CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 24, nº 1 (1983), p. 57-86

<http://www.numdam.org/item?id=CTGDC_1983__24_1_57_0>

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ET GÉOMÉTRIE DIFFÉRENTIELLE

Vol. XXIV-1 (1983)

MANIFOLDS OF SMOOTH MAPS IV: THEOREM OF DE RHAM

by P. MICHOR

Spaces of smooth mappings between finite dimensional manifolds are themselves manifolds modelled on nuclear (LF)-spaces in a canonical way. In this paper we develop the calculus of differential forms and use it to prove the theorem of de Rham for such infinite dimensional manifolds: the de Rham cohomology coincides with singular cohomology with real coefficients and in turn with sheaf cohomology with coefficients in the constant sheaf R. The essential point is the fact that (NLF)-manifolds (as we chose to call them - (NLF) for nuclear (LF)) are paracompact and admit smooth partitions of unity. Note, however, that (NLF)-manifolds are not compactly generated in general, so spaces of smooth mappings between them turn out to be not complete and the cotangent bundle does not exist. This drawback could be overcome by making all spaces compactly generated and using the calculus of U. Seip [20] devised for this setting. One would loose paracompactness however. In the last section we investigate the group of all diffeomorphisms of a locally compact manifold, connect its de Rham cohomology with the cohomology of the Lie algebra of all vector fields with compact support which has been investigated by Gel'fand, Fuks [5] and we make some observations on its exponential mapping and adjoint representation. It turns out that the exponential mapping is not analytic in the obvious sense.

1 . CALCULUS ON (NLF)-SPACES AND -MANIFOLDS.

1.1. DEFINITION. By an *(NLF)-space* we mean a nuclear (LF)-space, i.e. a locally convex vector space E which is the strict inductive limit of an increasing sequence of Frechet spaces

$$\dots \subset E_n \subset E_{n+1} \subset \dots \subset E,$$

and which is nuclear. So each E_n is nuclear and therefore separable (see Pietsch [18]).

Attention: E is not the inductive limit of the spaces E_n in the sense of topology ; it is so only in the category of topological vector spaces. For if it were so, it would be compactly generated ; but the space $\mathfrak{D}$ of test functions on R^n is not compactly generated (see Valdivia [23]).

We recall that a mapping $f: E \to F$ between locally convex spaces (or open subsets of these) is called C^1_c if

$$\lim_{t\to 0} \frac{1}{t} (f(x+ty) - f(x)) = Df(x)y$$

exists for all x, y in E , and $Df: E\times E \to F$ is jointly continuous ; f is called C^2_c if Df is C^1_c, and so on. See Keller [9] for a detailed account of this.

1.2. THEOREM. *Any (NLF)-space admits C^∞_c-partitions of unity. In particular it is paracompact.*

This result is proved in Michor [14] (8.6) for the space $\Gamma_c(E)$ of smooth sections with compact support of a smooth finite-dimensional vector bundle $E \to X$. But in the proof there only the following facts are needed: $\Gamma_c(E)$ is an (LF)-space and is nuclear. So the result above holds too.

1.3. DEFINITION. By an (NLF)*-manifold* we mean a Hausdorff topological space M that is a manifold in the C^∞_c-sense modelled on open subsets of (NLF)-spaces.

In Michor [13, 14], it is shown that the space $C^\infty(X, Y)$ of all smooth mappings $f: X \to Y$ between finite-dimensional manifolds is an (NLF)-manifold.

Note that (NLF)-manifolds admit C_c^∞-partitions of unity by 1.2.

The tangent bundle TM is again an (NLF)-manifold, but the natural transition functions for the cotangent bundle are not of class C_c^∞, not even continuous.

See Michor [14] (Section 9) for a short account of C_c^∞-manifolds. We will use notation from [14], which is largely self-explanatory.

1.4. *The algebra of C_c^∞-functions.* By $C_c^\infty(M)$ we denote the space of all C_c^∞-functions from an (NLF)-manifold into R. We put the «topology of uniform convergence on compact subsets in each derivative» on $C_c^\infty(M)$. So a net (f_i) converges to f iff $f_i \to f$ uniformly on each compact in M, $df_i \to df$ uniformly on each compact in TM, $ddf_i \to ddf$ uniformly on each compact in T^2M, etc. Here

$$df = pr_2 \circ Tf: TM \to T\mathrm{R} \to \mathrm{R}.$$

$C_c^\infty(M)$, equipped with this topology, is a locally convex vector space, even a locally-multiplicatively-convex algebra in the sense of Michael [12]. I suspect that $C_c^\infty(M)$ is not complete in general, since M is not compactly generated.

1.5. *Tangent vectors as continuous derivations.* Let $\xi_x \in T_x M$ be a tangent vector, then ξ_x defines a continuous derivation: $C_c^\infty(M) \to \mathrm{R}$ over ev_x by $f \mapsto \xi_x(f) = df(\xi_x)$. The converse is true on (NLF)-manifolds:

THEOREM. *Let M be an (NLF)-manifold, and let $A: C_c^\infty(M) \to \mathrm{R}$ be a continuous derivation over ev_x, i. e.*

$$A(f.g) = A(f).g(x) + f(x).A(g).$$

Then there is a unique tangent vector $\xi_x \in T_x M$ such that $A(f) = df(\xi_x)$ for all f.

PROOF. Let (U, u, E) be a chart of M with $x \in U$ and $u(x) = 0$ in E. Since there are C^∞-partitions of unity, $A(f)$ only depends on the germ of f at x. Now choose a C_c^∞-function ϕ which is 1 on a neighborhood of x and has support contained in U. Consider the mapping

$$a \mapsto A(\phi.(a \circ u)), \quad a \in E' \text{ (the dual of } E\text{)}.$$

This defines a linear functional on E'. We show that it is continuous. Suppose $a_i \to 0$ in E' in the topology of bounded convergence which coincides with the topology of compact convergence since E is nuclear. Then for each compact K in M, $K \cap supp\, \phi =: K_1$ is compact in U, so $u(K_1)$ is compact in E, so $a_i \mid u(K_1) \to 0$ uniformly, so $\phi.(a_i \circ u) \to 0$ uniformly on K. Now let $\hat{K}$ be compact in TM, then $supp(d\phi) \cap \hat{K} =: \hat{K}_1$ is compact in $\pi_M{}^{-1}(U)$, so $Tu(\hat{K}_1)$ is compact in $E \times E$, so

$$da_i = a_i \circ pr_2 : E \times E \to \mathrm{R}$$

converges to 0 uniformly on $Tu(\hat{K}_1)$, so

$$d(\phi.(a_i \circ u)) = d\phi.(a_i \circ u \circ \pi_M) + (\phi \circ \pi_M).(da_i \circ u) \to 0$$

uniformly on $\hat{K}$. This argument can be repeated and shows that

$$\phi.(a_i \circ u) \to 0 \quad \text{in} \quad C_c^\infty(M).$$

Thus $A(\phi.(a_i \circ u)) \to 0$. So the linear functional $a \mapsto A(\phi.(a_i \circ u))$ is continuous on E' and it is therefore represented by an element $\beta \in E$ since E is reflexive. We have

$$A(\phi.(a \circ u)) = \langle \beta, a \rangle \quad \text{for all} \quad a \in E'.$$

Claim: The tangent vector $\xi_x = (Tu)^{-1}(0, \beta) \in T_x M$ represents A. Let $f \in C_c^\infty(M)$. Then $\xi_x(f) = d(f \circ u^{-1})(0)\beta$. We have

$$d(f \circ u^{-1})(0) \in L(E, \mathrm{R}) = E',$$

and clearly

$$A(\phi.(d(f \circ u^{-1})(0) \circ u)) = \langle \beta, d(f \circ u^{-1})(0) \rangle = \xi_x(f).$$

So we have to prove that $A(f) = A(g)$ whenever $df_x = dg_x$. Note that by the derivation property $A(constant) = 0$, so it remains to show the following: if $f(x) = 0$ and $df_x = 0$, then $A(f) = 0$. For such an f let

$$g = f \circ u^{-1} : u(U) \to \mathrm{R} ;$$

this is a C_c^∞-function. By Taylor's Theorem (on R^1) we have:

$$g(y) = \int_0^1 (1-t) D^2 g(ty)(y,y)\, dt.$$

Now E is nuclear, so it has the approximation property, so $L(E,E) = E \hat{\otimes} E'$, and there is a net of finite-dimensional continuous linear operators

(L_i) in $L(E,E)$ converging to Id_E uniformly on compact subsets. Put

$$g_i(y) = \int_0^1 (1-t)D^2 g(ty)(L_i y, y)\,dt.$$

Then clearly $g_i \in C_c^\infty$. *Claim:* $g_i \to g$ in $C_c^\infty(u(U))$. Let K be compact in $u(U)$. The mapping $u(U) \to E'$ given by

$$y \mapsto \int_0^1 (1-t)D^2 g(ty)(.,y)\,dt$$

is continuous, so the image of K under this mapping is compact in $E' = L(E,\mathbb{R})$, so it is weakly bounded and thus equicontinuous, since E is barrelled (see Schaeffer [19], III, 4.2). This means that

$$\left|\int_0^1 (1-t)D^2 g(ty)(z,y)\,dt\right| < \epsilon$$

for all $y \in K$ and $z \in V$, a suitable neighborhood of 0 in E. Now let i_0 be such that $L_i y - y \in V$ for all $y \in K$ and $i \geq i_0$. Then

$$|g_i(y) - g(y)| < \epsilon \quad \text{for all} \quad y \in K.$$

So $g_i \to g$ uniformly on compacts of $u(U)$. Since derivatives with respect to y commute with the integral, the argument above can be repeated for all derivatives and the second claim is established. Now let

$$L_i = \sum_{j=1}^{N_i} e_{ij} \otimes e'_{ij} \in E \otimes E',$$

then we have

$$\begin{aligned} g_i(y) &= \int_0^1 (1-t)D^2 g(ty)\Big(\sum_{j=1}^{N_i} e_{ij}\langle y, e'_{ij}\rangle, y\Big)dt \\ &= \sum_{j=1}^{N_i} \langle y, e'_{ij}\rangle \int_0^1 (1-t)D^2 g(ty)(e_{ij}, y)\,dt. \end{aligned}$$

On the manifold M we have then

$$\phi^2.(g_i \circ u) \to \phi^2.(g \circ u) = \phi^2.f$$

in $C_c^\infty(M)$ by the second claim above. Since A is continuous, we get

$$A(\phi^2.(g_i \circ u)) \to A(\phi^2.f) = A(f),$$

since $\phi^2.f$ and f have the same germ at x. On the other hand:

$$\begin{aligned} &A(\phi^2.(g_i \circ u)) = \\ &= A\Big(\sum_{j=1}^{N_i} [\phi.(e'_{ij} \circ u)].[\phi.\int_0^1 (1-t)D^2 g(tu(.))(e_{ij}, u(.))\,dt]\Big) \end{aligned}$$

$$= \sum_{j=1}^{N_i} [A(\phi.(e'_{ij} \circ u)).\phi(x).\int_0^1 (1-t)D^2 g(0)(e_{ij},0)dt +$$
$$+ \phi(x)<0, e'_{ij}>.A(\phi.\int_0^1 (1-t)D^2 g(tu)(e_{ij}, u)dt)] = 0.$$

So $A(f) = 0$. qed

2. VECTOR FIELDS AND DIFFERENTIAL FORMS.

2.1. Let us denote the space of all vector fields on the (NLF)-manifold M by $\mathfrak{X}(M)$ as usual.

LEMMA. *$\mathfrak{X}(M)$ is a Lie-algebra, the bracket $[\xi, \eta]$ of two vector fields being given by*

$$[\xi,\eta](f) = \xi\eta(f) - \eta\xi(f) \quad \textit{for} \quad f \in C_c^\infty(M).$$

PROOF. Of course

$$f \mapsto [\xi,\eta](f) = \xi\eta(f) - \eta\xi(f)$$

is a continuous derivation of the algebra $C_c^\infty(M)$, so $f \mapsto [\xi,\eta](f)(x)$ is a continuous derivation over ev_M, so it is given by a tangent vector $\zeta_x \in T_x M$ by 1.5. It remains to show that $x \mapsto \zeta_x$ is a C_c^∞-mapping from M to TM. It suffices to check this on a local chart (U, u, E), and for the local representatives in U we have

$$\bar{\zeta}(x) = D\bar{\eta}(x).\bar{\xi}(x) - D\bar{\xi}(x).\bar{\eta}(x),$$

which is visibly C_c^∞. qed

We equip the space $\mathfrak{X}(M)$ with the topology of compact convergence in each derivative. Then it becomes a topological $C_c^\infty(M)$-module

2.2. *Differential forms.* By a differential form ω of degree p on M we mean a C_c^∞-mapping $TM \times_M \dots \times_M TM \to \mathbb{R}$ which is alternating and p-linear on each fibre $(T_x M)^p$. Let us denote the space of all p-forms by $\Omega^p(M)$.

For $\omega \in \Omega^p(M)$ and $\phi \in \Omega^q(M)$, define $\omega \wedge \phi \in \Omega^{p+q}(M)$ as usual by the formula

$$(\omega \wedge \phi)(\xi_1, \dots, \xi_{p+q}) =$$
$$= \frac{1}{p!\,q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} \operatorname{sign} \sigma \cdot \omega_x(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p)}) \cdot \phi_x(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)})$$

for $x \in M$, $\xi_i \in T_x M$, where $\mathfrak{S}_{p+q}$ is the full symmetric group of permutations of $p+q$ symbols. Clearly $\omega \wedge \phi$ is C_c^∞. Let

$$\Omega(M) = \bigoplus_{p \geq 0} \Omega^p(M),$$

a real graded algebra. The natural topology on $\Omega(M)$ is the direct sum of the topology of compact convergence in all derivatives.

Warning: It is not true that $C_c^\infty(M) = \Omega^0(M)$ and $\{df \mid f \in C_c^\infty(M)\}$ generate $\Omega(M)$. They generate a dense subalgebra, however, if each model space E of M has the property that $L(E, E)$ admits a bounded (= equicontinuous) finite dimensional approximate identity. This is not true for all nuclear spaces.

2.3. If $\omega \in \Omega^p(M)$ is a p-form, define the *exterior derivative of* ω by Palais's global formula

$$d\omega(\xi_0, \ldots, \xi_p) = \sum_{i=0}^{p} (-1)^i \xi_i (\omega(\xi_0, \ldots, \hat{\xi}_i, \ldots, \xi_p)) +$$
$$+ \sum_{0 \leq i < j \leq p} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_p),$$

where $\xi_i \in \mathfrak{X}(M)$.

LEMMA. *$d\omega$ is a $(p+1)$-form.*

PROOF. A purely combinatorial computation shows that $d\omega$ is $C_c^\infty(M)$-linear in each variable, and alternating, so on each fibre $(T_x M)^{p+1}$ it is given by a jointly continuous alternating $(p+1)$-linear functional. It remains to check that $d\omega$ is C_c^∞. For a local chart (U, u, E) on M the local representative of ω is a C_c^∞-mapping $\bar{\omega} : u(U) \times E^p \to \mathbb{R}$ which is alternating and p-linear in the last p variables. For $x \in u(U)$ and $y_i \in E$, considered as constant vector fields on $u(U)$ so that $[y_i, y_j] = 0$, we get the following local representative of $d\omega$:

$$d\bar{\omega}(x)(y_0, \ldots, y_p) = \sum_{i=0}^{p} (-1) \; D\bar{\omega}(x)(y_i)(y_0, \ldots, \hat{y}_i, \ldots, y_p),$$

which is clearly C_c^∞.

2.4. For $\omega \in \Omega^p(M)$ and $\xi \in \mathfrak{X}(M)$ define the Lie-derivative $\mathfrak{L}_\xi \omega \in \Omega^p(M)$ by the following formula:

$$(\mathcal{L}_\xi \omega)(\eta_1, \dots, \eta_p) =$$
$$= \xi(\omega(\eta_1, \dots, \eta_p)) - \sum_{i=0}^{p} \omega(\eta_1, \dots, [\xi, \eta_i], \dots, \eta_p).$$

LEMMA. *$\mathcal{L}_\xi \omega$ is again a p-form on M.*

PROOF. As usual the only problem is the differentiability. Using a local chart (U, u, E) and constant vector fields y_i on $u(U)$ (so

$$[\bar{\xi}, y_i](x) = -D\bar{\xi}(x)y_i)$$

one easily checks that $\mathcal{L}_\xi \omega$ has the following local representative on U :

$$\mathcal{L}_{\bar{\xi}}\bar{\omega}: u(U) \times E^p \to \mathbb{R},$$

$$(\mathcal{L}_{\bar{\xi}}\bar{\omega})_x(y_1, \dots, y_p) =$$
$$= D\bar{\omega}(x)(\bar{\xi}(x))(y_1, \dots, y_p) + \sum_{i=1}^{p} \bar{\omega}(x)(y_1, \dots, D\bar{\xi}(x)y_i, \dots, y_p).$$

This is clearly C_c^∞. qed

2.5. LEMMA. *For $\xi \in \mathfrak{X}(M)$ the mapping $\mathcal{L}_\xi : \Omega(M) \to \Omega(M)$ is a derivation, i. e.*

$$\mathcal{L}_\xi(\omega \wedge \phi) = \mathcal{L}_\xi \omega \wedge \phi + \omega \wedge \mathcal{L}_\xi \phi.$$

PROOF. A combinatorial computation.

2.6. If $\xi \in \mathfrak{X}(M)$ and $\omega \in \Omega^p(M)$, let

$$\xi \lrcorner \omega = i_\xi \omega \in \Omega^{p-1}(M)$$

be defined by

$$(i_\xi \omega)(\eta_2, \dots, \eta_p) = \omega(\xi, \eta_2, \dots, \eta_p)$$

for $\eta_i \in \mathfrak{X}(M)$.

LEMMA. $i_\xi(\omega \wedge \phi) = (i_\xi \omega) \wedge \phi + (-1)^{\deg \omega} \wedge (i_\xi \phi)$.

PROOF. A combinatorial computation.

2.7. If $f: M \to N$ is a C^∞-mapping between (NLF)-manifolds, then for any $\omega \in \Omega^p(M)$ define $f^*\omega \in \Omega^p(M)$ by

$$(f^*\omega)_x(\eta_1, \dots, \eta_p) = \omega_{f(x)}(T_x f \cdot \eta_1, \dots, T_x f \cdot \eta_p)$$

for $x \in M$ and $\eta_i \in T_x M$. The following diagram shows that $f^*\omega$ is a C_c^∞-mapping:

$$TM \underset{M}{\times} \dots \underset{M}{\times} TM \xrightarrow{Tf \underset{M}{\times} \dots \underset{M}{\times} Tf} TM \underset{M}{\times} \dots \underset{M}{\times} TM$$

$$f^*\omega \searrow \quad \mathbb{R} \quad \swarrow \omega$$

LEMMA. *$f^*: \Omega(N) \to \Omega(M)$ is an algebra-homomorphism.*

2.8. THEOREM. *We have the following formulas:*

1. $\mathcal{L}_\xi = i_\xi \circ d + d \circ i_\xi$.
2. $d(\omega \wedge \phi) = d\omega \wedge \phi + (-1)^{\deg\omega}\, \omega \wedge d\phi$.
3. $d^2 = d \circ d = 0$.
4. $d \circ \mathcal{L}_\xi = \mathcal{L}_\xi \circ d$.
5. $f^* \circ d = d \circ f^*$ *for* $f: M \to N$ *a* C^∞_c*-mapping.*

PROOF. 1. A combinatorial computation.

2. Use 1 and induction on $\deg\omega + \deg\phi$.

3. Follows from the local formula in 2.3, since any second derivative of a C^∞_c-mapping is symmetric (see Keller [9]).

4. Is immediate from 1.

5. Let (U, u, E), (V, v, F) be local charts on M, N respectively, such that $f(U) = V$. Denote local representatives by bars. Then for ω in $\Omega^p(N)$ we have

$$(\bar f^*\bar\omega)(x)(y_1, \dots, y_p) = \bar\omega(\bar f(x))(D\bar f(x)y_1, \dots, D\bar f(x)y_p),$$

$$d(\bar f^*\bar\omega)(x)(y_0, \dots, y_p) =$$

$$= \sum_{i=0}^{p} (-1)^i D(\bar\omega \circ \bar f)(x)(y_i)(D\bar f(x)y_0, \dots, \widehat{D\bar f(x)y_i}, \dots, D\bar f(x)y_p)$$

$$= \sum_{i=0}^{p} (-1)^i D\bar\omega(f(x))(D\bar f(x)y_i)(D\bar f(x)y_0, \dots, \widehat{D\bar f(x)y_i}, \dots, D\bar f(x)y_p)$$

$$= (\bar f^* d\bar\omega)(x)(y_0, \dots, y_p).$$

qed

2.9. Let $\xi \in \mathfrak{X}(M)$ be a vector field which has a local flow, i.e. there is a C^∞_c-mapping $\alpha: U \to M$, defined on an open neighborhood U of $M \times \{0\}$ in $M \times \mathbb{R}$ such that

$$\frac{d}{dt}\alpha(x, t) = \xi(\alpha(x, t)) \quad \text{for all } (x, t) \in U$$

and moreover

$$a(x,0)=x \quad \text{and} \quad a(a(x,t),s)=a(x,t+s)$$

whenever one side is defined. (In general, nothing is known about existence and uniqueness of nonlinear ordinary differential equations in non-normable locally convex spaces.)

LEMMA. *Let $\omega \in \Omega^p(M)$. With the assumptions above we may compute the Lie-derivative as follows:*

$$\mathcal{L}_\xi \omega = \frac{d}{dt}\Big|_{t=0} a_t^* \omega .$$

Here $a_t^*\omega$ can either be viewed as a C_c^∞-path in the sheaf of local p-forms on M, or the derivative above can be evaluated pointwise, since evaluation at a point is linear and continuous.

PROOF. For $f \in C_c^\infty(M)=\Omega^0(M)$ there is a global proof:

$$\left(\frac{d}{dt}\Big|_{t=0} a_t^* f\right)(x) = \frac{d}{dt}\Big|_{t=0} f(a_t(x)) = df\left(\frac{d}{dt}\Big|_{t=0} a(x,t)\right) =$$
$$= df(\xi(x)) = \xi(f)(x) = (\mathcal{L}_\xi f)(x).$$

Now let $\omega \in \Omega^p(M)$. Take any local chart (U,u,E) of M, let

$$\bar{\omega}: u(U)\times E^p \to \mathbb{R}, \quad \bar{\xi}: u(U)\to E, \quad \bar{a}(x,t)=u(a(u^{-1}(x),t))$$

be the local representations on U. Then we may compute as follows:

$$\frac{d}{dt}\Big|_{t=0} (\bar{a}_t^*\bar{\omega})(x)(y_1,\dots,y_p) =$$
$$= \frac{d}{dt}\Big|_{t=0} \bar{\omega}(\bar{a}(x,t))(D\bar{a}_t(x)y_1,\dots,D\bar{a}_t(x)y_p) =$$
$$= D\bar{\omega}(x)\left(\frac{d}{dt}\Big|_{t=0}\bar{a}(x,t)\right)(y_1,\dots,y_p) +$$
$$+ \bar{\omega}(x)\left(\frac{d}{dt}\Big|_{t=0} D\bar{a}_t(x)y_1, y_2,\dots,y_p\right) +$$
$$+ \bar{\omega}(x)\left(y_1,\dots,y_{p-1}, \frac{d}{dt}\Big|_{t=0} D\bar{a}_t(x)y_p\right) =$$
$$= D\bar{\omega}(x)(\bar{\xi}(x))(y_1,\dots,y_p) + \bar{\omega}(x)(D\bar{\xi}(x)y_1, y_2,\dots,y_p) + \dots$$
$$\dots + \bar{\omega}(x)(y_1,\dots,y_{p-1}, D\bar{\xi}(x)y_p),$$

since different partial derivatives commute, see Keller [9]. This is the local formula for $\mathcal{L}_\xi\omega$ of 2.4. qed

2.10. LEMMA. *If* $\xi \in \mathfrak{X}(M)$ *admits the local flow* a *as in 2.9, and if* $\eta \in \mathfrak{X}(M)$, *then we have*

$$\frac{d}{dt}\Big|_{t=0} a_t^* \eta = [\lambda, \eta] =: \mathfrak{L}_\xi \eta .$$

Here

$$a_t^* \eta = T a_{-t} \circ \eta \circ a_t \in \mathfrak{X}(M).$$

PROOF. Let (U, u, E) be a local chart on M, then for the local representatives we have

$$\frac{d}{dt}\Big|_{t=0} \bar{a}_t^* \bar{\eta}(x) = \frac{d}{dt}\Big|_{t=0} (D(\bar{a}_{-t})(x).\bar{\eta}(\bar{a}(x,t))) =$$

$$= \frac{d}{dt}\Big|_{t=0} D(\bar{a}_{-t})(x).\bar{\eta}(x) + \frac{d}{dt}\Big|_{t=0} (\bar{\eta}(\bar{a}(x,t))),$$

by the chain rule and the existence of partial derivatives, see [14], 8.3. This in turn equals

$$D(\frac{d}{dt}\Big|_{t=0} \bar{a}_{-t})(x).\bar{\eta}(x) + D\bar{\eta}(x).(\frac{d}{dt}\Big|_{t=0} \bar{a}(x,t)) =$$

$$= -D\bar{\xi}(x).\bar{\eta}(x) + D\bar{\eta}(x).\bar{\xi}(x) = [\bar{\xi}, \bar{\eta}](x).$$

qed

2.11. LEMMA. *Let* $\xi \in \mathfrak{X}(M)$ *admit the local flow* a. *Then for any* ω *in* $\Omega^p(M)$ *we have* $\frac{d}{dt} a_t^* \omega = a_t^* \mathfrak{L}_\xi \omega$ *on the open set where* a_t *is defined.*

PROOF. Let $x \in M$, $t \in \mathbb{R}$ be such that $a(x,t)$ is defined. Then we have

$$\frac{d}{dt} a_t^* \omega(x)(\eta_1, \dots, \eta_p) = \frac{d}{ds}\Big|_{s=0} (a_t^* a_s^* \omega)(x)(\eta_1, \dots, \eta_p) =$$

$$= \frac{d}{ds}\Big|_{s=0} (a_s^* \omega)_{a(x,t)}(T_x a_t \cdot \eta_1, \dots, T_x a_t \cdot \eta_p)$$

$$= (\mathfrak{L}_\xi \omega)_{a(x,t)}(T_x a_t \cdot \eta_1, \dots, T_x a_t \cdot \eta_p)$$

$$= (a_t^* \mathfrak{L}_\xi \omega)_x(\eta_1, \dots, \eta_p).$$

qed

2.12. LEMMA OF POINCARÉ. *A closed differential form on* M *is locally exact.*

PROOF. We have to show that for any $\omega \in \Omega^p(M)$ with $d\omega = 0$ and any $x \in M$ there is an open neighborhood U of x in M and a form $\phi \in \Omega^{p-1}(M)$

such that $d\phi = \omega | U$. Using a local chart (U, u, E) with $u(x) = 0$, we may assume that M is an absolutely convex open neighborhood of 0 in the (NLF)-space E. Consider the C_c^∞-mapping

$$a : M \times [-1, 1] \to M, \quad a(x, t) = t.x.$$

a is no local flow, so for $t \neq 0$,

$$\frac{d}{dt} a(x, t) = \xi(a(x, t), t)$$

for a time dependent vector field ξ, which is given by $\xi(x, t) = \frac{1}{t} x$. Put $\beta(x, t) = e^t . x$, then β is a local flow, defined for $-\infty < t \leq 0$, and the generating vector field is just Id_M. Now for $t > 0$ we have:

$$\frac{d}{dt} a_t^* \omega = \frac{d}{dt}(\beta_{log\,t}^* \omega) = \frac{d}{ds}\Big|_{s = log\,t} (\beta_s^* \omega) . \frac{d\,log\,t}{dt}$$

$$= \frac{1}{t} . \beta_{log\,t}^* \mathfrak{L}_{Id} \omega = \frac{1}{t} a_t^* (i_{Id} \circ d\omega + d \circ i_{Id} \omega)$$

$$= \frac{1}{t} a_t^* (d \circ i_{Id} \omega) = \frac{1}{t} d(a_t^* \circ i_{Id} \omega).$$

$$\frac{1}{t}(a_t^* \circ i_{Id} \omega)(y_2, \dots, y_p) = \frac{1}{t} \omega_{tx}(tx, ty_2, \dots, ty_p)$$

$$= \omega_{tx}(x, ty_1, \dots, ty_p) \qquad \text{if } p \geq 1.$$

So $\frac{1}{t} a_t^* i_{Id} \omega$ is a $(p-1)$-form for all $-1 \leq t \leq 1$, and is C_c^∞ in t. Furthermore $a_1^* \omega = \omega$, $a_0^* \omega = 0$. So

$$\omega = a_1^* \omega - a_0^* \omega = \int_0^1 \frac{d}{dt} a^* \omega \, dt$$

$$= \int_0^1 d(\frac{1}{t} a_t^* i_{Id} \omega) dt = d \int_0^1 \frac{1}{t} a_t^* i_{Id} \omega \, dt.$$

Choose

$$\phi = \int_0^1 \frac{1}{t} a_t^* i_{Id} \omega \, dt \in \Omega^{p-1}(M). \qquad \text{qed}$$

REMARK. See Papaghiuc [17] for a more elementary proof of this fact in general locally convex spaces.

3. COHOMOLOGY AND THE THEOREM OF DE RHAM.

3.1. Let M be a (NLF)-manifold. The de Rham cohomology of M is given by:

$$H_{dR}{}^{p}(M) = \frac{ker(d: \Omega^{p}(M) \to \Omega^{p+1}(M))}{im(d: \Omega^{p-1}(M) \to \Omega^{p}(M))}.$$

$H_{dR}{}^{p}(M)$ is a real vector space and $H_{dR}(M) = \bigoplus_{k \geq 0} H_{dR}{}^{k}(M)$ is an algebra, the product being induced by the exterior product Λ on $\Omega(M)$ (the exact forms are an ideal in the closed forms, by 2.8.2).

For any C_c^∞ mapping $f: M \to N$ between (NLF)-manifolds we get a cochain complex homomorphism $f^*: \Omega(N) \to \Omega(M)$ (by 2.8.5) and an induced homomorphism in cohomology

$$f^{\#} = H_{dR}(f): H_{dR}(N) \to H_{dR}(M),$$

which respects degrees and is an algebra homomorphism. $f \mapsto f^{\#}$ is clearly functorial.

3.2. THEOREM. *The de Rham cohomology of (NLF)-manifolds has the following properties:*

1. $H_{dR}(point) = 0$.

2. If $f, g: M \to N$ *are* C_c^∞*-homotopic mappings (i. e. there is a* C_c^∞*-mapping* $H: M \times \mathbb{R} \to N$ *with* $H(\,.\,,0) = f$ *and* $H(\,.\,,1) = g$*), then*

$$f^{\#} = g^{\#}: H_{dR}(N) \to H_{dR}(M).$$

3. If $M = \bigcup_{\alpha} M_\alpha$ *is a disjoint union of open submanifolds* M_α*, then* $H_{dR}{}^{p}(M) = \prod_{\alpha} H_{dR}{}^{p}(M_\alpha)$ *for all* $p \geq 0$.

4. (Mayer-Vietoris) If $M = U \cup V$, U, V *open, then there is a long exact sequence*

$$\dots \to H_{dR}{}^{p}(U) \oplus H_{dR}{}^{p}(V) \to H_{dR}{}^{p}(U \cap V) \xrightarrow{\delta} H_{dR}{}^{p+1}(M) \to \dots$$

which is natural in the obvious sense.

PROOF. 1 and 3 are obvious.

2. For $t \in \mathbb{R}$ let $j_t: M \to M \times \mathbb{R}$ be the embedding $j_t(x) = (x, t)$. For $\phi \in \Omega^{p}(M \times \mathbb{R})$ consider $j_t^* \phi \in \Omega^{p}(M)$. As a function of t, $j_t^* \phi$ is a C^∞ curve in the locally convex space $\Omega^{p}(M)$ with the topology of compact convergence in all derivatives. Since this space is probably not sequentially complete, the integral with respect to t need not exist. Therefore for $\phi \in \Omega^{p}(M \times \mathbb{R})$ and $\xi_i \in T_x M$ define

$$(I_0^1\phi)_x(\xi_1,\dots,\xi_p) = \int_0^1 (j_t^*\phi)_x(\xi_1,\dots,\xi_p)dt =$$
$$= \int_0^1 \phi_{(x,t)}((\xi_1,0_t),\dots,(\xi_p,0_t))dt.$$

Claim: $I_0^1\phi \in \Omega^p(M)$.

$$\phi : (TM \times TR) \underset{M\times R}{\times} \dots \underset{M\times R}{\times} (TM \times TR) \to \mathrm{R}$$

is of class C_c^∞. So for $\epsilon > 0$ there are open neighborhoods $U_{i,t}$ of ξ_i in TM, $V_{i,t}$ of 0_t in $T\mathrm{R}$ such that

$$|\phi((\eta_1,r_1),\dots,(\eta_p,r_p)) - \phi((\xi_1,0_t),\dots,(\xi_p,0_t))| < \epsilon$$

for all

$$(r_1,\dots,r_p) \in T\mathrm{R} \underset{\mathrm{R}}{\times} \dots \underset{\mathrm{R}}{\times} T\mathrm{R} \cap V_{1,t} \times \dots \times V_{p,t} \quad ,$$

$$(\eta_1,\dots,\eta_p) \in TM \underset{M}{\times} \dots \underset{M}{\times} TM \cap U_{1,t} \times \dots \times U_{p,t}. \quad .$$

Let $pr_1 : T\mathrm{R} = \mathrm{R}^2 \to \mathrm{R}$ be the projection, then $(pr_1(\bigcap_{i=1}^p V_{i,t}))_{t\in[0,1]}$, is an open cover of $[0,1]$, so there is a finite subcover

$$(pr_1(\bigcap_i V_{i,t_j}))_{j=1,\dots,N}.$$

Put $U_i = \bigcap_{j=1} U_{i,t_j}$. Then for all

$$(\eta_1,\dots,\eta_p) \in TM \underset{M}{\times} \dots \underset{M}{\times} TM \cap U_1 \times \dots \times U_p$$

we have

$$|\phi((\eta_1,0_t),\dots,(\eta_p,0_t)) - \phi((\xi_1,0_t),\dots,(\xi_p,0_t))| < \epsilon$$

uniformly for $t \in [0,1]$. Thus

$$I_0^1\phi : TM \underset{M}{\times} \dots \underset{M}{\times} TM \to \mathrm{R}$$

is continuous. Now the derivative, say

$$D\phi : T((TM\times T\mathrm{R}) \underset{M\times R}{\times} \dots \underset{M\times R}{\times} (TM\times T\mathrm{R})) \to \mathrm{R}$$

is continuous and the same method as above shows that $\int_0^1 j_t^* D\phi\, dt$ is continuous. A simple argument shows this expression is $D\int_0^1 j_t^*\phi\, dt$. This procedure may be repeated; it shows that $I_0^1\phi : TM \underset{M}{\times} \dots \underset{M}{\times} TM \to \mathrm{R}$ is C_c^∞.

So finally we may write $I_0^1\phi = \int_0^1 j_t^*\phi\, dt$, where the integral exists in $\Omega^p(M)$, and clearly the map $I_0^1 : \Omega^p(M\times\mathrm{R}) \to \Omega^p(M)$ is linear and continuous. From now on we may just repeat the finite-dimensional proof: let $T = \frac{\partial}{\partial t} \in \mathfrak{X}(M \times \mathrm{R})$, then T has the global flow

$$a : (M \times \mathrm{R}) \times \mathrm{R} \to M \times \mathrm{R}, \quad a((x,t),s) = (x, t+s),$$

and $j_{s+t} = a_t \circ j_s$. So we may compute

$$\frac{d}{ds} j_s{}^* \phi = \frac{d}{dt}\Big|_{t=0} (a_t \circ j_s)^* \phi = \frac{d}{dt}\Big|_{t=0} j_s{}^* a_t{}^* \phi =$$

$$= j_s{}^* \frac{d}{dt}\Big|_{t=0} a_t^* \phi = j_s^* \mathfrak{L}_t \phi$$

by 2.9. Here we use that $j_s^* : \Omega^p(M \times \mathrm{R}) \to \Omega^p(M)$ is linear and continuous.

Claim: $d \circ I_0^1 = I_0^1 \circ d$:

$$j_1^* - j_0^* = I_0^1 \circ \mathfrak{L}_T = d \circ I_0^1 \circ i_T + I_0^1 \circ i_T \circ d.$$

$$d \circ I_0^1 \phi = d \int_0^1 j_t^* \phi \, dt = \int_0^1 d \circ j_t^* \phi \, dt =$$

$$= \int_0^1 j_t{}^* d\phi \, dt = I_0^1 \circ d\phi.$$

$$j_1^* \phi - j_0^* \phi = \int_0^1 \frac{d}{dt} j_t^* \phi \, dt = \int_0^1 j_t^* \mathfrak{L}_T \phi \, dt =$$

$$= I_0^1 \circ \mathfrak{L}_T \phi = I_0^1 \circ (d \circ i_T + i_T \circ d)\phi.$$

Finally we may prove 2. Define the homotopy operator $h := I_0^1 \circ i_T \circ H^*$ where H is the homotopy connecting f and g. Then we have

$$g^* - f^* = (H \circ j_1)^* - (H \circ j_0)^* = (j_1^* - j_0^*) \circ H^* =$$

$$= d \circ I_0^1 \circ i_T \circ H^* + I_0^1 \circ i_T \circ H^* \circ d = d \circ h + h \circ d.$$

So $f^\# = g^\#$.

4. Can be proved without difficulty.

Consider the embeddings

$$\begin{array}{ccccc} & & M & & \\ & \overset{i_U}{\nearrow} & & \overset{i_V}{\nwarrow} & \\ U & & & & V \\ & \underset{j_U}{\nwarrow} & & \underset{j_V}{\nearrow} & \\ & & U \cap V & & \end{array}$$

and the sequence of cochain complexes

$$0 \to \Omega(M) \xrightarrow{\alpha} \Omega(U) \oplus \Omega(V) \xrightarrow{\beta} \Omega(U \cap V) \to 0,$$

where

$$\alpha\omega = (i_U^*\omega, i_V^*\omega), \quad \beta(\phi,\psi) = j_U^*\phi - j_V^*\psi.$$

This sequence is exact: on $\Omega(U\cap V)$ use a partition of unity on M subordinated to the cover $\{U, V\}$. As usual this gives the long exact cohomology sequence. qed

3.3. THEOREM. *Let M be a (NLF)-manifold. Then the de Rham cohomology of M coincides with the sheaf-cohomology of M with coefficients in the constant sheaf* R *on M.*

PROOF. Recall that M is paracompact.

$$\mathrm{R} \to \Omega^0 \to \Omega^1 \to \Omega^2 \to \dots$$

is a resolution of the constant sheaf R on M, where Ω^p denotes the sheaf of local p-forms in M. This is a resolution by the lemma of Poincaré. Since M admits C_c^∞-partitions of unity, each Ω^p is a fine sheaf, so the resolution above is acyclic, and by the general theory of sheaf cohomology the theorem follows. qed

3.4. THEOREM. *Let M be a (NLF)-manifold. The de Rham cohomology of M coincides with the singular cohomology with coefficients in* R, *an isomorphism being induced by integration of p-forms over C_c^∞-singular simplexes.*

PROOF. Denote by ζ_∞^k the sheaf which is generated by the presheaf of locally supported singular C_c^∞-cochains with coefficients in R. In more detail: let $S_\infty^k(U,\mathrm{R}) = \prod_\sigma \mathrm{R}$ where $\sigma: \Delta_k \to U$ is any mapping which extends to a C_c^∞-mapping from a neighborhood of the standard k-simplex Δ_k in R^{k+1} into U, U open in M. This defines a presheaf. The associated sheaf is denoted by ζ_∞^k. Then we have a sequence of sheaves

$$\mathrm{R} \to \zeta_\infty^0 \to \zeta_\infty^1 \to \zeta_\infty^2 \to \dots .$$

This sequence is a resolution for, if U is a small open set, say C_c^∞-diffeomorphic to an absolutely convex neighborhood of 0 in an (NLF)-space E, then U is C_c^∞-contractible to a point. Since C_c^∞-mappings clearly induce mappings in the S_∞^*-cohomology,

$$H^k(S^*_\infty(U, \mathrm{R}), d) = 0 \quad \text{for} \quad k > 0.$$

This implies that each associated sequence of stalks is exact, so the sequence above is a resolution. A standard argument of sheaf theory (using the axiom of choice) shows that each ζ^k_∞ is a fine sheaf, so we have an acyclic resolution, and $H^k(\zeta^*_\infty(M, \mathrm{R}), d)$ coincides with the sheaf cohomology with coefficients in the constant sheaf R.

Furthermore integration of p-forms over C^∞_c-singular p-simplexes in M defines a mapping of resolutions

$$\begin{array}{ccccccc} & & \Omega^0 & \to & \Omega^1 & \to & \Omega^2 \to \dots \\ & \nearrow & \downarrow & & \downarrow & & \downarrow \\ \mathrm{R} & & & & & & \\ & \searrow & \zeta^0_\infty & \to & \zeta^1_\infty & \to & \zeta^2_\infty \to \dots \end{array}$$

which induces an isomorphism

$$H_{dR}(M) \xrightarrow{\approx} H^*(\zeta^*_\infty(M, \mathrm{R}), d) = H^*(S^*_\infty(M, \mathrm{R}), d).$$

Now consider the resolution

$$\mathrm{R} \to \zeta^0 \to \zeta^1 \to \zeta^2 \to \dots$$

of the constant sheaf, where ζ^k is the usual sheaf induced by the locally supported singular cochains. Since M is paracompact and locally contractible, this is an acyclic resolution, and the embedding of C^∞_c-singular chains into all singular chains gives a mapping of resolutions

$$\begin{array}{ccccccc} & & \zeta^0 & \to & \zeta^1 & \to & \zeta^2 \to \dots \\ & \nearrow & \downarrow & & \downarrow & & \downarrow \\ \mathrm{R} & & & & & & \\ & \searrow & \zeta^0_\infty & \to & \zeta^1_\infty & \to & \zeta^2_\infty \to \dots \end{array}$$

which induces an isomorphism

$$H^*(S^*(M, \mathrm{R}), d) = H^*(\zeta^*(M, \mathrm{R}), d) \xrightarrow{\approx} H^*(\zeta^*_\infty(M, \mathrm{R}), d) =$$
$$= H^*(S^*_\infty(M, \mathrm{R}), d).$$

qed

3.5. REMARK. Note that the Alexander-Spanier cohomology and the Čech cohomology of a (NLF)-manifold coincide with the singular cohomology.

4. REMARKS ABOUT COHOMOLOGY OF DIFFEOMORPHISM GROUPS.

4.1. As shown in Michor [13, 14], the group $Diff(X)$ of all smooth diffeomorphisms of a finite dimensional manifold X is an (NLF)-manifold with C_c^∞-operations. The subgroup $Diff_c(X)$ of all diffeomorphisms with compact support is open in $Diff(X)$. The connected component $Diff_0(X)$ of the identity consists of all diffeomorphisms compactly diffeotopic to the identity.

4.2. The tangent space $T_{Id}Diff(X)$ is the space $\Gamma_c(TX)$ of all vectorfields with compact support on X, with its natural (NLF)-space topology. This is clearly a topological Lie-algebra. But one may define the Lie bracket on $\Gamma_c(TX)$ in another way: let $\xi, \eta \in \Gamma_c(TX)$; extend them to left invariant fields L_ξ, L_η on $Diff(X)$, and consider

$$[L_\xi, L_\eta] \in \mathfrak{X}(Diff(X))$$

and its value at Id. This gives the same Lie-algebra structure, up to sign on $\Gamma_c(TX)$, as we will show below.

4.3. For $\xi \in \Gamma_c(TX)$ denote the left invariant vector field on $Diff(X)$ generated by ξ by L_ξ, and call the right invariant one R_ξ. For $f \in Diff(X)$, we have

$$\begin{aligned} L_\xi(f) &= T_{Id}(\text{left translation by } f).\xi \\ &= T_{Id}(f_*).\xi = Tf \circ \xi, \qquad \text{by [14], 10.14.} \\ R_\xi(f) &= T_{Id}(\text{right translation by } f).\xi \\ &= T_{Id}(f^*).\xi = \xi \circ f. \end{aligned}$$

4.4. LEMMA. *For $\xi, \eta \in \Gamma_c(TX)$ we have*

$$[L_\xi, L_\eta] = -L_{[\xi,\eta]},\ [R_\xi, R_\eta] = R_{[\xi,\eta]},\ [L_\xi, R_\eta] = 0.$$

PROOF. Since the chart structure on

$$T\,Diff(X) = \mathfrak{D}_{Diff(X)}(X, TX)$$

is rather complicated (see [14], 10.13) we prefer to use 2.10.

Since ξ is a vector field with compact support, it has a global flow $a: X \times \mathbb{R} \to X$. Since a_t has compact support for each t, the mapping

$$t \mapsto a_t, \quad \mathbb{R} \to Diff(X),$$

is of class C_c^∞. This may be seen as follows: first note that this curve is continuous, $\frac{d}{dt}a_t = \xi \circ a_t$ exists in $\mathfrak{D}(X, TX)$ for all t and is again continuous in t, since a_t takes its values on the set of proper mappings in $C^\infty(X, X)$. By recursion $t \mapsto a_t$ is C_c^∞ - compact support is essential here, see [14], 11.9. Now define

$$a^L : Diff(X) \times \mathbb{R} \to Diff(X) \quad \text{by} \quad a^L(f, t) = f \circ a_t.$$

This is a C_c^∞-mapping. To compute $\frac{d}{dt}\, a^L(f, t)$ we may evaluate at $x \in X$ (see [14] 10.15). Then we have

$$\begin{aligned}\frac{d}{dt}a^L(f,t)(x) &= \frac{d}{dt}f(a(x,t)) = Tf(\frac{d}{dt}\, a(x,t)) \\ &= Tf \circ \xi(a(x,t)) = (L_\xi(a^L(f,t)))(x),\end{aligned}$$

since

$$\begin{aligned}L_\xi(a^L(f,t)) &= T(a^L(f,t)) \circ \xi = T(f \circ a_t) \circ \xi = \\ &= Tf \circ Ta_t \circ \xi = Tf \circ \xi \circ a_t.\end{aligned}$$

So a^L is the global flow for the left invariant vector field L_ξ. We can use Lemma 2.10 now to compute $[L_\xi, L_\eta]$. But first note that

$$a^L{}_t = (a_t)^*: Diff(X) \to Diff(X).$$

Thus

$$T(a^L{}_t) = \mathfrak{D}(a_t, TX), \quad T_f(a^L{}_t).s = s \circ a_t$$

for $s \in \mathfrak{D}_f(X, TX)$. Now we compute

$$\begin{aligned}[L_\xi, L_\eta](f) &= (\frac{d}{dt}\Big|_{t=0}(a^L{}_t)^* L_\eta)(f) = \\ &= \frac{d}{dt}\Big|_{t=0} T(a^L{}_{-t}) \circ L_\eta \circ a^L{}_t(f) = \frac{d}{dt}\Big|_{t=0} T(a^L_{-t})(L_\eta(f \circ a_t)) \\ &= \frac{d}{dt}\Big|_{t=0} T(a^L_{-t})(Tf \circ Ta_t \circ \eta) = \frac{d}{dt}\Big|_{t=0} Tf \circ Ta_t \circ \eta \circ a_{-t}) \\ &= \frac{d}{dt}\Big|_{t=0}(Tf) * (Ta_t \circ \eta \circ a_{-t}) = (Tf)_*(\frac{d}{dt}\Big|_{t=0} Ta_t \circ \eta \circ a_{-t}) \\ &= (Tf)_*(-[\xi, \eta]) = -Tf \circ [\xi, \eta] = -L_{[\xi, \eta]}.\end{aligned}$$

We have used that

$$(Tf)_*: \Gamma_c(TX) \to \mathfrak{D}_f(X, TX)$$

is linear and continuous.

For the proof of the second assertion first note that

$$Inv : Diff(X) \to Diff(X)$$

is C_c^∞ (by [14], 11.11), that $Inv^* L_\xi = R_{(-\xi)}$ and that

$$[R_\xi, R_\eta] = Inv^*[Inv^* R_\xi, Inv^* R_\eta] = Inv^*[L_\xi, L_\eta]$$
$$= -Inv^* L_{[\xi,\eta]} = R_{[\xi,\eta]}.$$

The last assertion is immediate since the flows of L_ξ, L_η commute (the flow of R_η is $\beta^R(f,t) = \beta_t \circ f$, where β_t is the flow of η). qed

4.5. Let us denote for the moment the right translation by $f \in Diff(X)$ $\rho_f : Diff(X) \to Diff(X)$, let similarly λ_f denote left translation.

A differential form $\omega \in \Omega^p(Diff(X))$ is called right invariant if $\rho_f^* \omega = \omega$ for all $f \in Diff(X)$.

The following results are easily seen to be true.

1. The subspace of all right invariant forms in $\Omega^p(Diff(M))$ is linearly and topologically isomorphic to the space $\Lambda^p(\Gamma_c'(TX))$ of all alternating p-linear jointly continuous mappings

$$\Gamma_c(TX) \times \ldots \times \Gamma_c(TX) \to \mathbb{R}$$

Similar for left invariant forms.

Note that we have to assume joint continuity, separate continuity is not enough if $\Gamma_c(TX)$ is not metrizable.

2. The subspace of right invariant forms in $\Omega(Diff(M))$ is stable under the exterior derivative d, since $d \circ \rho_f^* = \rho_f^* \circ d$. The exterior derivative induces the following operator on the space

$$\Lambda(\Gamma_c'(TX)) = \bigoplus_{k \geq 0} \Lambda^k(\Gamma_c'(TX)) :$$

$$d\omega(\xi_0, \ldots, \xi_p) =$$
$$= \sum_{0 \leq i < j \leq p} (\ 1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_p)$$

for $\omega \in \Lambda^p(\Gamma_c'(TX))$ and $\xi_i \in \Gamma_c(TX)$.

3. For $\xi \in \Gamma_c(TX)$ the space of right invariant forms in $\Omega(Diff(X))$ is invariant under the operators $\mathfrak{L}_{R_\xi}$, i_{R_ξ}, and these induce the following mappings on $\Lambda(\Gamma_c'(TX))$:

$$i_\xi \omega(\eta_2, \dots, \eta_p) = \omega(\xi, \eta_2, \dots, \eta_p).$$

$$\mathfrak{L}_\tau\, \omega(\xi_1, \dots, \xi_p) = \sum_{i=1}^{p} (-1)^i \omega([\xi, \xi_i], \xi_1, \dots, \hat{\xi}_i, \dots, \xi_p).$$

4. The results of Theorem 2.8 hold for these operators too.

4.6. The *exponential mapping of* $Diff(X)$ is the mapping

$$Exp : \Gamma_c(TX) = T_{Id} Diff(X) \to Diff(X),$$

which assigns to each vector field $\xi \in \Gamma_c(TX)$ with compact support the diffeomorphism with compact support

$$Exp(\xi) = Fl(\xi) = Fl(\xi)(\,.\,, 1),$$

where $Fl(\xi) : X \times \mathrm{R} \to X$ is the global flow of ξ.

THEOREM. *$Exp : \Gamma_c(TX) \to Diff(X)$ is C^∞.*

PROOF. The global flow $Fl(\xi) : X \times \mathrm{R} \to X$ of ξ is given by the ordinary differential equation

$$\frac{d}{dt} Fl(\xi)_t = \xi \circ Fl(\xi)_t = Comp(\xi, Fl(\xi)_t),$$

where

$$C_c^\infty(X, TX) \times Diff(X) \to \mathfrak{D}(X, TX)$$

is the composition mapping, which is C_c^∞ by [14], 11.4. The (NLF)-space $\Gamma_c(TX)$ is a splitting submanifold of $C^\infty(X, TX)$ by [14], 10.10, and for any $\xi, \eta \in \Gamma_c(TX)$ the tangent vector

$$\frac{d}{ds}(\xi + s\eta)\Big|_{s=0} \in \mathfrak{D}(X, T^2X) = TC^\infty(X, TX),$$

is given by

$$V \circ (\xi, \eta) : X \to T^2X, \quad \text{where } V : TX \times TX \to T^2X$$

is the vertical lift ([14], 1.15.3), since by ([14], 10.5) we may compute $\frac{d}{ds}(\xi + s\eta)\big|_{s=0}$ after evaluating at $x \in X$:

$$\left(\frac{d}{ds}\Big|_{s=0} \xi + s\eta\right)(x) = \frac{d}{ds}\Big|_{s=0} \xi(x) + s\eta(x) = V(\xi(x), \eta(x)) \in T^2X.$$

Recall that the canonical flip mapping $\kappa_X : T^2X \to T^2X$ satisfies

$$\frac{d}{ds}\frac{d}{dt}f(s,t) = \kappa_X \frac{d}{dt}\frac{d}{ds}f(s,t),$$

where $f: \mathbb{R}^2 \to X$ is any smooth mapping. Now we compute the tangent mapping of the ordinary differential equation above:

$$\kappa_X \frac{d}{dt}\frac{d}{ds}\Big|_{s=0} Fl(\xi+s\eta)_t = \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}Fl(\xi+s\eta)_t$$

$$= \frac{d}{ds}\Big|_{s=0}(\xi+s\eta)\circ Fl(\xi+s\eta)_t = \frac{d}{ds}\Big|_{s=0} Comp(\xi+s\eta, Fl(\xi+s\eta)_t)$$

$$= T_{(\xi, Fl(\xi)_t)} Comp.\left(\frac{d}{ds}\Big|_{s=0}(\xi+s\eta), \frac{d}{ds}\Big|_{s=0} Fl(\xi+s\eta)_t\right)$$

$$= T_{(\xi, Fl(\xi)_t)} Comp.\left(V_\bullet\circ(\xi,\eta), \frac{d}{ds}\Big|_{s=0} Fl(\xi+s\eta)_t\right)$$

$$= T_{(\xi, Fl(\xi)_t)} Comp.(V\circ(\xi,\eta), 0) +$$

$$+ T_{(\xi, Fl(\xi)_t)} Comp.\left(0, \frac{d}{ds}\Big|_{s=0} Fl(\xi+s\eta)_t\right)$$

$$= T_\xi(Fl(\xi)_t^*).\,V\circ(\xi,\eta) + T_{Fl(\xi)_t}(\xi_*).\,\frac{d}{ds}\Big|_{s=0} Fl(\xi+s\eta)_t$$

by [14], 10.14,

$$= V\circ(\xi,\eta)\circ Fl(\xi)_t + T\xi\circ\frac{d}{ds}\Big|_{s=0} Fl(\xi+s\eta)_t .$$

So the mapping $T_\xi(Fl(\,.\,)_t).\,\eta : X \to TX$ is given by the ordinary differential equation

$$\frac{d}{dt}(T_\xi(Fl(\,.\,)_t).\,\eta)(x) =$$

$$= \kappa_X(V\circ(\xi,\eta)\circ Fl(\xi)_t(x) + T\xi\circ(T_\xi(Fl(\,.\,)_t).\eta)(x))$$

with the initial condition

$$(T_\xi(Fl(\,.\,)_0).\,\eta)(x) = \frac{d}{ds}\Big|_{s=0} Fl(\xi+s\eta)_0(x) = 0 \quad .$$

This differential equation has a global solution for each x and is C_c^∞ in x, because we just differentiated a smooth family of global flows at $s = 0$. The solution is furthermore the global flow of a vector field. This is seen as follows: call

$$T_\xi(Fl(\,.\,)_t).\,\eta =: \alpha_t : X \to TX.$$

Then a_t satisfies

$$\frac{d}{dt}a_t(x) = \kappa_X(V\circ(\xi,\eta)\circ\pi_X\circ a\ (x) + T\xi\circ a_t(x)),$$

$a_0(x) = 0$. This is the flow equation of the vector field

$$\Theta(\xi,\eta) = \kappa_X\circ(V\circ(\xi,\eta)\circ\pi_X\circ a,\xi) : TX \to T^2X.$$

In a local chart on X this field is given by

$$(x,y) \mapsto (x,y;\bar{\xi}(x),\bar{\eta}(x)+d\bar{\xi}(x)y).$$

Since $\pi_X\circ a_t = Fl(\xi)_t$ we have

$$a_t \in \mathfrak{D}_{Fl(\xi)_t}(X,TX) \text{ and } a_t(x) = 0_x \text{ for } x\in X\setminus(supp\ \xi\cup supp\ \eta).$$

By the argument used in the beginning of the proof of 4.4 we may conclude that $a : \mathbb{R}\to C^\infty(X,TX)$ is of class C^∞_c .

After this detailed construction of the tangent to the mapping Fl, we return to the proof of the theorem. First note that

$$Exp : \Gamma_c(TX) \to Diff(X)$$

is continuous. If ξ is near ξ_1 then $Fl(\xi)_1$ is near $Fl(\xi_1)_1$ by the argument used below to prove 4.8. This holds for all derivatives with respect to X. Now $T\ Exp : \Gamma_c(TX)\times\Gamma_c(TX)\to\mathfrak{D}(X,TX)$ is given by

$$T\ Exp(\xi,\eta) = Fl(\Theta(\xi,\eta))_1\circ 0_X.$$

Θ is not continuous, but we need only its flow lines starting from 0_X , and $Fl(\Theta(\xi,\eta))_1\circ 0_X$ is indeed continuous. By recursion we get that Exp is C^∞_c . qed

4.7. It is known that $Exp : \Gamma_c(TX)\to Diff(X)$ does not contain any open neighborhood of Id in its image. There is a simple counterexample due to Omori [15] on $Diff(S^1)$. In contrast, the image of Exp still generates the connected component $Diff_0(X)$ of the identity in $Diff(X)$. A way to show this is indicated in Epstein [4]. We may suppose that X is connected (otherwise $Diff_0(X)$ is a direct sum of groups). Then by Epstein [4] the commutator group $[Diff_0(X), Diff_0(X)]$ is simple and coincides with $Diff_0(X)$ by Thurston [22]. The set $Exp(\Gamma_c(TX))$ is closed under conjugation in $Diff(X)$, so it generates a non trivial normal subgroup which coincides with $Diff_0(X)$. The same result holds for $Diff_0^k(X)$ (diffeomorphisms of class C^k) if $k \neq dim X + 1$. This has been shown by Mather, in [10].

A detailed proof of Thurston's result has not been published. In the following we prove a weaker result that suffices for our purpose by a simple argument.

4.8. LEMMA. *For any smooth finite dimensional (paracompact) manifold X, the image of the exponential mapping generates a dense subgroup of $Diff_0(X)$.*

PROOF. It suffices to prove this theorem for $X = R^n$, for any $f \in Diff_0(X)$ can be written in the form $f = f_1 \circ \ldots \circ f_k$, where $f_i \in Diff_0(X)$ has support contained in some chart. A proof of this fact that can be extended to the non compact case is in Palais-Smale [16], Lemma 3.1.

So let $f \in Diff_0(R^n)$. Take a smooth curve α from Id to f in $Diff_0(X)$, so α is a diffeotopy with compact support. Consider the time-dependent vector field $\xi : R^n \times [0, 1] \to R^n$ given by

$$\xi(\alpha(x,t),t) = \frac{d}{dt}\alpha(x,t).$$

ξ has compact support in $R^n \times [0,1]$. Now for $n \in N$, let

$$\xi_{k/n}(x) = \xi(x, \frac{k}{n}), \quad 0 \leq k \leq n-1.$$

These are vector fields with compact support. Let

$$f_{n,k} = Fl(\xi_{k/n})_{1/n} \in Diff_0(R^n),$$

and put

$$\begin{aligned} f_n := f_{n,n-1} \circ f_{n,n-2} \circ \ldots \circ f_{n,0} = \\ = Exp(\tfrac{1}{n}.\xi_{n-1/n}) \circ \ldots \circ Exp(\tfrac{1}{n}.\xi_0). \end{aligned}$$

We claim that $f_n \to f$ in $Diff_0(X)$. We will use the comparison theorem for (approximate) solutions of differential equations in the form of Dieudonné [2], 10.5.6. For that define $\alpha_{n,k} \in Diff_0(X)$ by $\alpha_{n,k} = \alpha_{n,k,k+1/n}$ where $\alpha_{n,k,t}$ is given by the differential equation

$$\frac{d}{dt} a_{n,k,t}(x) = \xi(a_{n,k,t}(x), t), \quad a_{n,k,k/n}(x) = x.$$

Let $\epsilon > 0$. Suppose that n is so large that

$$|\xi(x,t) - \xi(x, \tfrac{k}{n})| < \epsilon \quad \text{for all} \quad x \in \mathrm{R}^n \text{ and } \tfrac{k}{n} \leq t \leq \tfrac{k+1}{n},$$

$k = 0, 1, \ldots, n-1$. Put

$$M = max \{ \, |D(\xi(.,t)(x)| \;|\; x \in \mathrm{R}^n, \; 0 \leq t \leq 1 \}.$$

Then the comparison theorem mentioned above produces the following estimate:

$$|a_{n,k}(x) - f_{n,k}(y)| \leq |x-y| . e^{M/n} + \epsilon . \frac{e^{M/n} - 1}{M} .$$

Using this estimate we may compute as follows:

$$|f(x) - f_n(x)| =$$

$$= |a_{n,n-1} \circ a_{n,n-2} \circ \ldots \circ a_{n,0}(x) - f_{n,n-1} \circ f_{n,n-2} \circ \ldots \circ f_{n,0}(x)|$$

$$\leq |a_{n,n-2} \circ \ldots \circ a_{n,0}(x) - f_{n,n-2} \circ \ldots \circ f_{n,0}(x)| . e^{M/n} + \epsilon . \frac{e^{M/n} - 1}{M}$$

$$\leq \epsilon . \frac{e^{M/n} - 1}{M} . \sum_{k=0}^{n-1} (e^{M/n})^k = \frac{\epsilon}{M}(e^M - 1).$$

So $|f(x) - f_n(x)| \to 0$ uniformly for $x \in \mathrm{R}^n$.

The same argument may be repeated for each derivative with respect to x , as in the proof of 4.6. Since $f_n = f = Id$ off some compact set, $f_n \to f$ in $Diff(X)$. qed

4.9. DEFINITION. Let $H^*(\Gamma_c(TX))$ denote the cohomology of the Lie-algebra of vector fields with compact support with real coefficients, i.e., the homology of the cochain complex $\Lambda(\Gamma_c'(TX))$ described in 4.5. Extension of elements in $\Lambda(\Gamma_c'(TX))$ to right invariant differential forms on $Diff_0(X)$ gives an embedding $\Lambda(\Gamma_c'(TX)) \to \Omega(Diff_0(X))$ and this in turn induces a natural mapping in cohomology

$$H^*(\Gamma_c(TX)) \to H_{dR}(Diff_0(X)).$$

For a compact connected Lie-group this mapping turns out to be an isomorphism in cohomology - the proof uses invariant integration.

Note the following easy results:

1. $$H^0(\Gamma_c(TX)) = \mathrm{R} = H_{dR}(Diff_0(X)),$$

since $Diff_0(X)$ is connected.

2. $$H^1(\Gamma_c(TX)) = 0.$$

Let

$$\omega \in \Lambda^1(\Gamma_c'(TX)) = \Gamma_c'(TX) = L(\Gamma_c(TX), \mathrm{R})$$

with $d\omega = 0$. Then

$$d\omega(\xi_0, \xi_1) = -\omega([\xi_0, \xi_1]) = 0 \text{ for all } \xi_0, \xi_1 \in \Gamma_c(TX).$$

This implies $\omega = 0$ by the following

SUBLEMMA. *Any $\xi \in \Gamma_c(TX)$ can be represented as a finite sum*

$$\sum_{i=0}^{m} [\xi_{1,i}, \xi_{2,i}] \text{ for } \xi_{k,i} \in \Gamma_c(TX).$$

PROOF. By partition of unity let $\xi = \xi_1 + \ldots + \xi_p$, where each ξ_i has support in a chart neighborhood U_i of X. So suppose ξ has support in a chart (U, u) of X. Let

$$\xi = \sum_i f^i \frac{\partial}{\partial z^i} \quad \text{with } supp(f^i) \subset U.$$

Choose g, h smooth functions with compact support such that $g^i = u^i$, $h = 1$ on $supp(\xi)$. Then

$$[f^i \frac{\partial}{\partial u^i}, g^i \frac{\partial}{\partial u^i}] + [h \frac{\partial}{\partial u^i}, f^i g^i \frac{\partial}{\partial u^i}] = 2 f^i \frac{\partial}{\partial u^i}. \qquad \text{qed}$$

4.10. Substantial information about $H^*(\Gamma_c(TX))$ has been obtained by Gelfand-Fuks [5], who investigated this cohomology and got the following results:

If X is compact then $H^p(\Gamma_c(TX))$ is a finite dimensional real vector space for each p.

$H^*(\Gamma_c(TS^1))$ is the tensor product of the polynomial algebra over a generator in degree 2 and the exterior algebra over a generator in degree 3.

$H^*(\Gamma_c(TS^2))$ has ten generators and $H^*(\Gamma_c(T(S^1 \times S^1)))$ has 20 generators (with non trivial relations).

Since $Diff_0(S^2)$ contains $SO(3)$ as a strong deformation retract (see Smale [21]) the mapping $H^*(\Gamma_c(TS^2)) \to H^*_{cR}(Difff_c(S^2))$ cannot be injective.

4.11. The *adjoint representation of* $Diff(X)$ can be constructed as in the finite dimensional case, but then a curious thing happens:

$$Ad\,Exp: \Gamma_c(TX) \to L(\Gamma_c(TX), \Gamma_c(TX))$$

is *not* analytic. The construction follows:

1. Define conjugation

$$Conj: Diff(X) \to Aut(Diff(X)) \subset C_c^\infty(Diff(X), Diff(X))$$

by $Conj(f)(g) = f^{-1} \circ g \circ f$. This is a group anti-homomorphism (taken so to avoid a minus sign in the definition of *ad*, compare with 4.4).

$$Conj: Diff(X) \times Diff(X) \to Diff(X)$$

is a C_c^∞-mapping.

2. Define

$$Ad: Diff(X) \to L(\Gamma_c(TX), \Gamma_c(TX)) \text{ by } Ad(f) = T_{Id}(Conj(f)).$$

We have $Conj(f) = \lambda_{f^{-1}} \circ \rho_f$, where λ denotes left translation and ρ denotes right translation (as in 4.5). Thus we have

$$Ad(f) = T(\lambda_{f^{-1}}) \circ T(\rho_f) = T((f^{-1})_*) \circ T(f^*) = (Tf^{-1})_* \circ f^*,$$

$Ad(f)\xi = Tf^{-1} \circ \xi \circ f$. The mapping

$$Ad: Diff(X) \times \Gamma_c(TX) \to \Gamma_c(TX)$$

is C_c^∞.

3. Define $ad: \Gamma_c(TX) \to L(\Gamma_c(TX), \Gamma_c(TX))$ as the tangent vector part of $T_{Id}Ad$. We will see later that $ad(\xi)\eta = [\xi, \eta]$ as usual.

4. LEMMA. $\frac{d}{dt} Ad(Exp(t\xi))\eta = Ad(Exp(t\xi))[\xi, \eta]$.

PROOF.

$$\frac{d}{dt} Ad(Exp(t\xi))\eta = \frac{d}{dt}(T\,Fl(\xi)_{-t} \circ \eta \circ Fl(\xi)_t) =$$

$$= \frac{d}{dt} Comp(T\,Fl(\xi)_{-t} \circ \eta, Fl(\xi)_t) =$$

$$= T_{(T\,Fl(\xi)_{-t} \circ \eta,\, Fl(\xi)_t)} Comp(\tfrac{d}{dt} T\,Fl(\xi)_t \circ \eta, \tfrac{d}{dt} Fl(\xi)_t)$$

Now choose a smooth curve $c: \mathbb{R} \to X$ with $c'(0) = \eta(x)$. Then

$$\frac{d}{dt} T_x Fl(\xi)_{-t} . \eta(x) = \frac{d}{dt}\frac{d}{ds}|_{s=0} Fl(\xi)_{-t}(c(s)) =$$

$$= \kappa_X \frac{d}{ds}|_{s=0} \frac{d}{dt} Fl(\xi)_{-t}\, c(s) = \kappa_X \frac{d}{ds}|_{s=0}(-\xi \circ Fl(\xi)_{-t}\, c(s))$$

$$= -\kappa_X \circ T\xi \circ T\, Fl(\xi)_{-t}\, c'(0) = -\kappa_X\ T\xi \circ T\, Fl(\xi)_{-t}\, \eta(x),$$

where $\kappa_X : T^2 X \to T^2 X$ is the canonical flip map. So we may continue:

$$\frac{d}{dt} Ad(Exp(t\xi))\eta =$$

$$= T_{(T\, Fl(\xi)_{-t} \circ \eta,\, Fl(\xi)_t} Comp(-\kappa_X\, T\xi \circ T\, Fl(\xi)_{-t} \circ \eta,\, \xi \circ Fl(\xi)_t) =$$

$$= T_{(T\, Fl(\xi)_{-t} \circ \eta,\, Fl(\xi)_t)} Comp(-\kappa_X T\xi \circ T\, Fl(\xi)_{-t} \circ \eta, 0) +$$

$$+ T_{(T\, Fl(\xi)_{-t} \circ \eta,\, Fl(\xi)_t)} Comp(0, \xi \circ Fl(\xi)_t)$$

$$= -T_{T\, Fl(\xi)_{-t} \circ \eta}(Fl(\xi)_t^*)(\kappa_X \circ T\xi \circ T\, Fl(\xi)_{-t} \circ \eta) +$$

$$+ T_{Fl(\xi)_t}((T\, Fl(\xi)_{-t} \circ \eta))_*)(\xi \circ Fl(\xi)_t)$$

$$= -\kappa_X \circ T\xi \circ T\, Fl(\xi)_{-t} \circ \eta \circ Fl(\xi)_t + T^2(Fl(\xi)_{-t}) \circ T\eta \circ \xi \circ Fl(\xi)_t$$

$$\overset{(*)}{=} -\kappa_X \circ T^2(Fl(\xi)_{-t}) \circ T\xi \circ \eta \circ Fl(\xi)_t + T^2(Fl(\xi)_{-t}) \circ T\eta \circ \xi \circ Fl(\xi)_t$$

$$= T^2(Fl(\xi)_{-t}) \circ (-\kappa_X \circ T\xi \circ \eta + T\eta \circ \xi) \circ Fl(\xi)_t$$

$$= T^2(Fl(\xi)_{-t}) \circ V_X \circ (\eta, [\xi, \eta]) \circ Fl(\xi)_t$$

where $V_X : TX \underset{X}{\times} TX \to T^2 X$ is the vertical lift and

$$\kappa_X \circ T\xi \circ \eta - T\eta \circ \xi = V_X(\eta, [\eta, \xi]),$$

$$= V_X \circ (T\, Fl(\xi)_{-t} \underset{X}{\times} T\, Fl(\xi)_{-t}) \circ (\eta, [\xi, \eta]) \circ Fl(\xi)_t$$

$$= V_X \circ (T\, Fl(\xi)_{-t} \circ \eta \circ Fl(\xi)_t,\ T\, Fl(\xi)_{-t} \circ [\xi, \eta] \circ Fl(\xi)_t)$$

$$= V_X \circ (Ad(Exp(t\xi))\eta,\, Ad(Exp(t\xi))[\xi, \eta]).$$

Forget the base point $Ad(Exp(t\xi))\eta$ and the formula follows. qed

5. COROLLARY. $ad(\xi)\eta = [\xi, \eta]$.

PROOF. Let $t = 0$ in the formula of Lemma 4. qed

6. LEMMA. $Ad(Exp(t\xi)) \circ ad(\xi) = ad(\xi) \circ Ad(Exp(t\xi))$.

PROOF. We get

$$\frac{d}{dt} Ad(Exp(t\xi))\,\eta =$$

$$= -\kappa_X T^2(Fl(\xi)_{-t}) \circ T\xi \circ \eta \circ Fl(\xi)_t + T^2(Fl(\xi)_{-t}) \circ T\eta \circ \xi \circ Fl(\xi)_t$$

by the line (*) in the proof of Lemma 4

$$= -\kappa_X T^2 Fl(\xi)_{-t} \circ T\xi \circ T Fl(\xi)_t \circ T Fl(\xi)_{-t} \circ \eta \circ Fl(\xi)_t +$$

$$+ T^2 Fl(\xi)_{-t} \circ T\eta \circ T Fl(\xi)_t \circ T Fl(\xi)_{-t} \circ \xi \circ Fl(\xi)_t$$

$$= -\kappa_X \circ T(Ad(Exp(t\xi))\xi) \circ (Ad(Exp(t\xi))\,\eta) +$$

$$+ T(Ad(Exp(t\xi))\,\eta) \circ (Ad(Exp(t\xi))\xi)$$

$$= V_X \circ (Ad(Exp(t\xi))\eta,\ [\xi, Ad(Exp(t\xi))\xi, Ad(Exp(t\xi))\eta])$$

$$= V_X \circ (Ad(Exp(t\xi))\eta,\ [\xi, Ad(Exp(t\xi))\eta]).$$

Now combine with Lemma 4 and get the result. qed

7. The result of Lemma 4 can be interpreted as a differential equation for $t \mapsto Ad(Exp(t\xi)) \in L(\Gamma_c(TX), \Gamma_c(TX))$:

$$\frac{d}{dt} Ad(Exp(t\xi)) = Ad(Exp(t\xi)) \circ ad(\xi), \quad Ad(Exp(0)) = Id.$$

The solution of this differential equation ought to be the series

$$S(t,\xi) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (ad(\xi))^k,$$

which is the infinite Taylor expansion of $Ad(Exp(t\xi))$ too; this follows from repeated application of Lemma 4. But the series $S(t,\xi)$ does not converge in any sense, for the n^{th} term $\frac{t^n}{n!} ad(\xi)^n \eta(x)$ contains an n^{th} derivative of η at x and η can be chosen to have a (local) Taylor expansion at x whose coefficients go to infinity arbitrarily fast. Check this for $X = \mathbb{R}^1$.

REFERENCES.

1. D. BURGHELEA, The rational homotopy groups of *Diff(M)* and *Homeo(M)* in the stability range, *Lecture Notes in Math.* 763, Springer (1979), 604-626.

2. J. DIEUDONNE, *Foundations of modern Analysis*, I, Academic Press 1969.

3. E. DUBINSKY, The structure of nuclear Fréchet spaces, *Lecture Notes in Math.* 720, Springer (1979).

4. D.B.A. EPSTEIN, The simplicity of certain groups of homeomorphisms, *Comp. Math.* 22 (1970), 165-173.

5. I.M. GEL'FAND & D.B. FUKS, Cohomology of the Lie algebra of tangent vector fields of a smooth manifold, *Funct. Anal. Appl.*: I: 3 (1969), 194; II: 4 (1970).

6. R. GODEMENT, *Topologie algébrique et théorie des faisceaux*, Hermann.

7. W. GREUB, S. HALPERIN & R. VANSTONE *Connections, Curvature, Cohomology I, II, III*, Academic Press 1972, 1973, 1976.

8. A. GROTHENDIECK, Produits tensoriels topologiques et espaces nucléaires, *Mem. A.M.S.* 16 (1955).

9. H.H. KELLER, Differential calculus in locally convex spaces, *Lecture Notes in Math.* 417, Springer (1974).

10. J. MATHER, Commutators of diffeomorphisms, *Comm. Math. Helv.*, I: 49 (1974), 512-528; II: 50 (1975), 15-26.

11. D. MCDUFF, The homology of some groups of diffeomorphisms, *Comm. Math. Helv.* 55 (1980), 97-129.

12. E. MICHAEL, Locally multiplicatively convex algebras, *Mem. AMS* 11 (1952).

13. P. MICHOR, Manifolds of smooth maps, *Cahiers Top. et Géom. Diff.*: I: XIX (1978), 47-78; II: XXI (1980), 63-86; III: Id. 325-337.

14. P. MICHOR, *Manifolds of differentiable mappings*, Shiva Math. Ser. 3, Orpington, Kent, 1980.

15. H. OMORI, On the group of diffeomorphisms of a compact manifold, *Proc. Symp. Pure Math.* XV., A.M.S. 1970, 167-183.

16. J. PALAIS & S. SMALE, Structural stability theorems, *Proc. Symp. Pure Math.* XIV, A.M.S. 1970, 223-231.

17. N. PAPAGHIUC, Sur les formes différentielles dans les espaces localement convexes, le lemme de Pioncaré, *Anal. Stiint. Iaşi* 23 (1977).

18. A. PIETSCH, *Nukleare lokalkonvexe Raume*, Berlin 1955.

19. H.H. SCHAFER, *Topological vector spaces*, Springer GTM 3, 1971.

20. U. SEIP, A convenient setting for differential calculus, *J. Pure Appl. Algebra* 14 (1979), 73-100.

21. S. SMALE, Diffeomorphisms of the 2-sphere, *Proc. AMS* 10 (1959), 621-626.

22. W. THURSTON, Foliations and groups of diffeomorphisms *Bull. AMS* 80 (1974).

23. M. VALDIVIA, The space of distributions $\mathfrak{D}'(\Omega)$ is not B_r-complete, *Math. Ann.* 211 (1974), 145-149.

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