CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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On the categories $Sp(X)$ and $Ban(X)$

Cahiers de topologie et géométrie différentielle catégoriques, tome 24, nº 1 (1983), p. 97-112

<http://www.numdam.org/item?id=CTGDC_1983__24_1_97_0>

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ET GÉOMÉTRIE DIFFÉRENTIELLE

ON THE CATEGORIES *Sp(X)* AND *Ban(X)*

by Anthony Karel SEDA

1. INTRODUCTION.

Let $Ban(X)$ denote the category of Banach bundles and linear contractions over a fixed locally compact Hausdorff space X and $Spp(X)$ denote the category of spaces over X whose projections are proper mappings. Our objective, here, is to describe the construction of a pair of contravariant functors

$$\bar{A}: Spp(X) \to Ban(X) \quad \text{and} \quad \bar{S}: Ban(X) \to Spp(X)$$

which are adjoint on the right, but which do not determine an equivalence of categories. These ideas are a continuation of [6] and partially complement the work of several authors including Pelletier and Rosebrugh, Mulvey, Burden and Hofmann, see [5] and its references, in that they extend methods of classical functional analysis to the category $Ban(X)$. For example, the counit of our adjunction yields an isometric embedding ϵ_E : $E \to \bar{A}(\bar{S}(E))$ of any Banach bundle E in a Banach bundle whose fibres are spaces of continuous functions. Thus, we generalise the classical result of Alaoglu which gives an isometric embedding of a Banach space E in the space $\bar{A}(S^*)$ of continuous scalar valued functions, where S^* denotes the closed unit ball of the dual E^* endowed with the weak* topology. If we specialise to the case when X is a singleton set, then our embedding coincides with Alaoglu's but even in this special case the adjointness seems to be new *).

*) I am indebted to the referee for pointing out that the adjointness in this special case is not in fact new and has been discussed by Z. Semadeni in his article: « Categorical approach to extension problems », Proceedings of International Symposium on Extension theory of topological structures and its applications, VEB Deutsches Verlag der Wissenschaften, Berlin, 1969.

Actually, our usage of the term Banach bundle, and hence, of $Ban(X)$, is that of [1] and [2], which serve as references for definitions and basic properties, and differs somewhat from the usage of [5], or more precisely of that of Reference 11 in [5]. For one thing, we assume that the norm function is continuous rather than simply upper semi-continuous. Indeed, since the norm on $\bar{A}(Y)$ turns out to be continuous for any Y in $Spp(X)$ and in particular for $Y = \bar{S}(E)$, it follows from the continuity of ϵ_E that the norm on E is necessarily continuous and hence that our results do not in the main extend beyond $Ban(X)$. Finally, we note that, unless stated to the contrary, our scalar field is that of the complex numbers C.

2. THE FUNCTOR $\bar{A}: Sp(X) \to Ban(X)$.

Let $p: E \to X$ and $p': E' \to X$ be Banach bundles over X. A morphism $\Psi: (E, p) \to (E', p')$ in $Ban(X)$ is a fibre preserving continuous function $\Psi: E \to E'$ which is a linear contraction on fibres, that is, $\Psi_x = \Psi|_{E_x}$ is a linear operator on E_x with $||\Psi_x|| \leq 1$ (operator norm) for each $x \in X$, where E_x denotes the fibre of E over x.

An object in the category $Sp(X)$ of spaces over X is a continuous open surjection $q: Y \to X$, where Y is a locally compact Hausdorff space. If $q': Y' \to X$ is also an object in $Sp(X)$, then a morphism $\eta: (Y, q) \to (Y', q')$ is a continuous, proper and fibre preserving mapping $\eta: Y \to Y'$. The category $Sp(X)$ (or its objects ar least) is of course well studied. For example, James [3] has considered general topology in $Sp(X)$ somewhat in the spirit of this article. In fact, what we are showing is that general topology in $Sp(X)$ is related to functional analysis in $Ban(X)$ via $\bar{A}$ and $\bar{S}$ in the same way that (locally compact Hausdorff) topological spaces are related to Banach spaces.

We begin by stating a basic result in the theory due to Douady and dal Soglio-Hérault, see Appendix of [2] for proof.

THEOREM 1. *Let $p: E \to X$ be a Banach bundle over X and let $s \in E$. Then there exists a section σ of p such that $\sigma(p(s)) = s$.*

A section σ with the stated properties of Theorem 1 is said to *pass through* s and it follows that sets of the type

$$U(\sigma, V, \epsilon) = \{ t \in E \mid p(t) \in V, \ \| t - \sigma(p(t)) \| < \epsilon \}$$

form a neighborhood base at s as V ranges over neighborhoods of $p(s)$ in X and ϵ over positive real numbers. As an application of this fact one can prove the following proposition.

PROPOSITION 1. *Suppose $\Psi: E \to E'$ is a fibre preserving mapping such that Ψ_x is a bounded linear operator for all $x \in X$ and $\|\Psi_x\|$ is locally bounded on X. Suppose also that there is a vector space Γ of sections of E with the properties:*

(i) $\{ \gamma(x) \mid \gamma \in \Gamma \}$ is dense in E_x for each $x \in X$.

(ii) $\Psi \circ \gamma: X \to E'$ is a section of E' for each $\gamma \in \Gamma$.

Then Ψ is continuous.

We omit details but refer the reader to [6], Section 3 for the type of argument required.

With these preliminaries established we turn next to the description of $\bar{A}$. Let $q: Y \to X$ be an object in $Sp(X)$, let $Y_x = q^{-1}(x)$ be the fibre of q over x and let $A_x = C_0(Y_x)$ be the space of all scalar valued continuous functions on Y_x which vanish at infinity (a function f vanishes at infinity if for each $\epsilon > 0$ there exists a compact set in the domain of f on the complement of which $|f(x)| < \epsilon$). When endowed with the uniform norm $\| \ \|_x = \| \ \|$, A_x becomes a Banach space. Let $A = \bigcup_{x \in X} A_x$ and let $p: A \to X$ be the obvious projection with fibre A_x over x. For each function ϕ belonging to the space $\mathfrak{k}(Y)$ of scalar valued continuous functions on Y with compact support, define

$$\tilde{\phi}: X \to A \quad \text{by} \quad \tilde{\phi}(x) = \phi|_{Y_x} \in \mathfrak{k}(Y_x).$$

Let Γ denote the vector space $\{ \tilde{\phi} \mid \phi \in \mathfrak{k}(Y) \}$. In [6] we established the following theorem (see [1], Proposition 1.6).

THEOREM 2. *(i) For each $x \in X$ the set $\{ \tilde{\phi}(x) \mid \tilde{\phi} \in \Gamma \}$ is dense in A_x.*

(ii) For each $\tilde{\phi} \in \Gamma$ the numerical function $\|\tilde{\phi}(x)\|$ is continuous on X.

(iii) A is a Banach bundle over X when A is endowed with the topology determined by the sets

$$U(\tilde{\phi}, V, \epsilon) = \{ a \epsilon A \mid p(a) \epsilon V, \|a - \tilde{\phi}(p(a))\| < \epsilon \},$$

where $\tilde{\phi} \epsilon \Gamma$, *V is open in X and* $\epsilon > 0$. *Moreover, each element* $\tilde{\phi}$ *of* Γ *is continuous and is therefore a section of p; and this topology is unique with these properties.*

To define $\bar{A}$ on objects we set $\bar{A}(Y)$ to be the Banach bundle $p: A \to X$ described above. If $\eta : (Y, q) \to (Y', q')$ is a morphism in $Sp(X)$, we define

$$\bar{A}(\eta): \bar{A}(Y') \to \bar{A}(Y) \text{ by } \bar{A}(\eta)(f) = f \circ \eta_x ,$$

where $x = p'(f)$ and η_x denotes the restriction of η to Y_x. Because η is proper, $\eta^{-1}(C)$ is compact in Y for each compact set C in Y' and from this it follows that $f \circ \eta_x$ vanishes at infinity. Moreover, $\bar{A}(\eta)(f)$ is clearly continuous for each $f \epsilon \bar{A}(Y')_x$, which means that $\bar{A}(\eta)$ does map $\bar{A}(Y')$ into $\bar{A}(Y)$. It is clear that $\bar{A}(\eta)$ is fibre preserving and routine to verify linearity on fibres. Since

$$\|\bar{A}(\eta)_x(f)\| = \sup_{y \epsilon Y_x} |f \circ \eta_x(y)| \leq \sup_{y \epsilon Y'_x} |f(y)| = \|f\|$$

for all f, we have $\|\bar{A}(\eta)_x\| \leq 1$ for each $x \epsilon X$. Next we observe that if $\phi \epsilon \mathfrak{k}(Y')$, then

$$\phi \circ \eta \epsilon \mathfrak{k}(Y) \quad \text{and} \quad \widetilde{\phi \circ \eta} = \bar{A}(\eta)(\tilde{\phi}).$$

So by applying Proposition 1 and Theorem 2 we conclude that $\bar{A}(\eta)$ is continuous. It is routine to establish functoriality of $\bar{A}$ and we may summarise these conclusions as follows:

PROPOSITION 2. *$\bar{A}$ is a contravariant functor from $Sp(X)$ to $Ban(X)$.*

Let $q: Y \to X$ be an object in $Sp(X)$, and let $\bar{A}(Y) \times_X Y$ denote the fibred product

$$\{ (f, y) \mid p(f) = q(y) \}$$

regarded as a subspace of $\bar{A}(Y) \times Y$. There is a scalar valued function ρ defined on $\bar{A}(Y) \times_X Y$ by $\rho(f, y) = f(y)$ and called evaluation. The following result will not be needed until Section 4 but it is convenient to include it in this section.

PROPOSITION 3. *The evaluation map* ρ *is continuous.*

PROOF. Let $f \in \bar{A}(Y)_x$ and $y \in Y_x$. Given $\epsilon > 0$, let $\theta \in \Bbbk(Y)$ such that $\|\tilde{\theta}(x) - f\| < \epsilon/3$. For $(f', y') \in \bar{A}(Y) \times_X Y$ we have

$$|f'(y') - f(y)| \leq$$

$$|f'(y') - \tilde{\theta}(q(y'))(y')| + |\tilde{\theta}(q(y'))(y') - \tilde{\theta}(x)(y)| + |\tilde{\theta}(x)(y) - f(y)|$$

$$\leq \|f' - \tilde{\theta}(q(y'))\| + |\tilde{\theta}(q(y'))(y') - \tilde{\theta}(x)(y)| + \|\tilde{\theta}(x) - f\|.$$

Let O be a neighborhood of y in Y such that

$$|\theta(y') - \theta(y)| < \frac{\epsilon}{3} \quad \text{for all } y' \in O$$

and let $V = q(O)$, then V is a neighborhood of x in X since q is open. By definition of $\tilde{\theta}$ we have now that

$$|\tilde{\theta}(q(y'))(y') - \tilde{\theta}(x)(y)| = |\theta(y') - \theta(y)| < \frac{\epsilon}{3}$$

for all $y' \in O$. Put $U = U(\tilde{\theta}, V, \epsilon/3)$. Then, if $f' \in U$ and $y' \in O$ with $p(f') = q(y')$ we obtain

$$|f'(y') - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

by means of the inequality above and so ρ is continuous as required.

3. THE FUNCTOR $\bar{S}: Ban(X) \to Sp(X)$.

Let $p: E \to X$ be a Banach bundle over X, let E_x^* denote the dual of E_x endowed with the operator norm $\| \ \|_x = \| \ \|$ and let $E^* = \bigcup_{x \in X} E_x^*$ equipped with the obvious projection $p^*: E^* \to X$. Given a section σ of p we define

$$F_\sigma: E^* \to \mathbb{K} \quad \text{by} \quad F_\sigma(f) = f(\sigma(p^*(f))),$$

where $\mathbb{K}$ denotes the scalar field. Let $\Omega = \{ p^*, F_\sigma \mid \sigma \in \Sigma \}$, where Σ or $\Sigma(E)$ denotes the set of all sections of p, and give E^* the weak topology generated by Ω.

PROPOSITION 4. *a)* $p^*: E^* \to X$ *is continuous.*

b) E^* *is Hausdorff.*

c) For each $x \in X$ the induced topology on E_x^* is the weak topology.*

PROOF. a is obvious.

b) Since the range space of each function in Ω is Hausdorff it suffices to show that Ω separates points of E^*. If

$$p^*(f_1) = x \neq y = p^*(f_2),$$

then p^* separates f_1 and f_2. Otherwise, there is an element $s \in E_x$, where $p^*(f_1) = x = p^*(f_2)$, such that $f_1(s) \neq f_2(s)$. By Theorem 1 there is a section σ of p passing through s and then

$$F_\sigma(f_1) = f_1(s) \neq f_2(s) = F_\sigma(f_2)$$

and so Ω separates points of E^*.

c) This follows from Theorem 1 and the fact that subspaces in weak topologies have the weak topology generated by the restrictions of the functions in the generating family.

We denote by $\bar{S}(E)$ the subspace of E^* consisting of all those f with $\|f\| \leq 1$ and by q the restriction of p^* to $\bar{S}(E)$. Then $\bar{S}(E)$ is Hausdorff, q is continuous and $\bar{S}(E)_x = q^{-1}(x)$ is compact for each $x \in X$.

PROPOSITION 5. *The map $q: \bar{S}(E) \to X$ is open.*

PROOF. Let $f_0 \in \bar{S}(E)$, let $\sigma \in \Sigma$ and let $\epsilon > 0$. We define the set $S(\sigma, f_0, \epsilon)$ by

$$S(\sigma, f_0, \epsilon) = \{ f \in \bar{S}(E) \mid |F_\sigma(f) - F_\sigma(f_0)| < \epsilon \}$$

$$= \{ f \in \bar{S}(E) \mid |f(\sigma(q(f))) - f_0(\sigma(q(f_0)))| < \epsilon \}.$$

Sets of this type together with the sets $q^{-1}(U)$, U open in X, form a subbasis for $\bar{S}(E)$ and it suffices to show that $q(S(\sigma, f_0, \epsilon))$ is a neighborhood of $x_0 = q(f_0)$ in X. Let A be a scalar such that

$$f_0(\sigma(x_0)) = A \|\sigma(x_0)\|.$$

If $f_0(\sigma(x_0)) \neq 0$, then

$$|A| \leq \|f_0\| \leq 1$$

and if, on the other hand, $f_0(\sigma(x_0)) = 0$, then we choose $A = 0$ and so in any event $|A| \leq 1$. Since the norm on E is continuous and $\sigma \in \Sigma$, there is a neighborhood V of x_0 in X such that

$$\big| \|\sigma(x)\| - \|\sigma(x_0)\| \big| < \frac{\epsilon}{1+|A|}$$

for all $x \in V$. If $\sigma(x) = 0$, then the zero functional $f \in \bar{S}(E)_x$ has the property that $f(\sigma(x)) = A\|\sigma(x)\|$. If $\sigma(x) \neq 0$, the Hahn-Banach theorem shows that there is an $f \in \bar{S}(E)_x$ with $\|f\| = |A|$ and such that $f(\sigma(x)) = A\|\sigma(x)\|$. Hence, for each $x \in V$ there is an $f \in \bar{S}(E)_x$ such that

$$|f(\sigma(q(f))) - f_0(\sigma(q(f_0)))| = |A| \, \big|\|\sigma(x)\| - \|\sigma(x_0)\|\big| <$$
$$< \frac{|A|}{1+|A|}\epsilon < \epsilon$$

and it follows that $f \in S(\sigma, f_0, \epsilon)$. Therefore, $V \subset q(S(\sigma, f_0, \epsilon))$ and so q is open.

PROPOSITION 6. *The space* $\bar{S}(E)$ *is locally compact.* *)

PROOF. Let K_1 be a compact set in X and let $K_1' = q^{-1}(K_1)$; we show that K_1' is compact in $\bar{S}(E)$. If $\sigma \in \Sigma$ and $f \in K_1'$, then

$$|F_\sigma(f)| \leq \|f\| \, \|\sigma(q(f))\| \leq \sup_{x \in K_1} \|\sigma(x)\| < +\infty ;$$

let

$$C_\sigma = \{ z \mid z \text{ is a scalar and } |z| \leq \sup_{x \in K_1} \|\sigma(x)\| \}.$$

Since Ω separates points of E^* the evaluation map ev embeds K_1' in the product $K_1 \times \prod_{\sigma \in \Sigma} C_\sigma$ which is compact. Suppose G is an element of the closure of $ev(K_1')$ in $K_1 \times \prod_{\sigma \in \Sigma} C_\sigma$. Then G is a function on the index set $\{1\} \cup \Sigma$ and there is a net f_α in K_1' such that $ev(f_\alpha) \to G$. This latter statement is equivalent to

$$q(f_\alpha) = x_\alpha \to x = G(1)$$

in K_1 and $F_\sigma(f_\alpha) \to G(\sigma)$ in C_σ for each $\sigma \in \Sigma$, and this in turn is equi-

*) Kitchen and Robbins [4] have given a construction of a space K^* similar to but different from $\bar{S}(E)$ and a map $\pi^*: K^* \to X$. They show K^* is locally compact and that their projection π^* is continuous, but they do not show π^* is open. However their proof of local compactness is different from ours and so too are their objectives. Furthermore they do not show that π^* is a proper map which is essential for us and indeed is what we are really proving here about q, see Proposition 7.

valent to

(1) $x_\alpha \to x$ in K_1 and $f_\alpha(\sigma(x_\alpha)) \to G(\sigma)$ in C_σ for each $\sigma \in \Sigma$.

Now define $\hat{G}$ on E_x as follows:

(2) if $\sigma \in \Sigma$ and $\sigma(x) = s$, then $\hat{G}(s) = G(\sigma)$.

To see that $\hat{G}$ is well defined, suppose

$$w \in \Sigma \quad \text{and} \quad w(x) = \sigma(x) = s.$$

Then

$$|f_\alpha(\sigma(x_\alpha)) - f_\alpha(w(x_\alpha))| \leq ||\sigma(x_\alpha) - w(x_\alpha)|| \to 0$$

and so $f_\alpha(\sigma(x_\alpha))$ and $f_\alpha(w(x_\alpha))$ converge to the same limit. Hence, $G(\sigma) = G(w)$ by (1) and it follows that $\hat{G}$ is a well defined scalar valued function on E_x. Given $s \in E_x$ there is by the proof of [1], Proposition 1.5, a section $\gamma \in \Sigma$ passing through s with the property

$$||\gamma(y)|| \leq ||\gamma(x)|| \quad \text{for all } y \in K_1.$$

Hence

$$|\hat{G}(s)| = |\lim_\alpha f_\alpha(\gamma(x_\alpha))| \leq \lim_\alpha ||\gamma(x_\alpha)|| \leq ||s||$$

and so $||\hat{G}|| \leq 1$. One shows in like fashion that $\hat{G}$ is linear on E_x and it now follows that $\hat{G} \in \bar{S}(E)_x$. Since it is clear that $ev(\hat{G}) = G$, we now conclude that $ev(K_1')$ is closed in $K_1 \times \prod_{\sigma \in \Sigma} C_\sigma$ and finally that K_1' is compact as we require.

PROPOSITION 7. *The projection $q: \bar{S}(E) \to X$ is a proper map.*

PROOF. The proof of Proposition 6 shows that $q^{-1}(K)$ is compact in $\bar{S}(E)$ for each compact set K in X and so q is a proper map since X is Hausdorff.

The functor $\bar{S}: Ban(X) \to Sp(X)$ is now defined as follows: the image of (E, p) in $Ban(X)$ under the object function of $\bar{S}$ is $(\bar{S}(E), q)$ as described above. If $\Psi: E \to F$ is a morphism in $Ban(X)$, then $\bar{S}(\Psi)$ is the restriction $\Psi^*|_{\bar{S}(F)}$ of the «conjugate operator» Ψ^*. Thus,

$$\bar{S}(\Psi)(g) = \Psi^*(g) = g \circ \Psi_x \quad \text{for} \quad g \in \bar{S}(F)_x.$$

There are several things to check. Firstly, the expression $\Psi^*(g) = g \circ \Psi_x$

where $g \in F_x^*$, actually defines Ψ^* on all of F^* and determines a map $\Psi^*: F^* \to E^*$. We show in fact that Ψ^* is continuous on all of F^*. If p_E^* and p_F^* denote the respective projections on E^* and F^*, then we have $p_E^* \circ \Psi^* = p_F^*$ and so $p_E^* \circ \Psi^*$ is continuous. If $\sigma \in \Sigma(E)$, then $F_\sigma \circ \Psi^* = F_{\Psi\sigma}$ and so $F_\sigma \circ \Psi^*$ is also continuous. From this it follows that Ψ^* is continuous and hence that $\bar{S}(\Psi)$ is too. Next, if $g \in \bar{S}(F)_x$, then

$$\|\Psi^* g\| = \|g \circ \Psi_x\| \leq \|g\| \|\Psi_x\| \leq 1$$

and so $\bar{S}(\Psi)$ maps $\bar{S}(F)$ into $\bar{S}(E)$. Obviously $\bar{S}(\Psi)$ is fibre preserving. Thirdly, if $K \subset \bar{S}(E)$ is compact, than $q_F^{-1}(q_E(K))$ is compact by Proposition 7, where q_E denotes the projection on $\bar{S}(E)$, etc. But $\bar{S}(\Psi)^{-1}(K)$ is a closed subset of $q_F^{-1}(q_E(K))$ and is therefore compact. Thus, $\bar{S}(\Psi)$ is a proper map. Finally, one shows easily that $\bar{S}$ is functorial and the results of this section may be summarised as follows:

PROPOSITION 8. *$\bar{S}$ is a contravariant functor from $Ban(X)$ to $Sp(X)$.*

4. ADJOINTNESS OF $\bar{S}$ AND $\bar{A}$.

Let $Spp(X)$ denote the full subcategory of $Sp(X)$ in which objects $q: Y \to X$ have the extra property that q is a proper map. For such an object the fibres Y_x are compact spaces, and Proposition 7 shows that the receiving category of $\bar{S}$ is actually $Spp(X)$.

Our main objective is to demonstrate that the functors

$$\bar{S}: Ban(X) \to Spp(X) \quad \text{and} \quad \bar{A}: Spp(X) \to Ban(X)$$

are adjoint on the right. However, we prefer to frame this in terms of adjointness of covariant functors in the usual way. Thus, let

$$A: Spp(X)^{op} \to Ban(X) \quad \text{and} \quad S: Ban(X)^{op} \to Spp(X)$$

be defined by

$$A(\eta^{op}) = \bar{A}(\eta) \quad \text{and} \quad S(\Psi^{op}) = \bar{S}(\Psi)$$

respectively. Now let $A^{op}: Spp(X) \to Ban(X)^{op}$ be the dual of A defined by $A^{op}(\eta) = (\bar{A}\eta)^{op}$. One final piece of notation is required. For an object $q: Y \to X$ in $Sp(X)$ and $y \in Y_x$ let δ_y denote the point functional in $A(Y)_x^*$ defined by $\delta_y(f) = f(y)$. In this way we determine a map

$$\delta_Y : Y \to A(Y)^* \text{ defined by } \delta_Y(y) = \delta_y.$$

THEOREM 3. *The functor A^{op} is left adjoint to S via an adjunction whose unit δ has components δ_Y, $Y \in Spp(X)$, and whose counit ϵ has components ϵ_E^{op}, $E \in Ban(X)$, where $\epsilon_E : E \to A(S(E))$ is defined by*

$$\epsilon_E(s)(f) = f(s) \quad \textit{for} \quad f \in S(E) \textit{ and } s \in E.$$

PROOF. We shall display a bijection

$$\lambda_{Y,E} = \lambda : Spp(X)(Y, S(E)) \to Ban(X)^{op}(A^{op}(Y), E)$$

which is natural in Y and E. Thus, let $q: Y \to X$ and $p: E \to X$ be objects in $Spp(X)$ and $Ban(X)^{op}$ respectively. To clarify matters we break the proof into a sequence of separate steps.

1) *The definition of* λ. Given $\eta : Y \to S(E)$ in $Spp(X)$, we define

$$\lambda(\eta) = \Psi^{op} \text{ by } (\Psi_x(s))(y) = \eta(y)(s),$$

where $x \in X$, $s \in E_x$ and $y \in Y_x$. Clearly, $\Psi_x(s)$ is a scalar valued function on Y_x and is the composite $F_s \circ \eta$, where $F_s \in E_x^{**}$ denotes the functional induced by s which is continuous by definition of the weak* topology. Therefore, $\Psi_x(s)$ is continuous and hence is an element of $A(Y)_x$ since Y_x is compact. Linearity of Ψ_x is easily checked, and since $||\eta(y)|| \leq 1$ we have

$$||\Psi_x(s)|| = \sup_{y \in Y_x} |\eta(y)(s)| \leq \sup_{y \in Y_x} ||\eta(y)|| . ||s|| \leq ||s||.$$

Therefore Ψ_x is a contraction for each $x \in X$. Obviously Ψ is fibre preserving and so it remains to show that Ψ is continuous and this we do by applying Proposition 1. Thus, let σ be a section of p which, without loss of generality, can be supposed to have compact support $D \subset X$. Now, $(\Psi\sigma)(x)$ is the function θ_x where θ is defined by

$$\theta(y) = \eta(y)(\sigma(q(y)))$$

and is continuous. Moreover, the support of θ is contained in the set $q^{-1}(D)$ which is compact since q is proper. In other words, $\Psi \circ \sigma = \tilde{\theta}$, and it follows that Ψ is continuous. This establishes the mapping λ.

2) *The definition of* λ' $(=\lambda^{-1})$. The next step is to define a map

$$\lambda'_{Y,E} = \lambda' : Ban(X)^{op}(A^{op}(Y), E) \to Spp(X)(Y, S(E)).$$

Given Ψ^{op} in the left hand hom set, let $\Psi^*: A(Y)^* \to E^*$ denote the conjugate operator of Ψ. We define

$$\eta = \lambda'(\Psi^{op}) \text{ by } \eta(y) = \Psi^*(\delta_y) \text{ for } y \in Y.$$

Since

$$||\Psi^*(\delta_y)|| \leq ||\Psi^*||.||\delta_y|| = ||\Psi||.1 \leq 1,$$

we see that $\eta(y) \in S(E)$, that is, $\eta: Y \to S(E)$ and clearly η is fibre preserving. To check continuity of η, let $\bar{q}$ denote the projection on $S(E)$. Then $\bar{q} \circ \eta = q$ and so $\bar{q} \circ \eta$ is continuous. Next, let $\sigma \in \Sigma(E)$, then: $F_\sigma \circ \eta$ is the map

$$y \mapsto \eta(y)(\sigma(q(\eta(y)))) = \Psi^*(\delta_y)(\sigma q(y))) = (\Psi \circ \sigma)(q(y))(y)$$

and can be written as an obvious composite involving the evaluation map ρ of Section 2. From Proposition 3 it follows then that $F_\sigma \circ \eta$ is continuous and consequently that η is too by virtue of the topology on $S(E)$ as a weak topology. That η is proper is immediate since q is proper and this establishes the mapping λ'.

REMARK. Without the restriction that q be a proper map, it does not follow that η is proper as can be seen by considering the image $\lambda'(\Psi^{op})$ in the case that Ψ is the zero operator $E \to A(Y)$.

3) $\lambda^{-1} = \lambda'$. Let $\eta: Y \to S(E)$ in $Spp(X)$ and let

$$\Psi^{op} = \lambda(\eta) \text{ and } \lambda'(\Psi^{op}) = \theta.$$

Then for $y \in Y_x$ we have $\theta(y) = \Psi^*(\delta_y)$ and for $s \in E_x$ we have

$$(\Psi_x^*(\delta_y))(s) = \delta_y(\Psi_x(s)) = \Psi_x(s)(y) = \eta(y)(s).$$

Hence, $\theta(y) = \eta(y)$ for all $y \in Y$ and so $\lambda'\lambda$ is an identity. A similar calculation shows that $\lambda\lambda'$ is also an identity and therefore λ is a bijection.

It remains to show that λ_{Y_1, E_1} is natural in Y_1 and E_1.

4) λ_{Y_1,E_1} *is natural in* Y_1. Fix E_1 and consider variable arrows $\eta: Y' \to Y$ in $Spp(X)$. Naturality in Y_1 means equality of

$$\eta^{\square} \circ \lambda'_{Y, E_1} \quad \text{and} \quad \lambda'_{Y', E_1} \circ (A^{op}\eta)^{\square},$$

where the upper square, $\square$, means «compose on the right». Let

$$\Psi^{op} : A^{op}(Y) \to E_1 \quad \text{in} \quad Ban(X)^{op}.$$

Then $(\eta^{\square} \circ \lambda')(\Psi^{op}) = \lambda'(\Psi^{op})\eta$ and for $y' \in Y'_x$ we have

$$(\lambda'(\Psi^{op})\eta)(y') = \lambda'(\Psi^{op})(\eta(y')) = \Psi^*(\delta_{\eta(y')}).$$

For the other composite we have

$$(\lambda'(A^{op}\eta)^{\square})(\Psi^{op}) = \lambda'(\Psi^{op} A^{op}\eta) = \lambda'((\bar{A}\eta \circ \Psi)^{op}).$$

Hence, for $y' \in Y'_x$ we have

$$\lambda'((\bar{A}\eta \circ \Psi)^{op})(y') = (\bar{A}\eta \circ \Psi)^*(\delta_{y'}) = \Psi^*((\bar{A}\eta)^*(\delta_{y'})).$$

By a direct computation one verifies that $(\bar{A}\eta)^*(\delta_{y'}) = \delta_{\eta(y')}$ and so obtains the required equality to demonstrate naturality in Y_1.

5) λ_{Y_1, E_1} *is natural in* E_1. This time we fix Y_1 and consider variable arrows $\Psi^{op} : E \to E'$ in $Ban(X)^{op}$. Naturality in E_1 means equality

$$(S\Psi^{op})_{\square} \circ \lambda'_{Y_1, E} = \lambda'_{Y_1, E'} \circ \Psi^{op}_{\square},$$

where the lower square, $\square$, means «compose on the left». For

$$\Phi^{op} : A^{op}(Y_1) \to E \quad \text{in} \quad Ban(X)^{op}$$

and $y \in Y_{1x}$ both composites give $(\Phi\Psi)^*(\delta_y)$ and hence are equal showing naturality in E_1.

Finally, to compute the unit δ of the adjunction put $E = A^{op}(Y)$. Then the component δ_Y is given by $\delta_Y = \lambda'(I^{op}_{A^{op}(Y)})$ and is defined by

$$\delta_Y(y) = I^*(\delta_y) = \delta_y \circ I = \delta_y \quad \text{for all } y \in Y.$$

The counit is calculated similarly by computing the image of $I_{S(E)}$ under λ.

COROLLARY. *The map ϵ_E is an isometric isomorphism of E onto a Banach subbundle of $A(S(E))$.*

PROOF. Certainly ϵ_E is a morphism $E \to A(S(E))$ in $Ban(X)$ and since

$$\|\epsilon_E(s)\| = \|F_s\| = \|s\|,$$

where F_s is defined by $F_s(f) = f(s)$ for $s \in E_x$, $f \in S(E)_x$ and $x \in X$, it follows that ϵ_E is isometric and hence injective. Moreover, it is clear that

$$\epsilon_E(\sigma) = \widetilde{F_\sigma | S(E)} \quad \text{for each } \sigma \in \Sigma(E)$$

and that $\widetilde{F_\sigma | S(E)}$ is continuous, though it need not have compact support.

Let B denote $A(S(E))$ and let $\bar{p}$ denote the projection of B onto X. Given $s \in E_x$, let w be a section of $\bar{p}$ passing through $\epsilon_E(s)$, by Theorem 1, and let $U(w, V, \epsilon)$ be a basic neighborhood of $\epsilon_E(s)$ in B. If σ is a section of p passing through s, then $\widetilde{F_\sigma | S(E)}$ is a section of $\bar{p}$ passing through $\epsilon_E(s)$ and there is a neighborhood $V' \subset V$ of x in X such that

$$\| \tilde{F}_\sigma(y) - w(y) \| < \epsilon \quad \text{for all } y \in V'.$$

Therefore, for each $y \in V'$ the set $U = U(w, V, \epsilon)$ contains elements of $B' = \epsilon_E(E)$ which project under $\bar{p}$ onto y. That is, $V' \subset \bar{p}(B' \cap U)$ and so the restriction of $\bar{p}$ to B' is an open map. It now follows that B' is a Banach subbundle of B.

Next observe that in the subspace topology on B' we have

(i) $\| \tilde{F}_\sigma(x) \|$ is continuous on X for each $\sigma \in \Sigma(E)$,

(ii) the set $\{ \tilde{F}_\sigma(x) \mid \sigma \in \Sigma(E) \}$ is dense in (in fact equal to) $\epsilon_E(E_x)$ for each $x \in X$.

However, the sets

$$U(\tilde{F}_\sigma, V, \epsilon) = \epsilon_E(U(\sigma, V, \epsilon)),$$

where $\sigma \in \Sigma(E)$, V is open in X and $\epsilon > 0$, form a subbasis for a topology on B' in which ϵ_E is a homeomorphism and (i) and (ii) are both valid. By the uniqueness assertion of [1], Proposition 1.6, these two topologies coincide and so ϵ_E is a homeomorphism of E onto B' as required.

This result is, as we observed in the Introduction, the analogue in $Ban(X)$ of Alaoglu's theorem for Banach spaces. However, unlike the classical case, the image $\epsilon_E(E)$ of E need not be a closed set in $\bar{A}(\bar{S}(E))$. This is shown by the following example in which the scalar field is R rather than C.

EXAMPLE. Let X be the closed unit interval $[0, 1]$ in R and let E be the product bundle $X \times \mathrm{R}$. We form a Banach subbundle E' of E by setting $E'_x = E_x$ for all $x \neq 1/2$ and taking $E'_{1/2}$ to be the zero Banach space. Then E' is dense in E. Applying the constructions above to E' and using the notation of the proof of the Corollary, we find that $B = \bar{A}(\bar{S}(E'))$ is the subbundle of $X \times C([-1, 1])$ formed by replacing the fibre over $1/2$ by a copy of R. Here, $C([-1, 1])$ denotes the Banach space of all continuous real valued functions on the interval $[-1, 1]$. Then $\epsilon_{E'}$ embeds E' in B and the closure of $\epsilon_{E'}(E')$ in B contains a copy of E. Thus $\epsilon_{E'}(E')$ is not closed in B.

This example shows also that $\bar{S}$ and $\bar{A}$ do not determine an equivalence of categories, for E' is not $\bar{A}(Y)$ for any $Y \epsilon Spp(X)$. In fact, by suitably modifying this example we see that for no X do $\bar{S}$ and $\bar{A}$ detmine an equivalence.

One might consider the question of adjointness of $\bar{S}$ and $\bar{A}$ on the left, in other words determine whether or not A is left adjoint to S^{op}. In this respect, it is tempting to use the equation

$$F(\Psi(f)) = f(\eta(F)), \quad \text{where } f \epsilon \bar{A}(Y)_x \text{ and } F \epsilon \bar{S}(E)_x,$$

to define implicitly one of Ψ and η when given the other, thereby hopefully obtaining a correspondence between

$$Sp(X)^{op}(Y, S^{op}(E)) \quad \text{and} \quad Ban(X)(A(Y), E).$$

However, given η this determines a map Ψ of $\bar{A}(Y)$ into $E^{**} = \bigcup_{x \epsilon X} E_x^{**}$ rather than into E. If we assume reflexivity, then reflexivity of $\bar{A}(Y)$ forces the fibres of Y to be finite sets and this is too restrictive a condition to impose. It is my pleasure to record here my thanks to R. E. Harte for some illuminating comments on these matters.

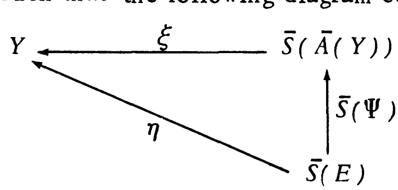
Interestingly enough, in circumstances where the equation above does determine η given Ψ and Ψ given η, the correspondence obtained is bijective and natural in Y and E. Nevertheless, we prove next that $\bar{S}$ and $\bar{A}$ are not in general adjoint on the left.

THEOREM 4. *In general A is not left adjoint to S^{op}.*

PROOF. Consider the case when X is a singleton set, thus $Ban(X)$ becomes the category Ban of Banach spaces and linear contractions and $Spp(X)$ becomes the category $Comp$ of compact Hausdorff spaces. Suppose A is left adjoint to S^{op} so that there is a bijection

$$Comp^{op}(Y, S^{op}(E)) \to Ban(A(Y), E)$$

natural in Y and E. Then the unit of this adjunction gives a universal arrow $\xi^{op}: Y \to S^{op} A(Y)$ from Y to S^{op} for each $Y \in Comp$. Thus, to each pair (E, η) with E in Ban and $\eta: \bar{S}(E) \to Y$ in $Comp$ there exists a unique $\Psi: \bar{A}(Y) \to E$ such that the following diagram commutes:



Now, take $E = \mathrm{R}$ so that $\bar{S}(E)$ can be identified with the interval $[-1, 1]$, take $Y = [-1, 1]$ also and define η by $\eta(y) = y_0$, where $y_0 \in Y$ is selected arbitrarily. For any functional $\Psi: \bar{A}[-1, 1] \to \mathrm{R}$ we have $\bar{S}(\Psi)$ defined by $\bar{S}(\Psi)(y) = Ty \circ \Psi$, where Ty denotes the functional «multiply by y». Then the commutativity of the diagram above means

$$\xi(Ty \circ \Psi) = y_0 \quad \text{for all } y \in Y.$$

Taking $y = 1$ we deduce that ξ is defined by $\xi(\Psi) = y_0$ for all Ψ, which is impossible since y_0 can be selected arbitrarily.

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