quad , \quad \begin{array}{ccc} & 1 & \\ 1 & \boxed{\;\ulcorner_f\;} & f \\ & f & \end{array}$$

are determined by restricting Δ to squares of $\square D_1$ of the form

$$\begin{array}{ccc} & f & \\ f & \boxed{\quad} & 1 \\ & 1 & \end{array} \quad \text{and} \quad \begin{array}{ccc} & 1 & \\ 1 & \boxed{\quad} & f \\ & f & \end{array} \quad \text{resp.}$$

The functions $\lrcorner$ and $\ulcorner$ are called connections and were first introduced in [3] to deal with double groupoids. Here we recall some of the properties of $\lrcorner$ and $\ulcorner$ described in [12].

(i) *Transport law:*

$$\begin{bmatrix} \lrcorner_f & \|_g \\ =_g & \lrcorner_g \end{bmatrix} = \lrcorner_{gf} \qquad\qquad \begin{bmatrix} \ulcorner_f & =_f \\ \|_f & \ulcorner_g \end{bmatrix} = \ulcorner_{gf}$$

(ii) $\lrcorner_{1_x} = \ulcorner_{1_x} = \square_x$.

(iii) $\ulcorner \underset{i}{+} \lrcorner = \square \quad (i = 1, 2).$

(iv) for $s = \begin{array}{ccc} & g & \\ f & \boxed{\;s\;} & p \\ & q & \end{array} \in \square D_1$,

$$\Delta(s) = \begin{bmatrix} =_f \\ \ulcorner_q \end{bmatrix} \underset{1}{+} \begin{bmatrix} \lrcorner_g \\ =_p \end{bmatrix}$$

DEFINITION 1.3. A double category D *is special* if $h(D)$ forms a groupoid under $\underset{1}{+}$.

For each $a \in h(D)$, $-a$ is used to denote its inverse. Here the word special has a different meaning from the literature, e. g. [6, 12].

From now on, we shall restrict our attention to *special double categories with connection.* Throughout this paper we shall use D to denote such an object. As examples, we have the special double categories with connection TT and CC of topological pointed spaces and chain complexes resp. In TT a square is the homotopy class of a homotopy from $\partial_1^1 a \partial_2^0 a$ to $\partial_2^1 a\, \partial_1^0 a$ and in CC, a is a chain homotopy from $\partial_1^1 a\, \partial_2^0 a$ to $\partial_2^1 a \partial_1^0 a$.

We recall from [12] that every 2-category determines a special double category with connection. From [12] we also recall the process of reflecting squares. For each

$$\begin{array}{ccc} & g & \\ f & \boxed{\quad a \quad} & p \\ & q & \end{array}$$

in D_2, let $T(a) \in h(D)$ denote the square

$$\ulcorner_g +_2 a +_2 \lrcorner_q \quad \text{(or more simply } \begin{bmatrix} \ulcorner \\ a \\ \lrcorner \end{bmatrix} \text{)}.$$

The function $r: D_2 \to D_2$ defined by

$$r(a) = \begin{bmatrix} =_f \\ \ulcorner_q \end{bmatrix} +_1 -T(a) +_1 \begin{bmatrix} \lrcorner_g \\ =_p \end{bmatrix}$$

is called the *reflection*.

PROPOSITION 1.1. *(i) Let*

$$\begin{array}{ccc} & g & \\ f & \boxed{\quad s \quad} & p \in \square D_1, \\ & q & \end{array}$$

then

$$\ulcorner_g +_2 \Delta(s) +_2 \lrcorner_q = =_{pq} = =_{qf},$$

$$\ulcorner_g +_1 r\Delta(s) +_1 \lrcorner_q = \|_{pq} = \|_{qf}.$$

(ii)

$$T(a +_1 b) = \begin{bmatrix} T(a) \\ =_{\partial_2^1 b} \end{bmatrix} +_1 \begin{bmatrix} =_{\partial_2^0 a} \\ T(b) \end{bmatrix},$$

$$T(a +_2 b) = \begin{bmatrix} =_{\partial_1^0 a} \\ T(b) \end{bmatrix} +_1 \begin{bmatrix} T(a) \\ =_{\partial_1^1 b} \end{bmatrix}$$

whenever $a +_1 b$ or $a +_2 b$ is defined.

(iii) For any $a, b \in D_2$, *if* $\partial_i^\alpha a = \partial_i^\alpha b$, $\alpha = 0, 1$, $i = 1, 2$, *and* $T(a) = T(b)$, *then* $a = b$.

PROOF. (i) and (ii) follow from definitions.

(iii) Since $T(a) = T(b)$, we have

$$\begin{bmatrix} \ulcorner \\ a \\ \lrcorner \end{bmatrix} = \begin{bmatrix} \ulcorner \\ b \\ \lrcorner \end{bmatrix}$$

and since a and b have the same edges we can compose both sides on the left by $=_{\partial_1^0} +_2 \ulcorner_{\partial_2^1}$ and on the right by $\lrcorner_{\partial_2^0} +_2 =_{\partial_1^1}$ resulting in

$$\begin{bmatrix} \square & \ulcorner & \lrcorner \\ = & a & = \\ \ulcorner & \lrcorner & \square \end{bmatrix} = \begin{bmatrix} \square & \ulcorner & \lrcorner \\ = & b & = \\ \ulcorner & \lrcorner & \square \end{bmatrix}$$

and hence $a = b$.

PROPOSITION 1.2 *(Théorème 2.4 of [1.2]).*

(i) $r(a +_1 b) = r(a) +_2 r(b)$.

(ii) $r(a +_2 b) = r(a) +_1 r(b)$.

(iii) $r^2 = id$.

(iv) $r : h(D) \to v(D)$ *is an isomorphism (in the sense of a double functor or a 2-functor).*

2. THIN SQUARES.

DEFINITION 2.1. $a \in D$ is *thin* (or degenerate in [12]) if a has a decomposition consisting of $\square$, $=_a$, $||_b$, $\lrcorner_c$, $\ulcorner_d$ *for some edges* a, b, c, d.

The concept of thin squares was first introduced by Brown , in the application of double groupoids to problems in topology, see for example [1, 3]. In TT, thin squares actually correspond to those with a constant homotopy filling. The essential feature of thin squares is described in the following proposition.

PROPOSITION 2.2.

(i) $a \in D_2$ *is thin iff* $a = \Delta(s)$ *for some* $s \in \square D_1$.

(ii) Let $s \in \square D_1$, *then* $\Delta(s)$ *is the unique thin square having the edges of* s.

PROOF. (i) Suppose $a \in D_2$ is thin so that

$$a = [\, a_{ij} \,], \quad \text{where} \quad a_{ij} = \square,\ \equiv,\ \lrcorner \ \text{ or } \ulcorner$$

and they are in $\square D_1$. Therefore

$$a = [\, \Delta(b_{ij})] \quad \text{where} \quad b_{ij} = \square,\ \equiv,\ ||,\ \lrcorner \text{ or } \ulcorner$$

and they are in $\square D_1$. By the double functor property of Δ,

$$a = \Delta\, [\, b_{ij} \,] \quad \text{and} \quad [\, b_{ij} \,] \in \square D_1\, ,$$

which means $a = \Delta(s)$ for some $s \in \square D_1$. The converse is obvious.

(ii) Proposition 1.1 of [12].

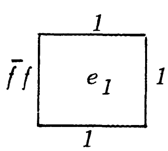
Since any composition of thin squares is thin and every thin square is determined by its edges it follows that one can identify a decomposition of thin squares by simple noting its edges. In each of the thin squares $\square$, $\equiv$, $||$, $\lrcorner$ or $\ulcorner$, the edges supplied represent identities. We shall sometimes omit the subscript of a thin square when it can be deduced from other information, for example, the edge of a neighbouring square. This notation was introduced by Brown in [1].

3. HOMOTOPIES AND HOMOTOPY COMMUTATIVE CUBES.

As indicated in [11], the familiar modulo homotopy category arising from a 2-category leads to a homotopy theory in a special double category with connection D, in which f and g in D_1 are said to be *homotopic* if there exists $a \in h(D)$ such that

$$\partial_1^0 a = f \quad \text{and} \quad \partial_1^1 a = g .$$

In this context, f is a *homotopy equivalence* if there is a homotopy inverse $\bar f$ and squares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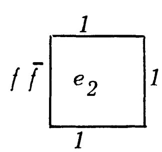


in $h(D)$; and furthermore, f is a *strong homotopy equivalence* if

$$\equiv_f \underset{2}{+} e_2 = e_1 \underset{2}{+} \equiv_{\bar f} \quad \text{and} \quad ||_{\bar f} \underset{2}{+} e_1 = e_2 \underset{2}{+} ||_f .$$

PROPOSITION 3.1. *f is a homotopy equivalence iff f is a strong homotopy equivalence.*

PROOF. Proposition 2.3 of [11].

PROPOSITION 3.2. *Let $f \in D_1$ and $g \boxed{a} h$, $g \boxed{b} h \in h(D)$. If*

$$a +_2 \boxminus_f = b +_2 \boxminus_f$$

and f is a homotopy equivalence, then $a = b$.

PROPOSITION 3.3. *Suppose $fg \boxed{b} fg \in h(D)$ with f a homotopy equivalence, then there exists a unique square*

$$a = g \boxed{a} h \in h(D) \quad \text{such that} \quad a +_2 \boxminus_f = b.$$

PROPOSITION 3.4. *(i) Let $a \in h(D)$. Then $\partial_1^0 a$ is a homotopy equivalence iff $\partial_1^1 a$ is a homotopy equivalence.*

(ii) Let $f, g \in D_1$. If any two edges from f, g, fg are homotopy equivalences, then the third one is also a homotopy equivalence.

Propositions 3.2, 3.3 and 3.4 are direct consequences of the corresponding results in the modulo homotopy category derived from the 2-category $h(D)$. However, we include below a proof of 3.2 as a simple demonstration of how we may work entirely with squares exploiting where appropriate the notion of thin square.

PROOF of Proposition 3.2. Let $\bar{f}$ be the homotopy inverse of f with homotopies

$$\bar{f}f \boxed{\;e_1\;} 1 \quad \text{and} \quad f\bar{f} \boxed{\;e_2\;} 1$$

Composing $a +_2 \boxminus_f = b +_2 \boxminus_f$ with $\boxminus_{\bar{f}}$, we have

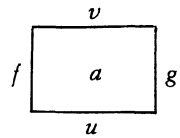
$$a +_2 \boxminus_{\bar{f}f} = b +_2 \boxminus_{\bar{f}f}$$

Whence

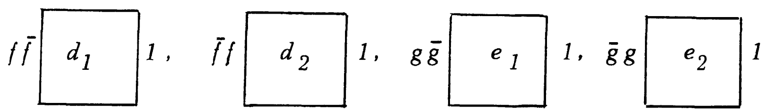
$$\begin{bmatrix} \square & a & \square \\ \cdot e_1 & \boxminus & e_1 \end{bmatrix} = \begin{bmatrix} \square & b & \square \\ \cdot e_1 & \boxminus & e_1 \end{bmatrix}$$

and so $a = b$.

Next we shall describe a process for vertically inverting squares of the form



where f and g are homotopy equivalences. To carry out this process we shall need the respective homotopy inverses $\bar{f}$ and $\bar{g}$ and homotopies

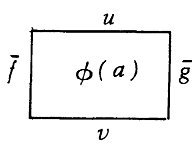


satisfying

$$\overline{-}_{f} \underset{2}{+} d_2 = d_1 \underset{2}{+} \overline{-}_{\bar{f}}, \quad \overline{-}_{\bar{f}} \underset{2}{+} d_1 = d_2 \underset{2}{+} \overline{-}_{f},$$

$$\overline{-}_{g} \underset{2}{+} e_2 = e_1 \underset{2}{+} \overline{-}_{\bar{g}}, \quad \overline{-}_{\bar{g}} \underset{2}{+} e_1 = e_2 \underset{2}{+} \overline{-}_{g}$$

(cf. Proposition 3.1). The resulting inverted square



is defined by

$$\phi(a) = \begin{bmatrix} \overline{-}_{\bar{f}} & \overline{-}_{\bar{f}} \\ \ulcorner_{v} & \overline{-}_{v} \\ ||_{v} & \cdot e_2 \end{bmatrix} \underset{1}{+} \begin{bmatrix} \overline{-}_{\bar{f}} \\ -T(a) \\ \overline{-}_{\bar{g}} \end{bmatrix} \underset{1}{+} \begin{bmatrix} d_1 & ||_{u} \\ \overline{-}_{u} & \lrcorner_{u} \\ \overline{-}_{\bar{g}} & \overline{-}_{\bar{g}} \end{bmatrix}$$

PROPOSITION 3.5. *(i)* $\bar{\phi}(\phi(a)) = a$, *where* $\bar{\phi}$ *is defined similarly to* ϕ *with* f *and* $\bar{f}$, g *and* $\bar{g}$, d_1 *and* d_2, e_1 *and* e_2, u *and* v *interchanged.*

(ii) $a \underset{2}{+} \phi(a) = d_2 \underset{1}{+} ||_{v} \underset{1}{+} \cdot e_2$,

(iii) $\phi(a) \underset{2}{+} a = d_1 \underset{1}{+} ||_{u} \underset{1}{+} \cdot e_1$.

(iv) If the square $\bar{a}$ *satisfies*

$$a \underset{2}{+} \bar{a} = d_2 \underset{1}{+} ||_{v} \underset{1}{+} \cdot e_2 \quad \text{or} \quad \bar{a} \underset{2}{+} a = d_1 \underset{1}{+} ||_{u} \underset{1}{+} \cdot e_1,$$

then $\bar{a} = \phi(a)$.

PROOF. We shall only prove the first part of (iv). Firstly, applying ∂^0_1 to the first equation, we see that $\partial^0_1 \bar{a} f = \bar{f} f$. Since, by Proposition 3.3, there exists a unique square $\partial^0_1 \bar{a} \boxed{e} \bar{f}$ in $h(D)$ such that $e \underset{2}{+} \ulcorner_f = \ulcorner_{f\bar{f}}$, e must be $\ulcorner_{\bar{f}}$, so that $\partial^0_1 \bar{a} = \bar{f}$. Similarly we have $\partial^1_1 \bar{a} = \bar{g}$. That $\partial^\alpha_2 \bar{a} = \partial^\alpha_2 \phi(a)$ follows immediately from the fact that $a \underset{2}{+} \bar{a}$ is defined and the first equation. Therefore, $\bar{a}$ and $\phi(a)$ have the same edges. Next we apply T:

$$T(a \underset{2}{+} \bar{a}) = \begin{bmatrix} \ulcorner_f \\ T(\bar{a}) \end{bmatrix} \underset{1}{+} \begin{bmatrix} T(a) \\ \ulcorner_{\bar{g}} \end{bmatrix} = \begin{bmatrix} d_2 \\ \ulcorner_v \\ \cdot e_2 \end{bmatrix}.$$

Composing both sides of the second equality at the top by $\ulcorner_{\bar{f}}$ we have

$$\begin{bmatrix} \ulcorner_{\bar{f}} \\ \ulcorner_f \\ T(\bar{a}) \end{bmatrix} = \begin{bmatrix} \ulcorner_{\bar{f}} \\ d_2 \\ \ulcorner_v \\ \cdot e_2 \end{bmatrix} \underset{1}{+} \cdot \begin{bmatrix} \ulcorner_{\bar{f}} \\ T(a) \\ \ulcorner_{\bar{g}} \end{bmatrix} = \begin{bmatrix} d_1 \\ \ulcorner_{v\bar{f}} \\ \cdot e_2 \end{bmatrix} \underset{1}{+} \cdot \begin{bmatrix} \ulcorner_{\bar{f}} \\ T(a) \\ \ulcorner_{\bar{g}} \end{bmatrix}.$$

Composing both sides on the left with $\cdot d_1 \underset{2}{+} \ulcorner_{v\bar{f}}$ gives

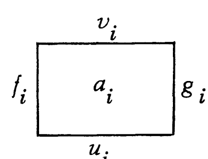
$$\begin{bmatrix} \cdot d_1 \\ T(\bar{a}) \end{bmatrix} = \begin{bmatrix} \ulcorner_{v\bar{f}} \\ \cdot e_2 \end{bmatrix} \underset{1}{+} \cdot \begin{bmatrix} \ulcorner_{\bar{f}} \\ T(a) \\ \ulcorner_{\bar{g}} \end{bmatrix}$$

and finally, composing on the right with $d_1 \underset{2}{+} \ulcorner_{\bar{g}u}$ we arrive at

$$T(\bar{a}) = \begin{bmatrix} \ulcorner_{v\bar{f}} \\ \cdot e_2 \end{bmatrix} \underset{1}{+} \begin{bmatrix} \ulcorner_{\bar{f}} \\ \cdot T(a) \\ \ulcorner_{\bar{g}} \end{bmatrix} \underset{1}{+} \begin{bmatrix} d_1 \\ \ulcorner_{g\bar{u}} \end{bmatrix} = T(\phi(a)).$$

Hence, since $\bar{a}$ and $\phi(a)$ have the same edges, by Proposition 1.1 (c), $\bar{a} = \phi(a)$, completing the proof.

Now, for $i = 1, 2$, let



be a square with homotopies

$$f_i\bar{f}_i \boxed{\quad d^i_1 \quad} 1, \quad \bar{f}_i f_i \boxed{\quad d^i_2 \quad} 1, \quad g_i\bar{g}_i \boxed{\quad e^i_1 \quad} 1, \quad \bar{g}_i g_i \boxed{\quad e^i_2 \quad} 1$$

satisfying the conditions for f_i and g_i to be strong homotopy equivalences. We then have corresponding inverted squares

$$\begin{array}{ccc} & u_i & \\ \bar{f}_i & \boxed{\quad \phi_i(a_i) \quad} & \bar{g}_i \\ & v_i & \end{array}$$

If $g_1 = f_2$, then

$$a_1 \underset{1}{+} a_2 = \begin{array}{ccc} & v_2 v_1 & \\ f_1 & \boxed{\quad b \quad} & g_2 \\ & u_2 u_1 & \end{array}$$

is defined and the homotopies d^1_α, e^2_α $(\alpha = 1, 2)$ determine the inverted square

$$\phi(b) = \begin{array}{ccc} & u_2 u_1 & \\ \bar{f}_1 & \boxed{\qquad} & \bar{g}_2 \\ & v_2 v_1 & \end{array}$$

We then have

COROLLARY 3.6. *$\phi(a_1 \underset{1}{+} a_2) = \phi\ (a_1) \underset{1}{+} \phi_2(a_2)$, provided $e^1_2 = d^2_2$.*

PROOF. The result follows directly from Proposition 3.5 (ii) and (iv) by considering

$$(a_1 \underset{1}{+} a_2) \underset{2}{+} (\phi(a_1) + \phi(a_2)).$$

Similarly if $u_1 = v_2$,

$$a_1 \underset{2}{+} a_2 = f_2 f_1 \begin{array}{ccc} & v_1 & \\ & \boxed{\quad c \quad} & g_2 g_1 \\ & u_2 & \end{array}$$

is defined and the homotopies

$$f_2 f_1 \bar{f}_1 \bar{f}_2 \boxed{\quad D_1 \quad} 1 = ({}^{-}_{\bar{f}_2} \underset{2}{+} d^1_1 \underset{2}{+} {}^{-}_{f_2}) \underset{1}{+} d^2_1 ,$$

$$\bar{f}_1 \bar{f}_2 f_2 f_1 \boxed{\quad D_2 \quad} 1 = ({}^{-}_{f_1} \underset{2}{+} d^2_2 \underset{2}{+} {}^{-}_{\bar{f}_1}) \underset{1}{+} d^1_2 ,$$

$$g_2 g_1 \bar{g}_1 \bar{g}_2 \boxed{\quad E_1 \quad} 1 = ({}^{-}_{\bar{g}_2} \underset{2}{+} e^1_1 \underset{2}{+} {}^{-}_{g_2}) \underset{1}{+} e^2_1$$

$$\bar{g}_1 \bar{g}_2 g_2 g_1 \boxed{\quad E_2 \quad} 1 = ({}^{-}_{g_1} \underset{2}{+} e^2_2 \underset{2}{+} {}^{-}_{\bar{g}_1}) \underset{1}{+} e^1_2$$

satisfy the conditions required to show $f_2 f_1$ and $g_2 g_1$ are strong homotopy equivalences. For example, we have

$$\begin{aligned}
{}^{-}_{\bar{f}_1\bar{f}_2} \underset{2}{+} D_2 &= ({}^{-}_{f_1\bar{f}_1\bar{f}_2} \underset{2}{+} d^2_2 \underset{2}{+} {}^{-}_{\bar{f}_1}) \underset{1}{+} ({}^{-}_{\bar{f}_1\bar{f}_2} \underset{2}{+} d^1_2) \\
&= ({}^{-}_{f_1\bar{f}_1\bar{f}_2} \underset{2}{+} d^2_2 \underset{2}{+} {}^{-}_{\bar{f}_1}) \underset{1}{+} ({}^{-}_{\bar{f}_2} \underset{2}{+} d^1_1 \underset{2}{+} {}^{-}_{\bar{f}_1}) \\
&= ({}^{-}_{\bar{f}_2} \underset{2}{+} d^1_1 \underset{2}{+} {}^{-}_{\bar{f}_1\bar{f}_2 f_2}) \underset{1}{+} ({}^{-}_{\bar{f}_2} \underset{2}{+} d^2_2 \underset{2}{+} {}^{-}_{\bar{f}_1}) \\
&= ({}^{-}_{\bar{f}_2} \underset{2}{+} d^1_1 \underset{2}{+} {}^{-}_{\bar{f}_1\bar{f}_2 f_2}) \underset{1}{+} (d^1_1 \underset{2}{+} {}^{-}_{\bar{f}_1\bar{f}_2}) \\
&= D_1 \underset{2}{+} {}^{-}_{\bar{f}_1\bar{f}_2} .
\end{aligned}$$

Hence, the homotopies D_α, E_α $(\alpha = 1, 2)$ determine the inverted square

$$\phi'(c) = \quad \bar{f}_1 \bar{f}_2 \overset{u_2}{\underset{v_1}{\boxed{\qquad\qquad}}} \bar{g}_1 \bar{g}_2$$

We may now state a further corollary to Proposition 3.5.

COROLLARY 3.7. $\phi'(a_1 \underset{2}{+} a_2) = \phi_2(a_2) \underset{2}{+} \phi_1(a_1)$.

PROOF.

$$a_1 \underset{2}{+} a_2 + \phi_2(a_2) + \phi_1(a_1) = a_1 \underset{2}{+} (d_2^2 \underset{1}{+} \|_{u_1} \underset{1}{+} \cdot e_2^2) \underset{2}{+} \phi_1(a_1),$$

(by Proposition 3.5 (ii))

$$= \begin{bmatrix} -f_1 & a_1 & -g_1 \\ d_2^2 & \|_{u_1} & \cdot e_2^2 \\ -\bar{f}_1 & \phi_1(a_1) & -\bar{g}_1 \end{bmatrix}$$

$$= (-f_1 \underset{2}{+} d_2^2 \underset{2}{+} -\bar{f}_1) \underset{1}{+} d_2^2 \underset{1}{+} \|_{v_1} \underset{1}{+} \cdot e_2^1 \underset{1}{+} (-g_1 \underset{2}{+} \cdot e_2^2 \underset{2}{+} -\bar{g}_1)$$

$$= D_2 \underset{1}{+} \|_{v_1} \underset{1}{+} \cdot E_2 .$$

Hence, by Proposition 3.5 (iv), we have the required result.

We shall devote the remainder of this section to a discussion of homotopy commutative cubes. The definition is modelled on the corresponding concept in topology as described in [11]. The importance of cubes in the setting of multiple categories was made evident in [8].

DEFINITION 3.8. The subset $\{a_0, a_1, b_1, b_2, b_3, b_4\}$ of D_2 is called a *cube* if:

(i) $\partial_1^0 b_1 = \partial_1^0 b_2$, $\partial_1^1 b_3 = \partial_1^1 b_4$, $\partial_1^1 b_1 = \partial_1^0 b_3$, $\partial_1^1 b_2 = \partial_1^0 b_4$,

(ii) $\partial_2^i b_1 = \partial_2^0 a_i$, $\partial_2^i b_2 = \partial_1^0 a_i$, $\partial_2^i b_3 = \partial_1^1 a_i$, $\partial_2^i b_4 = \partial_2^1 a_i$ $(i = 0, 1)$.

It is said to be *homotopy commutative* if

$$T\begin{bmatrix} b_1 & b_4 \\ a_1 & \lrcorner_{\partial_1^1 a_1} \end{bmatrix} = T\begin{bmatrix} \ulcorner_{\partial_1^0 a_0} & a_0 \\ b_2 & b_4 \end{bmatrix}$$

Such a cube is diagrammatically represented as:

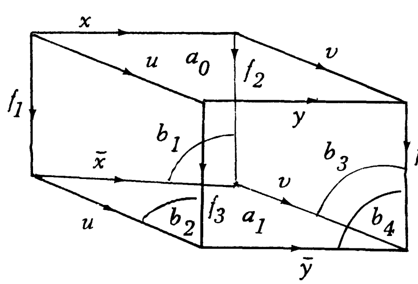


FIGURE 1

We shall frequently refer to the above cube.

PROPOSITION 3.9. *Let* $\{a_0, a_1, b_1, b_2, b_3, b_4\}$ *be a cube, then the following statements are equivalent:*

(i) $\{a_0, a_1, b_1, b_2, b_3, b_4\}$ *is homotopy commutative.*

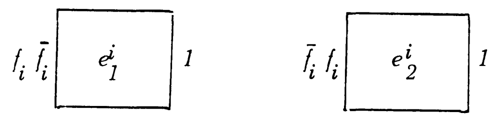
(ii) $$T\begin{bmatrix} a_0 & r(b_3) \\ b_4 & \lrcorner \end{bmatrix} = T\begin{bmatrix} \ulcorner & b_1 \\ r(b_2) & a_1 \end{bmatrix}$$

(iii) $$\begin{bmatrix} \ulcorner & b_1 +_1 \lrcorner \\ r(b_2 +_1 \lrcorner) & a_1 \end{bmatrix} = \begin{bmatrix} a_0 & r(b_3) \\ b_4 & \lrcorner \end{bmatrix}$$

(iv) $$(\ulcorner_{\bar f_2} + T(a_1)) +_1 (T(b_1) +_2 \lrcorner_{\bar v}) +_1 (\ulcorner_x +_2 T(b_3)) =$$
$$(T(b_2) +_2 \lrcorner_{\bar y}) + (\ulcorner_u +_2 T(b_4)) +_1 (T(a_0) +_2 \lrcorner_{\bar f_4}).$$

PROOF. This follows from direct simplifications of appropriate equations.

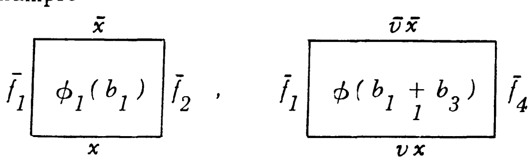
Let $\{a_0, a_1, b_1, b_2, b_3, b_4\}$ be a cube as shown in the figure, where f_i are homotopy equivalences with homotopy inverses $\bar f_i$ $(i = 1, \ldots, 4)$ and homotopies



satisfying the conditions for the f_i's to be strong homotopy equivalences. We may then use these homotopies to define the inverted squares

$$\phi_i(b_i)\ (i = 1, \ldots, 4),\quad \phi(b_1 +_1 b_3)\ \text{ and }\ \phi(b_2 +_1 b_4)$$

where for example



PROPOSITION 3.10. *If* $\{a_0, a_1, b_1, b_2, b_3, b_4\}$ *is a homotopy commutative cube then so is* $\{a_1, a_0, \phi_1(b_1), \phi_2(b_2), \phi_3(b_3), \phi_4(b_4)\}$.

PROOF. The condition of Definition 3.8 implies, with the help of Corollary 3.6,

$$\phi(b_1 +_1 b_3) +_2 (b_1 +_1 b_3) +_2 (\ulcorner_{\bar u} +_1 a_1 +_1 \lrcorner_{\bar v}) +_2 \phi(b_2 +_1 b_4) =$$
$$\phi(b_1 +_1 b_3) +_2 (\ulcorner_u +_1 a_0 +_1 \lrcorner_v) +_2 (b_2 +_1 b_4) +_2 \phi(b_2 +_1 b_4).$$

Hence, by Proposition 3.5 (ii) and (iii),

$$(e^1_1 +_1 \|_{\bar v \bar x} +_1 \cdot e^4_1) +_2 (\ulcorner +_1 a_1 +_1 \lrcorner) +_2 \phi(b_2 +_1 b_4) =$$
$$= \phi(b_1 +_1 b_3) +_2 (\ulcorner +_1 a_0 +_1 \lrcorner) +_2 (e^1_2 +_1 \|_{vx} +_1 \cdot e^4_2).$$

Rearrangement produces the equation:

$$\begin{bmatrix} e^1_1 & \ulcorner & a_1 & \lrcorner & \cdot e^4_1 \\ = & \phi_2(b_2) & \phi_4(b_4) & = & = \end{bmatrix} = \begin{bmatrix} = & = & \phi_1(b_1) & \phi_3(b_3) & = \\ e^1_2 & \ulcorner & a_0 & \lrcorner & \cdot e^4_2 \end{bmatrix}$$

whence, by the conditions for f_1 and f_4 to be strong homotopy equivalences, we have

$$\begin{bmatrix} \ulcorner & a_1 & \lrcorner \\ \phi_2(b_2) & \phi_4(b_4) & = \end{bmatrix} = \begin{bmatrix} = & \phi_1(b_1) & \phi_3(b_3) \\ \ulcorner & a_0 & \lrcorner \end{bmatrix}$$

from which the required result follows easily.

The next proposition says that vertical composition of homotopy commutative cubes preserves homotopy commutativity. Horizontal composition will similarly preserve homotopy commutativity.

PROPOSITION 3.11. *If*

$$\{a_1, a_2, b_1, b_2, b_3, b_4\} \text{ and } \{a_2, a_3, c_1, c_2, c_3, c_4\}$$

are homotopy commutative cubes, then

$$\{a_1, a_3, b_1 +_2 c_1, b_2 +_2 c_2, b_3 +_2 c_3, b_4 +_2 c_4\}$$

is a homotopy commutative cube.

PROOF. $$\begin{bmatrix} \ulcorner & (b_1 +_2 c_1) +_1 \lrcorner \\ r(b_2 +_2 c_2) +_1 \lrcorner & a_3 \end{bmatrix} =$$

$$\begin{bmatrix} \ulcorner & = & b_1 & \lrcorner & \| \\ \| & \ulcorner & c_1 & = & \lrcorner \\ r(b_2) & r(c_2) & a_3 & = & = \\ \lrcorner & \| & \| & \square & \square \\ = & \lrcorner & \| & \square & \square \end{bmatrix} = \begin{bmatrix} \ulcorner & = & b_1 +_1 \lrcorner \\ = & \ulcorner & c_1 +_1 \lrcorner \\ r(b_2 +_1 \lrcorner) & r(c_2 +_1 \lrcorner) & a_3 \end{bmatrix}$$

Here by homotopy commutativity of the second cube, the four terms in the lower right hand corner of the expression may be simplified to give:

$$\begin{bmatrix} \ulcorner & b_1 \underset{1}{+} \lrcorner & \vert\vert \\ r(b_2 \underset{1}{+} \square) & a_2 & r(c_3) \\ = & c_4 & \lrcorner \end{bmatrix}$$

which by homotopy commutativity of the first cube becomes

$$\begin{bmatrix} a_1 & r(b_3) & r(c_3) \\ b_4 & \lrcorner & \vert\vert \\ c_4 & = & \lrcorner \end{bmatrix} = \begin{bmatrix} a_1 & r(b_3 \underset{2}{+} c_3) \\ b_4 \underset{2}{+} c_4 & \lrcorner \end{bmatrix}$$

DEFINITION 3.12. The square a_1 is said to be *equivalent to* the square a_2, in symbols $a_1 \equiv a_2$, if there exist squares b_1, b_2, b_3 and b_4 whose vertical edges are homotopy equivalences and such that

$$\{a_1, a_2, b_1, b_2, b_3, b_4\}$$

is a homotopy commutative cube.

PROPOSITION 3.13. $\equiv$ *is an equivalence relation.*

PROOF. This follows from the above two propositions and the obvious fact that the trivial cube with $a_1 = a_0$ and thin vertical faces is homotopy commutative.

4. PULLBACK AND PUSHOUT SQUARES.

DEFINITION 4.1.

(i) A *pullback square* is a square a in D_2 such that for any square b in D with $\partial_1^1 b = \partial_1^1 a$ and $\partial_2^1 b = \partial_2^1 a$, there exist squares c_1 and c_2 with

$$\partial_2^0 c_1 = \partial_2^0 c_2 = \theta \text{ (say)}, \quad \partial_2^1 c_1 = 1, \quad \partial_2^1 c_2 = 1,$$

$$\partial_1^1 c_1 = \partial_1^0 a, \quad \partial_1^1 c_2 = \partial_2^0 a$$

such that

$$b = \begin{bmatrix} \ulcorner_\theta & r(c_2) \\ c_1 & a \end{bmatrix}$$

and, in addition, if

$$b = \begin{bmatrix} \ulcorner & r(c_2') \\ c_1' & a \end{bmatrix}$$

then there exists

$$\begin{array}{c|c|c} & 1 & \\ \hline \theta' & d & \theta \\ \hline & 1 & \end{array} \qquad \text{such that} \quad d \underset{1}{+} r(c_i) = r(c_i') \quad (i = 1, 2).$$

(ii) A *pushout square* is a square a in D_2 such that for any square $\bar{b}$ in D_2 with $\partial_2^0 \bar{b} = \partial_2^0 a$ and $\partial_1^0 \bar{b} = \partial_1^0 a$, there exist squares $\bar{c}_1, \bar{c}_2$ with

$$\partial_1^1 \bar{c}_1 = \partial_1^1 \bar{c}_2 = \ddot{\theta}, \quad \partial_2^0 \bar{c}_1 = \partial_2^1 a, \quad \partial_2^0 \bar{c}_2 = \partial_1^0 a,$$

$$\partial_1^0 \bar{c}_1 = 1, \quad \partial_1^0 \bar{c}_2 = 1$$

such that

$$\bar{b} = \begin{bmatrix} a & r(\bar{c}_2) \\ \bar{c}_1 & \lrcorner \bar{\theta} \end{bmatrix}$$

and, in addition, if

$$\bar{b} = \begin{bmatrix} a & r(\bar{c}_2') \\ \bar{c}_1' & \lrcorner \bar{\theta}' \end{bmatrix}.$$

then there exists

$$\begin{array}{c|c|c} & 1 & \\ \hline \bar{\theta}' & \bar{d} & \bar{\theta} \\ \hline & 1 & \end{array} \qquad \text{such that} \quad \bar{c}_i \underset{1}{+} \bar{d} = \bar{c}_i' \quad (i = 1, 2).$$

Here θ and $\bar{\theta}$ are called the *induced morphisms*.

At this point, we shall give two examples of pullback and pushout squares and they will be our sole concern.

EXAMPLE 1. Consider the special double category with connection TT.

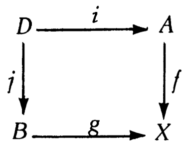
(a) Let A, B, X be topological spaces and $f: A \to X$, $g: B \to X$ continuous maps. Then the space

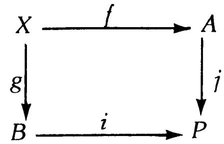
$$D = \{(a, l, b) \in A \times X^I \times B \mid f(a) = l(0),\ g(b) = l(1)\}$$

together with the inclusion maps $i: D \to A$, $j: D \to B$ and homotopy

$$c : fi \sim gj : D \to X \quad \text{defined by} \quad c(a, l, b, t) = l(t)$$

determines a pullback in TT.





(b) Let A, B, X be topological spaces and $f : X \to A$, $g : X \to B$ continuous maps. Then the double mapping cylinder $P = A \cup_f (X \times I) \cup_g B$ together with the inclusions $i : B \to P$, $j : A \to P$ and homotopy

$$d : jf \sim ig : X \to P \quad \text{defined by} \quad d(x, t) = [x, t]$$

determines a pushout square in TT.

EXAMPLE 2. Consider the special double category with connection CC.

(a) Let $f : A \to X$, $g : B \to X$ be chain maps. Define the chain complex $P = (P_n, \bar{\partial})$ where

$$P_n = \{(a, b, x, \bar{x}) \in A_n \oplus B_n \oplus X_n \oplus X_{n+1} \mid f(a) - g(b) = x + \partial \bar{x}\}$$

and $\bar{\partial} : P_n \to P_{n-1}$ is given by

$$\bar{\partial}(a, b, x, \bar{x}) = (\partial a, \partial b, 0, x).$$

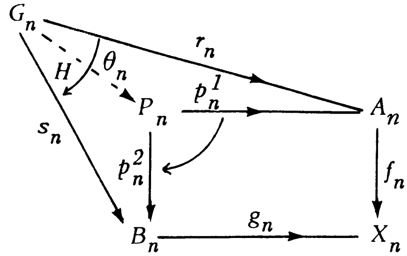
Let $p^1 : P \to A$, $p^2 : P \to B$ be the respective projections of chain complexes and $T : fp^1 \sim gp^2 : P \to X$ the chain homotopy defined by

$$T(a, b, x, \bar{x}) = \bar{x}.$$

One then verifies for each n,

$$f_n p_n^1 - g_n p_n^2 = T\bar{\partial} + \partial T.$$

The T then constitutes a pullback square in CC.

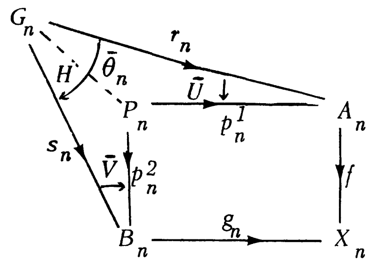


Suppose $r: G \to A$, $s: G \to B$ are chain maps such that $H: fr \sim gs$. Let

$$\theta: G \to P \quad \text{be} \quad \theta_n(g) = (r_n(g), s_n(g), H\partial(g), H(g));$$

we then have that θ is a chain map,

$$p^1\theta = r, \quad p^2\theta = s \quad \text{and} \quad T\theta = H.$$



Suppose $\bar{\theta}: G \to P$ is another chain map and $\bar{U}: r \sim p^1\theta$, $\bar{V}: s \sim p^2\theta$ are homotopies such that $[-g\bar{V} + T\bar{\theta} + f\bar{U}] = [H]$. (Here $[H]$ denotes the equivalence class of the chain homotopy H. Two chain homotopies H' and H are equivalent if for each n, there is a group homomorphism

$$M_n : G_n \to X_{n+2} \quad \text{such that} \quad H_n - H'_n = \partial_{n+2} M_n - M_{n-1}\partial_n.)$$

It is thus necessary to obtain a homotopy

$$R: \theta \sim \bar{\theta} \quad \text{such that} \quad [p^1R] = [\bar{U}] \quad \text{and} \quad [p^2R] = [\bar{V}].$$

For each g in G_n, let

$$R(g) = (\bar{U}(g), \bar{V}(g), (H - T\bar{\theta} - M\partial)(g), M(g)),$$

so that $R(g)$ is in P_{n+1} and $\bar{\partial}R + R\partial = \theta - \bar{\theta}$. Therefore $R: \theta \sim \bar{\theta}$. It is also clear that $[p^1R] = [\bar{U}]$ and $[p^2R] = [\bar{V}]$.

(b) Let $f: X \to A$, $g: X \to B$ be chain maps. Define the chain complex $Q = (Q_n, \partial)$ where $Q_n = (A_n \oplus B_n \oplus X_n \oplus X_{n-1})/F_n$ with

$$F_n = \{(f_n(x), -g_n(x), -x, -\partial x) \mid x \in X_n\} \quad \text{and}$$

$$\bar{\partial}: Q_n \to Q_{n-1} \quad \text{given by} \quad \bar{\partial}[a, b, x, \bar{x}] = [\partial a, \partial b, \bar{x}, 0].$$

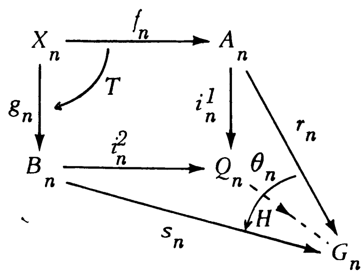
Here $[a, b, x, \bar{x}]$ denotes an element in Q_n with representative $(a, b, x, \bar{x})$. Let $i^1: A \to Q$, $i^2: B \to Q$ be the respective inclusions of chain complexes and $T: i^1f \sim i^2g: X \to Q$ the chain homotopy defined by

$$T(x) = [0, 0, 0, x].$$

One then verifies for each n,

$$i_n^1 f_n - i_n^2 g_n = \bar{\partial} T + T\partial .$$

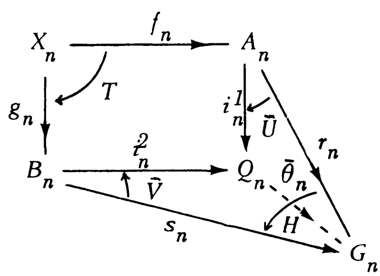
The T then constitutes a pushout square in CC.



Suppose $r: A \to G$, $s: B \to G$ are chain maps such that $H: rf \simeq sg$. Let $\theta: Q \to G$ be

$$\theta[a, b, x, \bar{x}] = r_n(a) + s_n(b) + \partial H(x) + H(\bar{x}).$$

We then have θ is a chain map, $\theta i^1 = r$, $\theta i^2 = s$ and $\theta T = H$.



Suppose $\bar{\theta}: Q \to G$ is another chain map and $\bar{U}: r \simeq \bar{\theta}i^1$, $\bar{V}: s \simeq \bar{\theta}i^2$ are homotopies such that $[-\bar{V}g + \bar{\theta}T + \bar{U}f] = [H]$. It is thus necessary to obtain a homotopy

$$R: \theta \simeq \bar{\theta} \text{ such that } [Ri^1] = [\bar{U}] \text{ and } [Ri^2] = [\bar{V}].$$

For each $[a, b, x, \bar{x}]$ in Q , let

$$R[a, b, x, \bar{x}] = \bar{U}(a) + \bar{V}(b) + (H - \bar{\theta}T - \partial L)(x) + L(\bar{x}),$$

where L is the homomorphism such that

$$-\bar{V}g + \bar{\theta}T + \bar{U}f - H = \bar{\partial}L - L\partial .$$

We then have

$$\partial R + R\bar{\partial} = \theta \cdot \bar{\theta}, \quad [R_i{}^1] = [\bar{U}] \quad \text{and} \quad [R_i{}^2] = [\bar{V}].$$

The following three propositions from [12] are recalled.

PROPOSITION 4.1 *(Proposition 3.4 of [12]). If a is a pullback (pushout) square, then so is $r(a)$ a pullback (pushout) square.*

PROPOSITION 4.2 *(Proposition 3.6 of [12]). Let a be a square such that one pair of opposite edges are homotopy equivalences. Then a is both a pullback and a pushout square.*

PROPOSITION 4.3 *(Proposition 3.7 of [12]). If a and b are pullback (pushout) squares, then $a \underset{i}{+} b$ is a pullback (pushout) square $(i = 1, 2)$.*

The next proposition includes the converse part of Proposition 3.2 in [12].

PROPOSITION 4.4. *Let a be a pullback square and a' a square with $\partial_i^1 a = \partial_i^2 a'$ $(i = 1, 2)$ and let θ be the induced morphism. Then a' is a pullback square iff θ is a homotopy equivalence.*

PROOF. We shall only prove the «if» part of the proposition. Since a is a pullback square, there are squares c_1 and c_2 such that

$$a' = \begin{bmatrix} \Gamma_\theta & r(c_2) \\ c_1 & a \end{bmatrix}$$

From Proposition 4.2, Γ_θ, c_1 and c_2 are pullback squares, so that from Proposition 4.1, Γ_θ, c_1 and $r(c_2)$ are pullback squares. Finally, from Proposition 4.3, we see that a is a pullback square. Of course, there should be a similar result for pushout squares; we will not write it down.

PROPOSITION 4.5. *Let the cube shown in Figure 1 be homotopy commutative and suppose a_1 is a pullback square. Then, if there are squares*

$$f_4 \begin{array}{|c|}\hline \overset{1}{} \\ c \\ \underset{1}{} \\ \hline\end{array} \bar{f}_4 \qquad \bar{f}_1 \begin{array}{|c|}\hline \overset{x}{} \\ \bar{b}_1 \\ \underset{\bar{x}}{} \\ \hline\end{array} f_2 \qquad \bar{f}_1 \begin{array}{|c|}\hline \overset{u}{} \\ \bar{b}_2 \\ \underset{u}{} \\ \hline\end{array} f_3$$

such that the cube

$$\{a_0, a_1, \bar{b}_1, \bar{b}_2, b_3 \underset{1}{+} c, b_4 \underset{1}{+} c\}$$

is homotopy commutative, there exists

$$\bar{f}_1 \begin{array}{|c|} \overset{1}{\hline} \\ \quad d \quad \\ \underset{1}{\hline} \end{array} f_1 \qquad \textit{such that } d \underset{1}{+} b_i = \bar{b}_i \ (i = 1, 2).$$

PROOF. We first have

$$\begin{bmatrix} a_0 & r(b_3) \\ b_4 & \lrcorner \end{bmatrix} = \begin{bmatrix} a_0 & = & = & r(b_3) \\ \| & \square & \square & r(c) \\ \| & \square & \square & r(-c) \\ b_4 & c & -c & \lrcorner \end{bmatrix} = \begin{bmatrix} a_0 & r(b_3 \underset{1}{+} c) \\ b_4 \underset{1}{+} c & \lrcorner \end{bmatrix}$$

Secondly, since the cubes

$$\{a_0, a_1, b_1, b_2, b_3, b_4\} \text{ and } \{a_0, a_1, \bar{b}_1, \bar{b}_2, b_3 \underset{1}{+} c, b_4 \underset{1}{+} c\}$$

are homotopy commutative,

$$\begin{bmatrix} \ulcorner & b_1 \underset{1}{+} \lrcorner \\ r(b_2 \underset{1}{+} \lrcorner) & a_1 \end{bmatrix} = \begin{bmatrix} a_0 & r(b_3) \\ b_4 & \lrcorner \end{bmatrix}$$

and

$$\begin{bmatrix} a_0 & r(b_3 \underset{1}{+} c) \\ b_4 \underset{1}{+} c & \lrcorner \end{bmatrix} = \begin{bmatrix} \ulcorner & \bar{b}_1 \underset{1}{+} \lrcorner \\ r(\bar{b}_2 \underset{1}{+} \lrcorner) & a_1 \end{bmatrix}$$

so that from the above equation, we have

$$\begin{bmatrix} \ulcorner & b_1 \underset{1}{+} \lrcorner \\ r(b_2 \underset{1}{+} \lrcorner) & a_1 \end{bmatrix} = \begin{bmatrix} \ulcorner & \bar{b}_1 \underset{1}{+} \lrcorner \\ r(\bar{b}_2 \underset{1}{+} \lrcorner) & a_1 \end{bmatrix};$$

hence by uniqueness of the pullback square a_1, there exists

$$\bar{f}_1 \begin{array}{|c|} \overset{1}{\hline} \\ \quad d \quad \\ \underset{1}{\hline} \end{array} f_1 \qquad \text{such that} \quad d \underset{1}{+} b_i \underset{1}{+} \lrcorner = \bar{b}_i \underset{1}{+} \lrcorner \ (i = 1, 2)$$

which means $d \underset{1}{+} b_i = \bar{b}_i \ (i = 1, 2)$.

PROPOSITION 4.6. *Given squares*

$$\begin{array}{c} x \\ u\,\boxed{\;a_0\;}\,v \\ y \end{array},\quad \begin{array}{c} \bar{x} \\ \bar{u}\,\boxed{\;a_1\;}\,\bar{v} \\ \bar{y} \end{array},\quad \begin{array}{c} v \\ f_2\,\boxed{\;b_3\;}\,f_4 \\ \bar{v} \end{array},\quad \begin{array}{c} y \\ f_3\,\boxed{\;b_4\;}\,f_4 \\ \bar{y} \end{array}$$

such that a_1 is a pullback square, there exist squares

$$\begin{array}{c} x \\ f_1\,\boxed{\;b_1\;}\,f_2 \\ \bar{x} \end{array},\quad \begin{array}{c} x \\ f_1\,\boxed{\;b_2\;}\,f_3 \\ \bar{x} \end{array}$$

such that $\{a_0, a_1, b_1, b_2, b_3, b_4\}$ is a homotopy commutative cube. Furthermore, suppose

$$\begin{array}{c} x \\ \bar{f}_1\,\boxed{\;\bar{b}_1\;}\,f_2 \\ \bar{x} \end{array},\quad \begin{array}{c} u \\ \bar{f}_1\,\boxed{\;\bar{b}_2\;}\,f_3 \\ \bar{u} \end{array}$$

are squares. Then $\{a_0, a_1, \bar{b}_1, \bar{b}_2, b_3, b_4\}$ is also homotopy commutative iff there exists

$$\begin{array}{c} 1 \\ \bar{f}_1\,\boxed{\;d\;}\,f_1 \\ 1 \end{array} \quad \text{such that} \quad d \underset{1}{+} b_i = \bar{b}_i \quad (i = 1, 2).$$

PROOF. Consider the square

$$\begin{bmatrix} a_0 & r(b_3) \\ b_4 & \lrcorner \end{bmatrix}$$

Since a_1 is a pullback square, there are squares

$$\begin{array}{c} f_1 \\ f_3 u\,\boxed{\;c_2\;}\,\bar{u} \\ 1 \end{array},\quad \begin{array}{c} f_1 \\ f_2 x\,\boxed{\;c_1\;}\,\bar{x} \\ 1 \end{array}$$

such that

$$\begin{bmatrix} a_0 & r(b_3) \\ b_4 & \lrcorner \end{bmatrix} = \begin{bmatrix} \ulcorner & r(c_1) \\ c_2 & a_2 \end{bmatrix}.$$

We shall refer to this as equation (a). Next let

$$b_1 = r((\ulcorner_x \underset{2}{+} \lrcorner_{f_2}) \underset{1}{+} c_1) \quad \text{and} \quad b_2 = r((\ulcorner_u \underset{2}{+} \lrcorner_{f_3}) \underset{1}{+} c_2).$$

One easily checks with the help of (a) that $\{a_0, a_1, b_1, b_2, b_3, b_4\}$ is a homotopy commutative square. To prove the latter part of the proposition, we proceed as follows: Suppose $\{a_0, a_1, \bar{b}_1, \bar{b}_2, b_3, b_4\}$ is a homotopy commutative cube. Therefore

$$\begin{bmatrix} \ulcorner & \bar{b}_1 \underset{1}{+} \lrcorner \\ r(\bar{b}_2 \underset{1}{+} \lrcorner) & a_1 \end{bmatrix} = \begin{bmatrix} a_0 & r(b_3) \\ b_4 & \lrcorner \end{bmatrix}$$

so that by equation (a), it is equal to

$$\begin{bmatrix} \ulcorner & r(c_1) \\ c_2 & a_2 \end{bmatrix}.$$

Hence by uniqueness of the pullback property of a_1, there is

$$\bar{f}_1 \overset{1}{\underset{1}{\boxed{\quad d \quad}}} f_1 \qquad \text{such that} \qquad d \underset{1}{+} r(c_i) = \bar{b}_i \underset{1}{+} \lrcorner \quad (i = 1, 2),$$

which, by definition of b_1 and b_2 is equivalent to the equation

$$d \underset{1}{+} b_i = \bar{b}_i \quad (i = 1, 2).$$

Conversely, suppose there is a square

$$\bar{f}_1 \overset{1}{\underset{1}{\boxed{\quad d \quad}}} f_1 \qquad \text{such that} \qquad d \underset{1}{+} b_i = \bar{b}_i \quad (i = 1, 2).$$

As $\{a_0, a_1, b_1, b_2, b_3, b_4\}$ is a homotopy commutative cube, we have the equal squares

$$T((a_0 \underset{1}{+} r(b_3)) \underset{2}{+} (b_4 \underset{1}{+} \lrcorner)) \quad \text{and} \quad T \begin{bmatrix} \ulcorner & b_1 \\ r(b_2) & a_1 \end{bmatrix}$$

the latter of which can be subdivided into

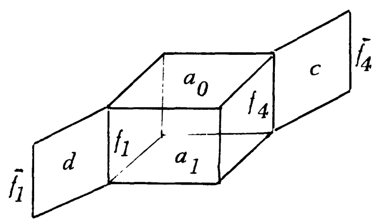
$$T\begin{bmatrix} \Gamma & \cdot d & d & b_1 \\ r(\cdot d) & \square & \square & \| \\ r(d) & \square & \square & \| \\ r(b_2) & = & = & a_1 \end{bmatrix}$$

and consequently is equal to

$$T\begin{bmatrix} \Gamma & d \underset{1}{+} b_1 \\ r(d \underset{1}{+} b_2) & a_2 \end{bmatrix}$$

Hence $\{a_1, a_2, \bar{b}_1, \bar{b}_2, b_3, b_4\}$ is a homotopy commutative cube.

These two propositions express the fact that in the homotopy commutative cube



if the bottom square a_1 is a pullback square, then each $\{f_4\}$ determines a unique $\{f_1\}$, where the bracket denotes the homotopy class of a morphism. The dual result is that if a_0 is a pushout square, then each $\{f_1\}$ determines a unique $\{f_4\}$. Hence when a_0 is a pushout square and a_1 is a pullback square, there is a bijection between the sets of the classes $\{f_1\}$ and $\{f_4\}$.

Proposition 4.6 leads to a number of interesting results.

COROLLARY 4.7. *Let*

$$u\ \begin{array}{|c|}\hline x \\ a_0 \\ y \\ \hline\end{array}\ v, \quad \bar{u}\ \begin{array}{|c|}\hline \bar{x} \\ a_1 \\ \bar{y} \\ \hline\end{array}\ \bar{v}, \quad f_2\ \begin{array}{|c|}\hline v \\ b_3 \\ \bar{v} \\ \hline\end{array}\ f_4, \quad f_3\ \begin{array}{|c|}\hline y \\ b_4 \\ \bar{y} \\ \hline\end{array}\ f_4$$

be squares in which f_2, f_3, f_4 *are homotopy equivalences. If* a_0, a_1 *are* *pullback squares, then there exist squares*

$$\begin{array}{c} x \\ f_1 \boxed{\quad b_1 \quad} f_2 \\ \bar{x} \end{array} \quad \textit{and} \quad \begin{array}{c} u \\ f_1 \boxed{\quad b_2 \quad} f_3 \\ \bar{u} \end{array}$$

such that f_1 *is a homotopy equivalence and* $\{a_0, a_1, b_1, b_2, b_3, b_4\}$ *is a homotopy commutative cube.*

PROOF. By Proposition 4.6, there are squares

$$\begin{array}{c} x \\ f_1 \boxed{\quad b_1 \quad} f_2 \\ \bar{x} \end{array} , \qquad \begin{array}{c} u \\ f_1 \boxed{\quad b_2 \quad} f_3 \\ \bar{u} \end{array}$$

$$\begin{array}{c} \bar{x} \\ \bar{f}_1 \boxed{\quad b'_1 \quad} \bar{f}_2 \\ x \end{array} , \qquad \begin{array}{c} \bar{u} \\ \bar{f}_1 \boxed{\quad b'_2 \quad} \bar{f}_3 \\ u \end{array}$$

such that

$$\{a_0, a_1, b_1, b_2, b_3, b_4\} \text{ and } \{a_1, a_0, b'_1, b'_2, \phi(b_3), \phi(b_4)\}$$

are homotopy commutative cubes. Consider the homotopy commutative cubes

$$\{a_0, a_0, b_1 \underset{2}{+} b'_1, b_2 \underset{2}{+} b'_2, b_3 \underset{2}{+} \phi(b_3), b_4 \underset{2}{+} \phi(b_4)\},$$
$$\{a_0, a_0, \mid\,\mid \underset{1}{+} \cdot e_3, \mid\,\mid \underset{1}{+} \cdot e_2, b_3 \underset{2}{+} \phi(b_3), b_4 \underset{2}{+} \phi(b_4)\}$$

where

$$\bar{f}_2 f_2 \boxed{\quad e_2 \quad} 1 , \qquad \bar{f}_3 f_3 \boxed{\quad e_3 \quad} 1$$

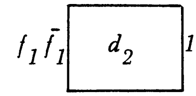
are homotopies. By Proposition 4.5, there is a homotopy

$$\bar{f}_1 f_1 \boxed{\quad d_1 \quad} 1$$

such that

$$d_1 \underset{1}{+} \mid\,\mid \underset{1}{+} \cdot e_3 = b_2 \underset{2}{+} b'_2 \quad \text{and} \quad d_1 \underset{1}{+} \mid\,\mid \underset{1}{+} \cdot e_2 = b_1 \underset{2}{+} b'_1 .$$

Similarly, by considering the other pair of homotopy commutative cubes, the homotopy



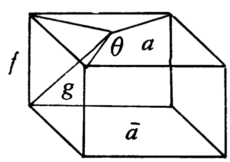
is obtained.

COROLLARY 4.8. *Let* $\{a_0, a_1, b_1, b_2, b_3, b_4\}$ *be a homotopy commutative cube as shown in Figure 1. If* a_0 *and* a_1 *are pullback squares and* f_2, f_3, f_4 *are homotopy equivalences, then* f_1 *is also a homotopy equivalence.*

PROOF. This follows directly from Corollary 4.7 and Proposition 4.6.

COROLLARY 4.9. *Let* a *and* $\bar{a}$ *be squares such that* a *is equivalent to* $\bar{a}$. *If* $\bar{a}$ *is a pullback (pushout) square, then* a *is a pullback (pushout) square.*

PROOF. Here we have to assume pullback or pushout squares exist in D. Let b_1, b_2, b_3 and b_4 be squares with vertical edges homotopy equivalences. We then have the homotopy commutative cube $\{a, \bar{a}, b_1, b_2, b_3, b_4\}$ below.



Let a_1 be the pullback square of $\partial_1^1 a$ and $\partial_2^1 a$ so that

$$a = (\Gamma_\theta \underset{1}{+} r(c_2)) \underset{2}{+} (c_1 \underset{1}{+} a_1))$$

for some squares c_1 and c_2. By Corollary 4.7, there are squares $\bar{b}_1, \bar{b}_2$ such that $\partial_1^0 \bar{b}_1 = \partial_1^0 \bar{b}_2 = g$ (say) is a homotopy equivalence and

$$\{a_1, \bar{a}, \bar{b}_1, \bar{b}_2, b_3, b_4\}$$

is a homotopy commutative cube. One then verifies that

$$\{a_1, \bar{a}, c_1 \underset{1}{+} r(\bar{b}_1), c_2 \underset{1}{+} r(\bar{b}_2), b_3, b_4\}$$

is also a homotopy commutative cube. Hence by Proposition 4.6, there is the square

$$\begin{array}{ccc} & 1 & \\ f & \boxed{\quad d \quad} & g\theta \\ & 1 & \end{array}$$

so that by Proposition 3.4, θ is a homotopy equivalence. Therefore, by Proposition 4.4, a is a pullback square.

The following proposition follows easily from Definition 3.8.

PROPOSITION 4.10. *Let* a *be a pullback square and let*

$$\{a_0, a_1, b_1, b_2, b_3, b_4\}$$

be the homotopy cube of Figure 1. Suppose

$$f_2 \boxed{\quad S_2 \quad} \bar{f}_2 \quad \text{and} \quad f_3 \boxed{\quad S_3 \quad} \bar{f}_3$$

are homotopies. Then the cube

$$\{a_0, a_1, b_1 +_1 S_2, b_2 +_1 S_3, -S_2 +_1 b_3, -S_3 +_1 b_4\}$$

is homotopy commutative.

Now we obtain from Proposition 4.6 that for the cube in Figure 1, when f_2 and f_3 are replaced by homotopic edges, the induced edge from $\partial_0^0 \partial_1^0 a_0$ to $\partial_0^0 \partial_1^0 a_1$ is homotopic to f_1.

Hence we have the following refinement of Proposition 4.5.

COROLLARY 4.11 *(refer to Figure 1). Let* a_1 *be a pullback square and* a_0, b_3, b_4 *given squares. Then there are squares* b_1 *and* b_2 *such that the cube* $\{a_0, a_1, b_1, b_2, b_3, b_4\}$ *is homotopy commutative. Furthermore, if* f_2, f_3 *and* f_4 *are replaced by homotopic edges, the induced edge is homotopic to* f_1.

Finally, we consider results converse to Proposition 3.6 of [12]. Proposition 4.12 is well known for ordinary pullbacks, i.e. for pullback squares in $\square D$, whenever f is a fibration. Proposition 4.13 provides a necessary and sufficient condition for the converse of Proposition 4.12.

By this result and Propositions 3.7 and 3.8 of [12] in any category satisfying the condition (for example, according to [11], in the category of CW-complexes), the algebra of pullback and pushout squares takes on a particularly simple form.

PROPOSITION 4.12. *Let*

$$\begin{array}{c|c|c} & h & \\ \hline g & a & f \\ \hline & k & \end{array}$$

be a pullback square. If f is a homotopy equivalence, then g is a homotopy equivalence.

PROOF. Let $\bar{f}$ be the homotopy inverse of f with strong homotopies

$$f\bar{f}\,\boxed{\;e\;}\;,\qquad \bar{f}f\,\boxed{\;d\;}\;.$$

Consider the square

$$b = \Vert_k \underset{1}{+} ((\Vert_{\bar{f}} \underset{1}{+} \ulcorner_f) \underset{2}{+} r(d)).$$

Since a is a pullback square, there exist squares

$$\begin{array}{c|c|c} & \bar{g} & \\ \hline 1 & c & g \\ \hline & 1 & \end{array}\;,\qquad \begin{array}{c|c|c} & \bar{g} & \\ \hline \bar{f}k & c & h \\ \hline & 1 & \end{array}$$

such that

$$b = \begin{bmatrix} \ulcorner & r(c_2) \\ c_1 & a \end{bmatrix}$$

Therefore, $-T(c_1)$ is a homotopy from $g\bar{g}$ to 1. To obtain the other homotopy, let

$$c'_1 = \lrcorner_g \underset{1}{+} c_1,\quad c'_2 = r\{=_{\bar{g}g} \underset{1}{+} (a \underset{2}{+} (\Vert_k \underset{1}{+} \ulcorner_{\bar{f}}) \underset{2}{+} r(c_2)) \underset{1}{+} e\}.$$

Then

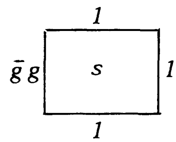
$$\begin{bmatrix} \ulcorner & r(c'_2) \\ c'_1 & a \end{bmatrix} = \begin{bmatrix} \ulcorner & = & a \underset{2}{+} (\Vert \underset{1}{+} \ulcorner) & e \\ \Vert & \ulcorner & r(c_2) & \square \\ \lrcorner & c_1 & a & = \end{bmatrix}$$

$$= \begin{bmatrix} a & - \\ \mid\mid & \ulcorner \\ \mid\mid & r(d) \end{bmatrix} \underset{1}{+} \begin{bmatrix} e \\ - \\ -_f \end{bmatrix} = a \cdot (-_f \underset{2}{+} d) \underset{1}{+} (e \underset{2}{+} -_f) = a .$$

However, we have

$$\begin{bmatrix} \ulcorner & r(-) \\ - & a \end{bmatrix} = a .$$

Hence, by uniqueness of the pullback property of a, there exists



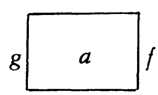
such that

$$s \underset{1}{+} r(c_1') = \mid\mid_g \quad \text{and} \quad s \underset{1}{+} r(c_2') = \mid\mid_h .$$

Therefore, s is the homotopy from $\bar{g}g$ to 1.

PROPOSITION 4.13. *The following two statements are equivalent:*

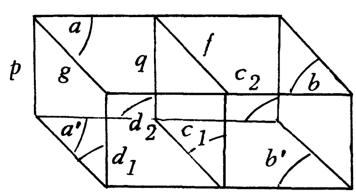
(1) For any pullback square



if g is a homotopy equivalence then f is a homotopy equivalence.

(2) For any squares a, b, if a and $a \underset{1}{+} b$ are pullback squares then b is a pullback square.

PROOF. (1) ⇒ (2). Suppose a, b are squares such that a, $a \underset{1}{+} b$ are pullback squares.



Let b' be the pullback square of $\partial_1^1 b$ and $\partial_2^1 b$. Therefore, there are squares c_1 and c_2 such that $\{ b, b', c_1, c_2, \mid\mid_{\partial_2^1 b}, \mid\mid_{\partial_1^1 b} \}$ is a homotopy commutative cube. Let a' be the pullback square of $\partial_2^1 a$ and $\partial_1^0 b'$. Again there are squares d_1 and d_2 such that $\{a, a', d_1, d_2, \shortparallel_{\partial_2^1 a}, c_1\}$ is a homotopy commutative cube. Composing the two cubes, we see that

$$\{a \underset{1}{+} b,\ a' \underset{1}{+} b',\ d_1,\ d_2 \underset{1}{+} c_2,\ \shortparallel_{\partial_2^1 (a \underset{1}{+} b)},\ \shortparallel_{\partial_1^1 b}\}$$

is also a homotopy commutative cube. Since both $a \underset{1}{+} b$ and $a' \underset{1}{+} b'$ are pullback squares, Corollary 4.8 implies $p\ (= \partial_1^0 d_1 = \partial_1^0 d_2)$ is a homotopy equivalence. On the other hand, since $\{a, a', d_1, d_2, \shortparallel_{\partial_2^1 a}, c_1\}$ is a homotopy commutative cube, we obtain

$$d_2 \underset{2}{+} a' = -Tr(d_1) \underset{1}{+} a \underset{1}{+} Tr(c_1).$$

As a is a pullback square and composition of pullback squares is a pullback square, we thus have $d_2 \underset{2}{+} a'$ is a pullback square. By Proposition 3.8 of [12], d_2 is a pullback square. By assumption, because p is a homotopy equivalence, $q = \partial_1^1 d_2$ is also a homotopy equivalence. Finally, we see that b is equivalent to b', so that b is also a pullback square.

(2) $\Rightarrow$ (1). Suppose a is a pullback square such that g is a homotopy equivalence. Since the vertical edges of $(a \underset{1}{+} \lrcorner_f)$ are homotopy equivalences, Proposition 3.6 of [12] implies that it is a pullback square. Therefore, by assumption, $\lrcorner_f$ is a pullback square. Finally, by Proposition 4.12, we have f is a homotopy equivalence.

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