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THE DERIVED FUNCTORS OF *lim* AND PROTORSION MODULES

by Timothy PORTER

The amount that is known about the values taken by the derived functors of *lim* is quite limited. One has information on $lim^{(1)}$ (compare Warfield and Huber [15]) and also on conditions which guarantee that certain of the $lim^{(i)}$ are zero (see Jensen [9], Gruson and Jensen [3] and Porter [12]).

In this paper we extend the methods of [12] to give limited information on values taken by all $lim^{(i)}$ in systems of modules of finite (bounded) Krull-Gabriel dimension. We prove that if M is an inverse system of finitely generated modules of Krull-Gabriel dimension $\leq n$, then $lim^{(i)}M$ can be constructed from limits of finitely generated modules of Krull-Gabriel dimension less than or equal to $n-i$ by a well controlled process of extensions, countable direct unions and quotients. (The exact statement of the results is more technical but this gives the idea of what they state.) Corollaries of this result apply to certain inverse systems of linearly topologised modules and continuous maps generalising a result of Jensen [8].

The protorsion modules of the title were introduced by Lambek in [10]. Although they have exceedingly nice properties, they do not seem to be closed under countable direct unions and quotients and the class we shall be considering seems, therefore, to be larger. However their properties allow one to gain some knowledge of this larger class as these protorsion modules do form the basic building blocks for this class, hence we have included a brief résumé of their properties. It is clear that further effort is needed in their study, at least in the special cases considered here which relate to p. f. g. modules of finite Krull-Gabriel dimension.

1. KRULL-GABRIEL DIMENSION AND PSEUDO FINITELY GENERATED MODULES.

Given an associative ring A, we will write $\mathrm{A} = Mod\text{-}A$ for the category of right modules over A. A is filtered by an ordinal indexed sequence of localising subcategories $\{ A_\alpha \}$, called the Krull-Gabriel filtration of A, defined as follows:

$A_{-1} = \{0\}$ - the zero subcategory,

If $\alpha = \beta + 1$ and $T_\beta : A \to A/A_\beta$ is the β^{th} quotient functor, A_α is the smallest localising subcategory of A containing the class

$$\{ M \mid M \in A,\ T_\beta(M) \text{ has finite length in } A/A_\beta \}.$$

If α is a limit ordinal, A_α is the smallest localising subcategory of A containing $\bigcup_{\beta<\alpha} A_\beta$.

The torsion radical associated to A_α will be denoted r^α. We shall examine r^α more closely in a moment.

We say that an object M *has KG-dimension* α if $M \in A_\alpha$ but $M \notin A_\beta$ for all $\beta < \alpha$. We write $KG\text{-}dim\, M = \alpha$.

We shall be only interested in finite α and will construct $r^n(M)$ step by step by recursion on n.

If $n = 0$, set

$$r^0_1(M) = soc(M) = \oplus \{ S \mid S \subset M,\ S \text{ simple} \}.$$

If $\alpha = \beta + 1$ and $r^0_\beta(M)$ is defined then $r^0_\alpha(M)$ is given by

$$\frac{r^0_\alpha(M)}{r^0_\beta(M)} = r^0_1\left(\frac{M}{r^0_\beta(M)}\right).$$

If α is a limit ordinal,

$$r^0_\alpha(M) = \bigcup_{\beta<\alpha} r^0_\beta(M).$$

Finally $r^0(M) = \bigcup_\alpha r^0_\alpha(M)$.

If $M \in A_0$, $r^0(M) = M$. In any case $r^0_\alpha(M) = r^0(M)$ for some ordinal α and the minimal such α will be called the (simple) *0-length of* M.

We next assume that $r^{n-1}(M)$ is constructed for some $n \geq 1$. We need an analogue of «simple». We say M *is n-simple* if $r^{n-1}(M) = 0$ and for all $N \subset M$, $M/N \in A_{n-1}$. Now define for any M with $r^{n-1}(M) = 0$

$$\tau_1^n(M) = \Sigma\{ S \mid S \subset M,\ S \text{ is } n\text{-simple}\}.$$

(This sum is not direct in general but it is an essential extension of a direct sum of n-simples as these latter are coirreducible (compare Popescu [11] Chapter 5).) Note

$$T_{n-1}\tau_1^n = \tau_1^0 T_{n-1}.$$

Defining τ^n by recursion as above for the case $n = 0$, we get the concept of (simple) n-length. To handle the general case when $\tau^{n-1}(M)$ may be non-zero, we specify

$$\frac{\tau^n(M)}{\tau^{n-1}(M)} = \tau^n\left(\frac{M}{\tau^{n-1}(M)}\right)$$

and extend the notion of n-length similarly.

So much is fairly standard. We next need the notion of pseudofinitely generated modules as defined first in [12] (page 44) from which we take the following. (All pseudo-finitely generated (p. f. g.) modules will be of finite Krull-Gabriel dimension.)

If $n = -1$, all objects of A_n are p. f. g. Assume therefore that the term is defined up to dimension $n-1$. If M is a sum of n-simple objects, then M is p. f. g. if it satisfies the conditions:

(i) $T_{n-1}(M)$ is a direct sum of finitely many simples in A/A_{n-1}.

(ii) If $N \subset M$ is such that M/N is in A_{n-1} then M/N is p. f. g.

In general M in A_n is p. f. g. if

(iii) $\tau^{n-1}(M)$ is p. f. g., and

(iv) for each α, writing $\bar{M} = M/\tau^{n-1}(M)$, one has $\tau_1^n(\bar{M}/\tau_\alpha^n(\bar{M}))$ is p. f. g. in the earlier sense.

We proved in [11] the following proposition.

PROPOSITION. *If M is a Noetherian object of finite Krull-Gabriel dimension, then M is p. f. g.* .

In fact Noetherian objects have even more structure and we shall briefly turn to this next.

If M is Noetherian for each n, M has finite n-length, also, in the terms $r_1^n(M/r_\alpha^n(\bar{M}))$, one has «essentially» only finitely many direct n-simple summands (i. e. it is an essential extension of such a finite direct sum).

For each n, we generalise length in the classical sense as follows

If $n = 0$, length is to be interpreted in the old sense. However this equals the sum

$$long^0(M) = \sum_\alpha l_\alpha^0(M),$$

where $l_\alpha^0(M)$ is the number of simple summands in $r_1^0(\frac{M}{r_\beta^0(M)})$ if $\alpha = \beta+1$. (As M is supposed Noetherian, limit ordinals do not need to be considered and the sum is finite.)

For general n, again with Noetherian M, write $\bar{M} = M/r^{n-1}(M)$ and set, for $\alpha = \beta+1$,

$l_\alpha^n(M) =$ the number of simple direct summands in $T_{n-1}(r_1^n(\frac{\bar{M}}{r_\beta^n(\bar{M})}))$.

As $T_{n-1}(M)$ is Noetherian (Popescu [11] page 372), $l_\alpha^n(M)$ is finite and non-zero for only finitely many α. Let

$$long^n(M) = \sum_\alpha l_\alpha^n(M).$$

To distinguish (simple) n-length from this more specialised notion, we shall call this latter notion *composite n-length.*

Clearly if M is Noetherian, then M has finite composite n-length for each n. It is by no means clear that the converse holds.

2. PROTORSION MODULES AND LIMIT TORSION CLASSES.

Lambek [10] has introduced the term «generalised torsion theory» to signify a class of modules which is closed under isomorphic images, finite direct sums and submodules. For instance, the Noetherian or the p. f. g. objects of a given KG-dimension form a generalised torsion class in this sense.

Given any generalised torsion theory C in A one can topologise right A-modules by using the «dense» submodules (with respect to C) as a fundamental system of neighborhoods of zero: $D \subset M$ is dense (with respect to C) if $M/D \in C$.

We will denote by $L(M)$ the module M considered as a topological right $L(A)$-module. We denote by $CL(M)$ the Cauchy completion of M in this topology.

An other construction on modules (relative to a fixed generalised torsion theory C) is the protorsion completion. An inverse limit of a projective system of modules in C is called a *protorsion module.* Giving each module in C the discrete topology, or, what is equivalent, the topology coming from C as above, we give the inverse limit the subspace topology of the product as usual.

The *protorsion completion of a module* M is the protorsion module:

$$F(M) = \lim \{ M/D \mid D \subset M,\ M/D \in C \}.$$

Any protorsion module is a topological $L(A)$-module and as such is complete. In particular $F(A)$ is a complete topological ring which will be denoted by $\hat{A}$. Each protorsion module is in fact a complete topological $\hat{A}$-module.

If one denotes by J the forgetful functor from protorsion modules to complete $L(A)$ (or $\hat{A}$)-modules, then Lambek [10] proves that $JF(M)$ and $CL(M)$ are isomorphic.

Thus the protorsion modules are fairly easy to handle. However we need to work with a larger class. The class of protorsion modules is almost certainly not closed under quotient by closed submodules. More importantly, the results of these operations may or may not be complete. Thus to obtain a useful class of topological $\hat{A}$-modules, we close up the class of complete modules under the operations of taking:

(i) isomorphic images by continuous maps,

(ii) closed submodules,

(iii) finite unions,

(iv) quotients by closed submodules,

(v) extensions

$$0 \longrightarrow E \xrightarrow{\ i\ } F \longrightarrow G \longrightarrow 0$$

where E, G are in the class and $i(E)$ is closed in F.

We call the result the *limit torsion class of* C and will denote it $Lim(C)$.

We shall also use a second larger class where (iv) is weakened to include all quotients, (iii) to include countable unions and (ii) is weakened to include all subobjects. We denote this class by $wLim(C)$.

3. SPECIAL DIRECT LIMITS.

We shall denote by $pro(\mathrm{A})$ the category of projective systems in A. The theory of procategories is too large to be sketched adequately here so we shall assume that the reader has some knowledge of the basic results and refer to [12] for more detailed information on localisations in procategories.

Although, of course, A has exact direct limits, $pro(\mathrm{A})$ in general does not. However there is a useful class of «special direct systems» on which «exactness» does hold.

Let J be a directed set and A^J the category of J-indexed directed systems in A. (In fact we will only need ordered J but will give the wider definition.) Suppose given a proobject $M: I \to \mathrm{A}^J$. M can be considered as a J-indexed directed system in $pro(\mathrm{A})$. If for each i in I the directed system $M(i)$ consists of monomorphisms then we shall say M *is a special direct system* and $colim_J M$ *a special direct limit.*

EXAMPLE. Let $\{r^n_\alpha\}$ be the indexed family of subfunctors of the identity whose union is r^n (as described in Section 1). Then for any promodule M, $r^n_\alpha(M)$ is a special direct system whose direct limit is $r^n(M)$.

The advantage of special direct limits is the exactness of *colim* on them. It is this feature which plays the decisive role in the construction of the associated spectral sequence whose existence was shown in [12], page 47.

One constructs from a special direct system $M: I \to \mathrm{A}^J$ a double complex $\Sigma\Pi(M)$ and two spectral sequences. (We follow the spectral sequence convention of Hilton & Stammbach [7].)

$$_I E_1^{p,q} = \begin{cases} lim_I^{(-p)} colim_J^{(q+p)} M & p \le 0,\ q \ge -p \\ 0 & \text{otherwise} \end{cases}$$

and

$$ {}_2E_1^{p,q} = \begin{cases} colim_J^{(-p)} lim_I^{(q+p)} M & p \leq 0, \quad q \geq -p \\ 0 & \text{otherwise} \end{cases} $$

which arise from the natural filtrations of $B = Tot(\Sigma\Pi(M))$. As the colimits are all exact, one has considerable simplification of these to give

$$ {}_1E_1^{p,q} = \begin{cases} lim_I^{(-p)} colim_J M & p = -q \\ 0 & \text{otherwise} \end{cases} $$

and

$$ {}_2E_1^{p,q} = \begin{cases} colim_J \, lim_I^{(q)} M & p = 0, \quad q \geq 0 \\ 0 & \text{otherwise.} \end{cases} $$

The second filtration of $Tot(\Sigma\Pi(M))$ is both complete and cocomplete (in the terminology of Hilton & Stammbach [7]) so

$$ H^q(B) = colim_J \, lim_I^{(q)} M , \qquad q \geq 0 . $$

The first filtration is unfortunately not complete but one does have that the zero-th term of that filtration ${}_1F_0B$ is all of B. Hence

$$ Im(H^q({}_1F_0B) \longrightarrow H^q(B)) = H^q(B). $$

Now each ${}_1E_\infty^{p,q}$ back as far as ${}_1E_\infty^{-q,q}$ is zero so we have

$$ Im(H^q({}_1F_pB) \longrightarrow H^q(B)) = H^q(B) $$

for $p = -1, -2, \dots, -q$ and so

$$ {}_1E_\infty^{-q,q} \cong \frac{H^q(B)}{Im(H^q(F_{-(q+1)}B) \to H^q(B))} $$

and we have proved:

PROPOSITION 3.1. *If* M *is a special direct system in* $pro(A)$ *then there are natural epimorphisms (for each* $q \geq 0$ *)*

$$ colim(lim^{(q)} M) \longrightarrow lim^{(q)} colim M . $$

We shall later be considering linear topologies on these modules, but, in the cases studied in this paper, we seem to have no control over whether the kernel of the above epimorphism is closed. This fact is the reason why we have considered weak limit torsion classes above. It seems just possible that a closer inspection of the spectral sequence will enable this weakness to be removed. We shall obtain our best results by avoiding it.

4. LOCALISATIONS IN PROCATEGORIES.

We include here a brief résumé of the results from [12] on localisations. We limit attention to the results and definitions necessary for understanding of the subsequent development of this paper.

Suppose C is a full subcategory of A, then we say that an object X in $pro(\mathrm{A})$ is *essentially of type* C if the following equivalent conditions hold:

(i) X is isomorphic in $pro(\mathrm{A})$ to an object in the subcategory $pro(\mathrm{C})$.

(ii) If $X: I \to \mathrm{A}$ represents the proobject X then for any i in I there is a morphism $a: j \to i$ in I such that the transition morphism $X(a): X(j) \to X(i)$ factors through an object of C.

If C is a localising subcategory of A then the subcategory $\mathrm{E}(\mathrm{C})$ of those proobjects essentially of type C is localising in $pro(\mathrm{A})$ and $\mathrm{E}(\mathrm{C})$ is the kernel of the canonical functor $pro(\mathrm{A}) \to pro(\mathrm{A}/\mathrm{C})$. Thus there is a functor induced:

$$T': pro(\mathrm{A})/\mathrm{E}(\mathrm{C}) \to pro(\mathrm{A}/\mathrm{C}).$$

The quotient functors

$$T: \mathrm{A} \to \mathrm{A}/\mathrm{C}, \quad \overline{T}: pro(\mathrm{A}) \to pro(\mathrm{A})/\mathrm{E}(\mathrm{C})$$

have «sections» and the images of the sections are respectively the C-closed objects of A and the $\mathrm{E}(\mathrm{C})$-closed objects of $pro(\mathrm{A})$. We denote the corresponding local subcategories by «C-closed» and «$\mathrm{E}(\mathrm{C})$-closed». Then one has

$$\mathrm{E}(\text{C-closed}) = \mathrm{E}(\mathrm{C})\text{-closed}.$$

The localising subcategory C determines a torsion subradical r. If $F: \mathrm{A} \to \mathrm{B}$ is any functor then there is a proextension of F,

$$pro(F): pro(\mathrm{A}) \to pro(\mathrm{B}),$$

defined by:

$$\text{if } X: I \to \mathrm{A},\ pro(F)(X) = F \circ X: I \to \mathrm{B}.$$

Taking $F = \tau : \mathrm{A} \to \mathrm{A}$, one has : *$pro(\tau)$ is naturally equivalent to the torsion subradical associated to* E(C).

Finally if $L : \mathrm{A} \to \mathrm{A}$ is the localisation functor associated with C and $\psi : 1 \to L$ the corresponding natural transformation, then for E(C) one has (up to natural equivalence):

$pro(L) : pro(\mathrm{A}) \to pro(\mathrm{A})$ is the localisation functor and $pro(\psi)$, the corresponding natural transformation $pro(\psi) : 1 \to pro(L)$.

We will use the notation $\bar{\tau} = pro(\tau)$, $\bar{T} = pro(T)$, *etc.* Because of the natural equivalences concerned, this slight abuse of notation with $\bar{T}$ should not cause any confusion. (For the basic ideas of localisation theory, we refer the reader to the original source: Gabriel [2] or for a more recent treatment Popescu [11]. The author has also found Hacque [5, 6] to be extremely useful. For localisations in procategories and in particular for the proofs of the above results we refer the reader again to [12].)

5. THE VALUES OF $lim^{(i)}$ ON $\mathrm{E}(\mathrm{A}_{n,p.f.g.})$.

We shall denote by $\mathrm{A}_{n,p.f.g.}$ the full subcategory of A_n defined by the p. f. g. modules.

THEOREM 5.1. *Let M be a projective system in* A *which is essentially of type $\mathrm{A}_{n,p.f.g.}$, then:*

(i) If $0 \le k \le n+1$, $lim^{(k)} M \in wLim(\mathrm{A}_{n-k,p.f.g.})$,

(ii) If $k \ge n+1$, $lim^{(k)} M = 0$.

REMARK. Case (ii) is that previously handled in [12] and it drops out of this proof in exactly the way it was proved there. Of course, the case $k = n+1$ is repeated but since $\mathrm{A}_{-1} = \{0\}$ there is no problem here. We shall not provide a repetion of the proof of (ii).

PROOF OF THEOREM 5.1 (i). The case $n = -1$ is trivial. In fact, the case $n = 0$ is only slightly less trivial as the case of $lim^{(1)}$ is essentially classical here, whilst the other interesting case $k = 0$ states merely that $lim\, M \in wLim(\mathrm{A}_{0,p.f.g.})$, which is immediate from the definitions. We shall thus assume the result for all $M \in \mathrm{E}(\mathrm{A}_{r,p.f.g.})$ for $r < n$ and work with a projective system M in $\mathrm{E}(\mathrm{A}_{n,p.f.g.})$.

We start with the simplest possible kind of M namely we assume

(a) $\bar{\tau}^{n-1}(M) = 0$,

(b) $T_{n-1}(M)$ is isomorphic in $pro(A/A_{n-1})$ to a system in which each $T_{n-1}(M)(i)$ is a direct sum of at most l simple objects for some fixed but arbitrary l.

Then $T_{n-1}(M)$ is isomorphic in $pro(A/A_{n-1})$ to a finite direct sum of simple objects of A/A_{n-1} (considered via the embedding

$$h: A/A_{n-1} \to pro(A/A_{n-1})$$

as constant projective systems). Thus $\bar{T}_{n-1}(M) \approx h(\bigoplus_{i=1}^{m} S_i)$, S_i simple in A/A_{n-1}. By the definition of p. f. g., we can choose each S_i to be the image of some p. f. g. n-simple N_i. As the sum is finite and T_{n-1} is exact, we have

$$\bar{T}_{n-1}(M) \approx h(\bigoplus_{i=1}^{m} S_i) \approx h(T_{n-1}(\bigoplus_{i=1}^{m} N_i)) \approx \bar{T}_{n-1}(h(\bigoplus_{i=1}^{m} N_i)).$$

Using the description of isomorphisms in localised categories, one obtains (cf. [11], pages 52-53) that this composite isomorphism can be represented by a pair of monomorphisms

$$M \xleftarrow{\ s\ } M' \xrightarrow{\ f\ } h(N)$$

where we have written N for $\bigoplus_{i=1}^{m} N_i$ and where $Coker\, s$ and $Coker\, f$ are in $E(A_{n-1})$ and as M and $h(N)$ are p. f. g., we have that $Coker\, s$ and $Coker\, f$ are in $E(A_{n-1,p.f.g.})$ and thus are covered by the induction hypothesis.

To link $lim^{(k)}M$ and $lim^{(k)}h(N)$, we use the long exact sequences corresponding to

(c) $\qquad 0 \longrightarrow M' \longrightarrow M \longrightarrow Coker\, s \longrightarrow 0$

and

(d) $\qquad 0 \longrightarrow M' \rightarrow h(N) \rightarrow Coker\, f \longrightarrow 0.$

From (d) we have

(e) $\quad 0 \longrightarrow lim\, M' \longrightarrow lim\, h(W) \longrightarrow lim\, Coker\, f \longrightarrow lim^{(1)}M' \longrightarrow 0$

and

(f) for $k > 0$, $\quad lim^{(k)}Coker\, f \approx lim^{(k+1)}M'$.

If we put on $lim^{(k+1)}M$ the linear topology corresponding to

$A_{n-(k+1), p.f.g.}$, we have that the isomorphism in (f) is continuous ($lim^{(k)} Coker f$ already has a topology coming from the fact that it is in $wLim(A_{n-1-k, p.f.g.})$). Thus we have that

$$lim^{(k+1)} M' \in wLim(A_{n-(k+1), p.f.g.}), \quad 0 < k \leq n+1.$$

For $k = 0$ one gives $lim^{(1)} M'$ the A_{n-1}-topology, then as $lim\, h(N)$ is protorsion, $lim^{(1)} M'$ is the quotient of a (complete) $pro(A_{n-1})$-torsion module by a closed submodule, hence

$$lim^{(1)} M' \in Lim(A_{n-1, p.f.g.}) \subset wLim(A_{n-1, p.f.g.}).$$

Of course

$$lim\, M' \in Lim(A_{n, p.f.g.}) \subset wLim(A_{n, p.f.g.})$$

so this particular case is finished.

Next turning to (c) we find

$$0 \longrightarrow lim\, M' \longrightarrow lim\, M \longrightarrow lim\ Coker\, s \longrightarrow lim^{(1)} M' \longrightarrow \ldots .$$

Each exact segment

$$lim^{(k)} M' \xrightarrow{\alpha} lim^{(k)} M \xrightarrow{\beta} lim^{(k)} Coker\, s$$

has both ends in $wLim(C_{n-k, p.f.g.})$. $Ker \beta$ is a closed submodule of $lim^{(k)}(M)$ and $Im \beta$ is a closed submodule of $lim^{(k)} Coker\, s$ in the relevant topologies: $A_{n-k, p.f.g.}$ for $lim^{(k)} M$ and $A_{n-k-1, p.f.g.}$ for $lim^{(k)} Coker\, s$. Thus since $wLim(A_{n-k, p.f.g.})$ is closed under extensions of this kind

$$lim^{(k)} M \in wLim(A_{n-k, p.f.g.}) \quad \text{for all} \quad 0 \leq k \leq n+1.$$

Now we turn to a more general type of projective system M. Namely we require that each $T_{n-1}(M(i))$ is still a finite direct sum of n-simples, but now the number of direct summands need not be bounded. Such an M is a special direct limit of its subsystems of the type just considered. Thus there is an epimorphism

$$colim\ lim^{(k)} M \longrightarrow lim^{(k)} colim\, M = lim^{(k)} M$$

where M is the special direct system in question.

We can put the $A_{n-k, p.f.g.}$-topology on both sides, but unfortunately as mentioned previously, we do not know if the kernel of this morphism is closed. In any case as $colim(lim^{(k)}M) \in wLim(A_{n-k,\,p.f.g.})$, the same is true for $lim^{(k)}M$.

Passing to a yet more general M with

$$M \in E(A_{n,\,p.f.g.}) \text{ and } \bar{\tau}^{n-1}(M) = 0,$$

we find each factor used in the construction of $\tau^n(M)$ is of a form already considered. For limit ordinals, one will of course have to use the epimorphism

$$\underset{\alpha<\beta}{colim}\, lim^{(k)}\tau^n_\alpha(M) \longrightarrow lim^{(k)}\tau^n_\beta(M)$$

but the end result will again be that

$$lim^{(k)}M \in wLim(A_{n-k,\,p.f.g.}).$$

Finally if M is arbitrary in $E(A_{n,\,p.f.g.})$ then

$$(g) \qquad 0 \longrightarrow \tau^{n-1}(M) \longrightarrow M \longrightarrow M/\tau^{n-1}(M) \longrightarrow 0$$

is exact. $\tau^{n-1}(M) \in E(A_{n-1,\,p.f.g.})$ and $M/\tau^{n-1}(M)$ has just been handled as it satisfies $\tau^{n-1}(M/\tau^{n-1}(M)) = 0$. The long exact sequence corresponding to (g) together with extensions completes the proof of the theorem.

The weakness of the class concerned, $wLim(A_{n-k,\,p.f.g.})$ lies in the lack of topological conditions on subobjects and quotients. If one does not take direct limits in the argument for a particular M then for each step in the calculation, one finds that the topological modules involved are separated (i. e. $\{0\}$ is closed) and hence, kernels of morphisms are closed subobjects. Thus one is led to the following more restrictive, but stronger, version of the theorem.

THEOREM 5.2. *Let M be a projective system in* A *which is essentially of type $A_{n,\,p.f.g.}$ and such that the numbers*

$$\sup_{i\in I}\{long^k(M(i))\} < \infty \quad \text{for } k = 0, 1, \ldots, n,$$

then for each $0 \leq k \leq n+1$,

$$lim^{(k)}M \in Lim(A_{n-k,\,p.f.g.}).$$

REMARK. As completeness is not in general inherited by quotients (even when the corresponding submodule is closed), this class is slightly less good than it might at first appear.

6. APPLICATIONS.

(a) As mentioned in [12], it is fairly easy to use the vanishing of $lim^{(i)}$ to obtain results on the vanishing of $Ext^i_A(M,N)$ when M is flat and N is p. f. g. of Krull dimension n (cf. 6.11 of [12]). The result below applies the same arguments to obtain other information on the $Ext^i_A(M,N)$ in this situation.

PROPOSITION 6.1. *Let A be a commutative ring, M a flat A-module and N a p. f. g. A-module of Krull-Gabriel dimension n, then*

(i) if $0 \leq k \leq n+1$, $Ext^{(k)}_A(M,N) \in wLim(A_{n-k,\, p.f.g.})$.

(ii) if $k \geq n+1$, $Ext^{(k)}_A(M,N) = 0$.

Writing $M = colim\, L_\alpha$, L_α free of finite type, we have from Jensen [9] a spectral sequence

$$E_1^{p,q} = \varprojlim{}^{(p)} Ext^{(q-p)}_A(L_\alpha, N) \Rightarrow Ext^{(n)}_A(M,N)$$

which, since the L_α are free, degenerates to a sequence of isomorphisms

$$lim^{(p)} Hom(K_\alpha, N) \approx Ext^{(n)}_A(M,N).$$

If $rank\, L_\alpha = r_\alpha$, $Hom(L_\alpha, N) = N^{r_\alpha}$ so we are in a situation in which 5.1 can be used. This gives the result.

In Gruson and Raynaud [4] one finds the statement that the infinimum of those n for which $Ext^{n+1}_A(M,N)$ is zero for all flat M is the same as the pure injective dimension of N. One finds a proof of this in Gruson and Jensen [3]. Using this we obtain:

COROLLARY 6.2. *If A is a commutative ring, any p. f. g. A-module N with Krull-Gabriel dimension $\leq n$ has pure injective dimension $\leq n$.*

REMARK. If N is Noetherian then as mentioned before, the (composite) k-length of N is finite for all $k \leq n$. If then M is a flat module such that the L_α's in the above proof can be chosen to have bounded rank then one can replace $wLim$ by Lim in the statement of the proposition.

(b) The modules in $A_{0,p.f.g.}$ are Artinians so the protorsion completion of a module with respect to $A_{0,p.f.g.}$ is a strictly linearly compact module in the $A_{0,p.f.g.}$-topology (which is the inverse limit topology as well).

Jensen proves in [8] that derived functors of *lim* vanish on systems of strictly linearly compact modules (with continuous «bonding» maps).

Supposing that $\{M(i) \mid i \in I\}$ is the given system, he uses the strict continuity of the transition morphisms $p_i^j : M(j) \to M(i)$ to find a partially ordered set S and submodules $U_{\lambda,i} \subset M(i)$, $\lambda \in S$, $i \in I$ such that:

(1) $U_{\lambda,i}$, $\lambda \in S$, form a fundamental system of neighborhoods of 0 in $M(i)$ such that $M(i)/U_{\lambda,i}$ is Artinian;

(2) $U_{\lambda,i} \subset U_{\mu,i}$ for all $i \in I$ and all $\lambda \leq \mu$;

(3) $p_i^j(U_{\lambda,j}) \subset U_{\lambda,i}$ for all $\lambda \in S$ and $j \to i$.

Each $M(i)$ satisfies

$$M(i) = \lim_S (M(i)/U_{\lambda,i})$$

and one has

$$\lim_I{}^{(q)} M(i) = \lim_I{}^{(q)} \lim_S (M(i)/U_{\lambda,i})$$

Roos [13] gives the spectral sequence

$$E_1^{p,q} = \begin{cases} \lim_I{}^{(p)}(\lim{}^{(q-p)}(M(i)/U_{\lambda,i})) & p \geq 0, \quad q \geq p \\ 0 & \text{otherwise} \end{cases}$$

which converges to $\lim_{I \times S}{}^{(n)}(M(i)/U_{\lambda,i})$.

As each $M(i)/U_{\lambda,i}$ is Artinian, these limit terms for $n > 0$ and E_1-terms other than for $p = q$ vanish and one reads off:

$$\lim{}^{(p)} M(i) = 0 \quad p > 0.$$

We shall use a version of this argument but with «Artinian» replaced by «in $A_{n,p.f.g.}$» to obtain information on derived limits of systems of certain types of protorsion modules.

We shall say that a protorsion module M *is countably protorsion* if it has a countable cofinal family of open neighborhoods U of 0 such that M/U is «torsion». Thus M is countably protorsion iff it is an inverse limit of a sequence of torsion modules linked by epimorphisms. (It is well known that on inverse sequences with epimorphic transition morphisms the $lim^{(i)}$, $i > 0$, vanish cf. Jensen [9] Chapter 2.)

If $M : I \to \mathrm{A}$ is a projective system of countably protorsion modules (relative to some generalized torsion class C) and strict morphisms (cf. Bourbaki [1], III, 2.8) then we can repeat Jensen's argument to find a system $\{U_{\lambda,i} \mid i \in I,\ \lambda \in S\}$ satisfying his conditions except that, now, $M(i)/U_{\lambda,i} \in \mathrm{C}$.

In the spectral sequence we have

$$E_1^{1,p} = lim^{(p)}(lim\, M(i)/U_{\lambda,i}) = lim^{(p)} M(i)$$

and zero otherwise. The spectral sequence, as before, converges to $lim^{(n)} M(i)/U_{\lambda,i}$.

PROPOSITION 6.3. *Let* $M : I \to \mathrm{A}$ *be a projective system of countably protorsion modules (for* $\mathrm{C} = \mathrm{A}_{n,\,p.f.g.}$*) and strict morphisms then*

(i) if $0 \le k \le n+1$, $lim^{(k)} M \in wLim(\mathrm{A}_{n-k,\,p.f.g.})$,

(ii) if $k \ge n+1$, $lim^{(k)} M = 0$.

The rest of the proof comes from feeding of 5.1 into the limit term of the spectral sequence. The collapse of the spectral sequence gives

$$lim^{(k)} M \approx \lim_{I \times S}{}^{(k)}(M(i)/U_{\lambda,i}).$$

REMARK. Clearly by bounding the composite k-lengths of the $M(i)/U_{\lambda,i}$ one can obtain

$$lim^{(k)} M \in Lim(\mathrm{A}_{n-k,\,p.f.g.}).$$

The same basic argument as used above works whenever one has a system M which is expressible as a limit over one factor of a product, i.e. $M' : I \times S \to \mathrm{A}$, $M = \lim_S M' : I \to \mathrm{A}$. An important instance of this occurs when completing with respect to an n-saturated ideal.

For each n, there is a complete modular lattice

$$C_n(A) = \{L \subset A \mid \tau^n(A/L) = 0\}$$

of right ideals (see Stenström [14] and Popescu [11]). These ideals are called n-saturated. ($C_n(A)$ is never empty since A is always there, but it can happen that A is the only ideal in $C_n(A)$.)

For simplicity, throughout the rest of this paper we shall assume A is right Noetherian. If $J \subset A$ is n-saturated then for any finitely generated right A-module M and any $k > 0$, $M/J^k M$ is in $\mathrm{A}_{n, p.f.g.}$.

If we have a projective system $M: I \to \mathrm{A}$, we can thus build a new one $M: I \times \mathrm{N} \to \mathrm{A}$ with $M(i, k) = M(i)/J^k M(i)$. Taking the pointwise limit over N we get the J-adic completion $\hat{M}: I \to \mathrm{A}$ of M. An obvious adaptation of the proof of 6.3 then gives (with the above notation):

PROPOSITION 6.4.

(i) If $0 \leq k \leq n+1$, $lim^{(k)}\hat{M} \in wLim(\mathrm{A}_{n-k, p.f.g.})$.

(ii) If $k \geq n+1$, $lim^{(k)}\hat{M} = 0$.

The interest of this result is of course that by passing to the completion we have «killed off» all of the higher order $lim^{(k)}$. To study this process better one needs to have a lot of information on the kernel and cokernel of the completion morphism $\phi: M \to \hat{M}$. If J is contained in the n-radical of A, i.e. the intersection of the maximal elements of $C_n(A)$, then $Ker\, \phi \subset r^{n-1}(M)$ and thus is fairly easily handled. This result need other methods than those developed here and will be proved in a subsequent note. With this last result (6.4) one begins to see the possibility of extending the neat result of Jensen ([9], 8.1) in which the finite products of complete local rings are characterised amongst the commutative Noetherian rings by means of criteria relating to the vanishing of $lim^{(i)}$ and $Ext_A^{(i)}$. As this result is the prototype of other interesting results of Gruson and Jensen [3], an extension to rings complete in other topologies would be of some interest.

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