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S. KASANGIAN
G. M. KELLY
F. ROSSI

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COFIBRATIONS AND THE REALIZATION OF NON-DETERMINISTIC AUTOMATA

by S. KASANGIAN*, G.M. KELLY** *and* F. ROSSI*

ABSTRACT.

We first consider modules (= profunctors) between $\mathcal{V}$-categories for a monoidal biclosed $\mathcal{V}$, and show that any module $\theta: \mathfrak{A} \rightarrow \mathcal{C}$ with small $\mathfrak{A}$ has a canonical decomposition as $\theta = G^* Y_*$, where $Y: \mathfrak{A} \Rightarrow P\mathfrak{A}$ is the Yoneda embedding and $G: \mathcal{C} \Rightarrow P\mathfrak{A}$ is the $\mathcal{V}$-functor corresponding to θ; observing further that G^* is in fact the «right-extension» $[Y_*, \theta]$ of θ along Y_* in the bicategory of modules. We then apply this to show that the forgetful 2-functor from the cofibrations $\mathfrak{A} \rightarrow \mathcal{C}$ to the modules $\mathfrak{A} \rightarrow \mathcal{C}$ has a «weak» right adjoint. We observe finally that this includes, as a special case, a theorem of Betti and Kasangian on the existence of a weak right adjoint - which we may call «canonical realization» - to the «behaviour» 2-functor defined on non-deterministic automata.

1. INTRODUCTION.

It is a banality that the mathematical structures of any given species, along with the appropriate morphisms, form a category. A much more striking observation is due to Lawvere [8] : namely, that many of the basic structures of mathematics are *themselves* categories; while others are functors between categories.

It is clear that monoids, groups, groupoids, ordered sets, and equivalence relations, are simply ordinary categories of special kinds. A topological space is essentially a special case of a topos, which is a category. An algebraic theory was exhibited by Lawvere [7] as a category, and

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its models as functors. A ring R is a one-object *additive* category - that is, one enriched over the category Ab of abelian groups; and an R-module is an additive functor $R \to Ab$. In [8], Lawvere exhibits metric spaces as (symmetric) categories enriched over the category of extended positive real numbers. Joyal treats species of combinatorial structures in [4] as endofunctors of the category of finite sets and permutations. Walters has emphasized the notion, in [11] and [12], of categories enriched over a *bicategory* B, and has identified the sheaves on a site as the Cauchy-complete symmetric B-categories for a suitable B; with the promise of a similar description of manifolds and of other structures formed by «patching models together». The list could be continued: suffice it to recall that a scheme can be identified with a functor from commutative rings to sets.

It is of course of no value to identify some structure with a category or a functor or some more general categorical situation, unless the natural morphisms between such structures, and the important operations on them, themselves turn out to be those appropriate to the categorical context - which is indeed the case in the examples above.

Our present purpose is to point out one more example, of a somewhat novel kind. Betti and Kasangian [3], in the context of non-deterministic automata, have described a canonical realization functor that is a very weak kind of right adjoint to the behaviour functor. Our observation is that this relation between automata and behaviours is but a special case of a similar relation between cofibrations and codiscrete cofibrations (in the sense of Street [9]) in the 2-category of $\mathcal{V}$-categories.

To encompass the case of automata, we must allow the $\mathcal{V}$ here to be a not-necessarily-symmetric biclosed monoidal category. Because there is no sufficient account of profunctors - which we prefer to call *modules* - in the non-symmetric case, we give one briefly in Section 2; a still more general account, with $\mathcal{V}$ now a biclosed bicategory, is given in a forthcoming paper [10] of Street, but this does not contain all that we need. In Section 3 we establish the weak adjunction between cofibrations and codiscrete cofibrations, and then in Section 3 we give the application to automata and re-find the Betti-Kasangian result.

2. $\mathcal{V}$-MODULES FOR A MONOIDAL BICLOSED $\mathcal{V}$.

2.1. We consider a locally-small monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ which is *biclosed*, in the sense that we have natural isomorphisms

$$(1) \qquad \mathcal{V}_0(X,[Y,Z]) \approx \mathcal{V}_0(X \otimes Y, Z) \approx \mathcal{V}_0(Y, \{X,Z\}).$$

As in Sections 1.2 and 1.3 of [5], we have the 2-category $\mathcal{V}$-CAT of $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations, and the 2-functor $(\ )_0 : \mathcal{V}\text{-}CAT \to CAT$ represented by the unit $\mathcal{V}$-category $\mathcal{I}$; this much requires no symmetry. We here denote a $\mathcal{V}$-functor by a double arrow, as in $T: \mathcal{A} \Rightarrow \mathcal{B}$, keeping single arrows for $\mathcal{V}$-modules, which occur below more than $\mathcal{V}$-functors. In the absence of symmetry we have neither the opposite $\mathcal{V}$-category $\mathcal{A}^{op}$ nor the tensor-product $\mathcal{V}$-category $\mathcal{A} \otimes \mathcal{B}$ of [5] Section 1.4, so that we have neither contravariant $\mathcal{V}$-functors in general, nor $\mathcal{V}$-functors of two variables.

We do, of course, have as in [5] Section 1.5 various isomorphisms concerning $\mathcal{V}$ itself, obtained by specializing (1). Thus, writing V for $\mathcal{V}_0(I, -) : \mathcal{V}_0 \to Set$, we have

$$V[X,Y] \approx V\{X,Y\} \approx \mathcal{V}_0(X,Y).$$

We also have $[I,Z] \approx \{I,Z\} \approx Z$; and besides the isomorphism

$$[X \otimes Y, Z] \approx [X,[Y,Z]]$$

and its dual $\{X \otimes Y, Z\} \approx \{Y, \{X,Z\}\}$, we also have

$$\{X,[Y,Z]\} \approx [Y,\{X,Z\}].$$

2.2. Symmetry was in fact not needed for the observation in [5] Section 1.6 that there is a $\mathcal{V}$-category called $\mathcal{V}$, with the same objects as $\mathcal{V}_0$ and with $\mathcal{V}(X,Y) = [X,Y]$, whose underlying ordinary category is (canonically isomorphic to) $\mathcal{V}_0$; and that each object A of a $\mathcal{V}$-category $\mathcal{A}$ determines a representable $\mathcal{V}$-functor $\mathcal{A}(A,-) : \mathcal{A} \Rightarrow \mathcal{V}$ sending B to $\mathcal{A}(A,B)$.

Now, however, there is also a $\mathcal{V}$-category called $\bar{\mathcal{V}}$, with the same objects as $\mathcal{V}_0$ and with $\bar{\mathcal{V}}(X,Y) = \{Y,X\}$, whose underlying ordinary category is (canonically isomorphic to) $\mathcal{V}_0{}^{op}$. Although we do not have contravariant $\mathcal{V}$-functors in general, we may think of a $\mathcal{V}$-functor $\mathcal{A} \Rightarrow \bar{\mathcal{V}}$ as a contravariant functor from $\mathcal{A}$ to $\mathcal{V}$; and each $B \in \mathcal{A}$ determines a representable $\mathcal{A}(-,B): \mathcal{A} \to \bar{\mathcal{V}}$ sending A to $\mathcal{A}(A,B)$.

The ordinary functors

$$\mathcal{A}(A,-)_0 : \mathcal{A}_0 \to \mathcal{V}_0 \quad \text{and} \quad \mathcal{A}(-,B)_0 : \mathcal{A}_0 \to \mathcal{V}_0{}^{op}$$

are easily verified to be the partial functors of a bifunctor

$$hom_{\mathcal{A}} : \mathcal{A}_0{}^{op} \times \mathcal{A}_0 \to \mathcal{V}_0$$

sending (A,B) to $\mathcal{A}(A,B)$, whose value on morphisms we write as $\mathcal{A}(f,g)$. We still find the expressions (1.31) and (1.32) of [5] for $\mathcal{A}(A,g)$ and $\mathcal{A}(f,B)$, and hence the alternative expression [5] (1.39) of $\mathcal{V}$-naturality. Then, after the easy verification that $\mathcal{V}(f,g) = [f,g]$, we get essentially as in [5] Section 1.9 the Yoneda lemma, giving for $F: \mathcal{A} \Rightarrow \mathcal{V}$ a natural bijection between $\mathcal{V}$-natural transformations $\mathcal{A}(A,-) \to F$ and maps $I \to FA$. There is also a dual Yoneda lemma for transformations $G \to \mathcal{A}(A,B)$ where $G: \mathcal{A} \Rightarrow \bar{\mathcal{V}}$.

Although it no longer makes sense to speak, as in [5] Section 1.6, of a $\mathcal{V}$-functor $Ten: \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$, it is still the case that $- \otimes Z: \mathcal{V}_0 \to \mathcal{V}_0$ has an evident enrichment to a $\mathcal{V}$-functor $- \otimes Z: \mathcal{V} \Rightarrow \mathcal{V}$; while $Z \otimes -$ has a similar enrichment to a $\mathcal{V}$-functor $Z \otimes - : \bar{\mathcal{V}} \Rightarrow \bar{\mathcal{V}}$.

2.3. Given $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, by a *module* (or $\mathcal{V}$-*module*) $\phi: \mathcal{A} \to \mathcal{B}$ we mean what has been variously called a *profunctor*, a *distributor*, or a *bimodule*. It is a function assigning to each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$ an object $\phi(A,B)$ of $\mathcal{V}$, together with *actions*

$$\mathcal{B}(B,B') \otimes \phi(A,B) \to \phi(A,B'), \quad \phi(A,B) \otimes \mathcal{A}(A',A) \to \phi(A',B),$$

subject to five axioms: the evident associativity and unit axioms for the left action of $\mathcal{B}$ on ϕ, which equivalently assert that each $A \in \mathcal{A}$ gives a $\mathcal{V}$-functor $\phi(A,-): \mathcal{B} \Rightarrow \mathcal{V}$; the corresponding associativity and unit axioms for the right action of $\mathcal{A}$ on ϕ, which equivalently assert that each $B \in \mathcal{B}$ gives a $\mathcal{V}$-functor $\phi(-,B): \mathcal{A} \Rightarrow \bar{\mathcal{V}}$; and the axiom asserting the equality of the two maps

$$\mathcal{B}(B,B') \otimes \phi(A,B) \otimes \mathcal{A}(A',A) \to \phi(A',B').$$

For a symmetric $\mathcal{V}$, it is immediate from [5] Section 1.4 that such a module is the same thing as a $\mathcal{V}$-functor $\mathfrak{A}^{op} \otimes \mathfrak{B} \Rightarrow \mathcal{V}$. See in particular [1].

A morphism $\alpha : \phi \to \psi : \mathfrak{A} \to \mathfrak{B}$ of modules is simply a family

$$\alpha_{AB} : \phi(A, B) \to \psi(A, B)$$

of maps which commute with the two actions. It comes to the same thing to say that each $\alpha_{A\,-} : \phi(A, -) \to \psi(A, -)$ and each $\alpha_{-B} : \phi(-, B) \to \psi(-, B)$ is a $\mathcal{V}$-natural transformation. Thus we have a category $MOD(\mathfrak{A}, \mathfrak{B})$ of modules from $\mathfrak{A}$ to $\mathfrak{B}$.

If $\mathfrak{I}$ is the unit $\mathcal{V}$-category with one object 0, a module $\phi : \mathfrak{I} \to \mathfrak{B}$ is essentially a $\mathcal{V}$-functor $\phi : \mathfrak{B} \Rightarrow \mathcal{V}$, and we may write ϕB for $\phi(0, B)$; and moreover $MOD(\mathfrak{I}, \mathfrak{B})$ is the category $\mathcal{V}\text{-}CAT(\mathfrak{B}, \mathcal{V})$. Similarly a module $\phi : \mathfrak{A} \to \mathfrak{I}$ is essentially a $\mathcal{V}$-functor $\phi : \mathfrak{A} \Rightarrow \overline{\mathcal{V}}$; while a module $\phi : \mathfrak{I} \to \mathfrak{I}$ is just an object ϕ of $\mathcal{V}$, and $MOD(\mathfrak{I}, \mathfrak{I})$ is $\mathcal{V}_0$.

2.4. Consider $\mathcal{V}$-functors $\phi, \theta : \mathfrak{A} \Rightarrow \mathcal{V}$ and $\psi, \chi : \mathfrak{A} \Rightarrow \overline{\mathcal{V}}$ (or the corresponding modules). We need below a notion of $\mathcal{V}$-naturality for families

(2) $\alpha_A : \psi A \otimes \phi A \to Z,\quad \beta_A : X \to [\psi A, \chi A],\quad \gamma_A : Y \to \{\phi A, \theta A\}.$

Under the isomorphisms (1), there is a bijection between such families α_A, families $\alpha'_A : \psi A \to [\phi A, Z]$, and families $\alpha''_A : \phi A \to \{\psi A, Z\}$. Since ψ and $[\phi -, Z]$ are both $\mathcal{V}$-functors $\mathfrak{A} \Rightarrow \overline{\mathcal{V}}$, we know what it means for α' to be $\mathcal{V}$-natural; and since ϕ and $\{\psi -, Z\}$ are both $\mathcal{V}$-functors $\mathfrak{A} \Rightarrow \mathcal{V}$, we know what it means for α'' to be $\mathcal{V}$-natural. An easy calculation shows that the $\mathcal{V}$-naturality of α' is equivalent to that of α'', and equivalent to the commutativity for all $A, B \in \mathfrak{A}$ of the diagram

$$(3)\qquad \begin{array}{ccccc} & & \psi A \otimes \phi A & & \\ & \nearrow & & \searrow{\scriptstyle \alpha_A} & \\ \psi B \otimes \mathfrak{A}(A, B) \otimes \phi A & & & & Z\,; \\ & \searrow & & \nearrow{\scriptstyle \alpha_B} & \\ & & \psi B \otimes \phi B & & \end{array}$$

which now becomes our *definition* of $\mathcal{V}$-naturality for α. Similarly the $\mathcal{V}$-naturality of β is defined as that of the corresponding family $X \otimes \psi A \to \chi A$, or equivalently as that of the corresponding family $\psi A \to \{X, \chi A\}$, which turns out to mean the commutativity for all A, B of the diagram

$$(4) \qquad \begin{array}{ccc} X & \xrightarrow{\beta_B} & [\psi B, \chi B] \\ & & \downarrow {\scriptstyle - \otimes \mathcal{A}(A,B)} \\ \beta_A \downarrow & & [\psi B \otimes \mathcal{A}(A,B), \chi B \otimes \mathcal{A}(A,B)] \\ & & \downarrow \\ [\psi A, \chi A] & \longrightarrow & [\psi B \otimes \mathcal{A}(A,B), \chi A] . \end{array}$$

Finally, $\mathcal{V}$-naturality of γ is defined as that of $\phi A \otimes Y \to \theta A$ or equally that of $\phi A \to [Y, \theta A]$, which gives a diagram analogous to (4).

To say that α is the *universal* $\mathcal{V}$-natural family with domain $\psi A \otimes \phi A$ is to say that (3), taken for all $A, B \in \mathcal{A}$, is a colimit diagram. We then call Z the *coend* $\int^A \psi A \otimes \phi A$ of $\psi A \otimes \phi A$. If the coend exists but Z is arbitrary, to give a $\mathcal{V}$-natural α_A as in (2) is exactly to give a map $\alpha : \int^A \psi A \otimes \phi A \to Z$. We henceforth suppose $\mathcal{V}_0$ to be cocomplete; so that the coend certainly exists if $\mathcal{A}$ is small.

Similarly β is universal exactly when (4) is a limit diagram, whereupon we call X the *end* $\int_A [\psi A, \chi A]$. If the end exists but X is arbitrary, to give a $\mathcal{V}$-natural β_A as in (2) is to give a map $\beta : X \to \int_A [\psi A, \chi A]$. We henceforth suppose $\mathcal{V}_0$ to be complete; so that the end certainly exists if $\mathcal{A}$ is small.

Again, in the case of γ, we define the end $\int_A \{\phi A, \theta A\}$ as the limit of a diagram analogous to (4), which certainly exists if $\mathcal{A}$ is small.

Whether $\mathcal{A}$ is small or not, we have the results

$$(5) \qquad \int^A \mathcal{A}(A,B) \otimes \phi A \approx \phi B,$$

$$(6) \qquad \int^A \psi A \otimes \mathcal{A}(B,A) \approx \psi B,$$

$$(7) \qquad \int_A [\mathcal{A}(A,B), \chi A] \approx \chi B,$$

$$(8) \qquad \int_A \{\mathcal{A}(B,A), \theta A\} \approx \theta B.$$

Each of these follows directly from the Yoneda lemma; for instance, in the case of (5), a $\mathcal{V}$-natural $\mathfrak{A}(A,B)\otimes\phi A\to Z$ corresponds to a $\mathcal{V}$-natural $\mathfrak{A}(A,B)\to[\phi A,Z]$ and hence to a map $\phi B\to Z$. So we may call (5)-(8) *Yoneda isomorphisms.*

2.5. Given modules $\phi:\mathfrak{A}\to\mathfrak{B}$ and $\psi:\mathfrak{B}\to\mathcal{C}$, we define their *composite* $\psi\phi:\mathfrak{A}\to\mathcal{C}$ by

$$(9)\qquad (\psi\phi)(A,C)=\int^B\psi(B,C)\otimes\phi(A,B),$$

if the coend exists; otherwise the composite is undefined. The module-structure of $\psi\phi$ is inherited from those of ϕ and ψ using the universal property of the colimit and the preservation of colimits by $X\otimes-$ and $-\otimes Y$. In so far as we suppose $\mathcal{V}_0$ to have *chosen* colimits, the composite is well defined; but equally it does no harm, where convenient, to replace the right side of (9) by an isomorph.

The preservation of colimits by $X\otimes-$ and $-\otimes Y$, and the commutativity of colimits with colimits, easily give associativity isomorphisms $(\chi\psi)\phi\approx\chi(\psi\phi)$ whenever the inner composites and one of the outer ones exist. For any $\mathfrak{A}$ we have the module

$$1_{\mathfrak{A}}:\mathfrak{A}\to\mathfrak{A}\quad\text{given by}\quad 1_{\mathfrak{A}}(A,A')=\mathfrak{A}(A,A');$$

and if $\phi:\mathfrak{A}\to\mathfrak{B}$ is any module we have by (5) and (6) unit isomorphisms

$$(10)\qquad 1_{\mathfrak{B}}\phi\approx\phi\approx\phi 1_{\mathfrak{A}}.$$

The universal property of $\psi\phi$ tells us how to define

$$\sigma\rho:\psi\phi\to\psi'\phi'\ \text{ for }\ \rho:\phi\to\phi'\ \text{ and }\ \sigma:\psi\to\psi';$$

and easy verifications confirm that modules and their morphisms (which we may now call *2-cells*) «constitute a bicategory *MOD* in so far as composition is defined». We of course get a true bicategory *Mod* if we consider only modules between *small* $\mathcal{V}$-categories.

2.6. Consider modules $\phi:\mathfrak{A}\to\mathfrak{B}$, $\psi:\mathfrak{B}\to\mathcal{C}$, and $\theta:\mathfrak{A}\to\mathcal{C}$. We define a module $[\phi,\theta]:\mathfrak{B}\to\mathcal{C}$ by

$$(11)\qquad [\phi,\theta](B,C)=\int_A[\phi(A,B),\theta(A,C)]$$

whenever this end exists - its module structure being inherited from those of ϕ and θ. Similarly we define $\{\psi, \theta\} : \mathcal{A} \to \mathcal{B}$ by

$$\{\psi, \theta\}(A, B) = \int_C \{\psi(B, C), \theta(A, C)\}. \tag{12}$$

If $\psi\phi$ and $[\phi, \theta]$ both exist, a 2-cell $\alpha : \psi\phi \to \theta$ corresponds to a family

$$\alpha_{ABC} : \psi(B, C) \otimes \phi(A, B) \to \theta(A, C)$$

that is $\mathcal{V}$-natural in all variables, and hence by 2.4 to a family

$$\alpha'_{ABC} : \psi(B, C) \to [\phi(A, B), \theta(A, C)]$$

that is $\mathcal{V}$-natural in all variables, and so finally to a 2-cell $\alpha' : \psi \to [\phi, \theta]$. Similarly α corresponds to a 2-cell $\alpha'' : \phi \to \{\psi, \theta\}$ if the latter exists.

It follows that *Mod* is a *biclosed bicategory*, while *MOD* is «partially» one. We have an *evaluation* $[\phi, \theta]\phi \to \theta$ only when the domain here exists; and similarly for the evaluation $\psi\{\psi, \theta\} \to \theta$. Isomorphisms of the kinds

$$[\psi\phi, \theta] \approx [\psi, [\phi, \theta]], \quad \{\psi\phi, \theta\} \approx \{\phi, \{\psi, \theta\}\},$$
$$\text{and } \{\psi, [\phi, \theta]\} \approx [\phi, \{\psi, \theta\}],$$

which are automatic in a true biclosed bicategory such as *Mod*, need to be explicitly proved in *MOD*, under the hypotheses that the inner limits or colimits and one of the outer ones exist; we shall not explicitly use them. There is, however, no problem with the isomorphisms

$$[1_{\mathcal{A}}, \theta] \approx \theta \approx \{1_{\mathcal{C}}, \theta\}, \tag{13}$$

which follow from (7) and (8).

2.7. When the $\mathcal{V}$-category $\mathcal{A}$ is small, we can represent modules $\theta : \mathcal{A} \to \mathcal{C}$ as $\mathcal{V}$-functors $G : \mathcal{C} \Rightarrow P\mathcal{A}$, there being an isomorphism of categories

$$MOD(\mathcal{A}, \mathcal{C}) \approx \mathcal{V}\text{-}CAT(\mathcal{C}, P\mathcal{A}) \tag{14}$$

for a suitable $\mathcal{V}$-category $P\mathcal{A}$. The case $\mathcal{C} = \mathcal{I}$ shows that an object ξ of $P\mathcal{A}$ must be a module $\xi : \mathcal{A} \to \mathcal{I}$ or equally a $\mathcal{V}$-functor $\xi : \mathcal{A} \Rightarrow \bar{\mathcal{V}}$. We set

$$(P\mathcal{A})(\xi, \zeta) = [\xi, \zeta] = \int_A [\xi A, \zeta A], \tag{15}$$

which as a module $\mathcal{I} \to \mathcal{I}$ is an object of $\mathcal{V}$. The composition in $P\mathcal{A}$ is the obvious one deriving from the biclosed structure of *Mod* via the evaluation $[\xi, \zeta]\xi \to \zeta$. A morphism $\xi \to \zeta$ in the underlying ordinary category $(P\mathcal{A})_0$ of $P\mathcal{A}$, being a map $I \to (P\mathcal{A})(\xi, \zeta) = [\xi, \zeta]$, is equivalently a map $\xi \approx I\xi \to \zeta$, so that $(P\mathcal{A})_0 \approx Mod(\mathcal{A}, \mathcal{I})$, which is the special case $\mathcal{C} = \mathcal{I}$ of (14).

For the general case of (14), we leave the reader to verify that if we set

(16) $$(GC)A = \theta(A, C),$$

then the data and axioms making G into a $\mathcal{V}$-functor $\mathcal{C} \Rightarrow P\mathcal{A}$ correspond exactly to those making θ into a module $\mathcal{A} \to \mathcal{C}$; and that $\rho_C : GC \to G'C$ are the components of a $\mathcal{V}$-natural $\rho : G \to G'$ if and only if *their* components $\rho_{CA} : (GC)A \to (G'C)A$ constitute a 2-cell $\rho : \theta \to \theta'$.

In particular the identity module $1_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ corresponds to a $\mathcal{V}$-functor $Y : \mathcal{A} \to P\mathcal{A}$ (the *Yoneda embedding*) sending A to $YA = \mathcal{A}(-, A)$. From (15) and (7) we have a natural isomorphism

(17) $$(P\mathcal{A})(YA, \zeta) \approx \zeta A ;$$

from which it follows in particular that

(18) *the $\mathcal{V}$-functor $Y : \mathcal{A} \Rightarrow P\mathcal{A}$ is fully faithful.*

In the case of a symmetric $\mathcal{V}$, it is immediate that

$$P\mathcal{A} \approx [\mathcal{A}^{op}, \mathcal{V}] ;$$

and then (14) is just the remark of 2.3 that a module $\mathcal{A} \to \mathcal{C}$ is a $\mathcal{V}$-functor $\mathcal{A}^{op} \otimes \mathcal{C} \Rightarrow \mathcal{V}$ or $\mathcal{C} \Rightarrow [\mathcal{A}^{op}, \mathcal{V}]$.

Dually to the above, we have for small $\mathcal{C}$ a $\mathcal{V}$-category $P'\mathcal{C}$ whose objects are modules $\xi : \mathcal{I} \to \mathcal{V}$ or equivalently $\mathcal{V}$-functors

$$\xi : \mathcal{C} \Rightarrow \mathcal{V}, \quad \text{with} \quad (P'\mathcal{C})(\xi, \zeta) = \{\zeta, \xi\}.$$

In place of (14) we have

$$MOD(\mathcal{A}, \mathcal{C}) \approx (\mathcal{V}\text{-}CAT(\mathcal{A}, P'\mathcal{C}))^{op} ;$$

and, in the case of a symmetric $\mathcal{V}$ we have $P'\mathcal{C} \approx [\mathcal{C}, \mathcal{V}]^{op}$.

2.8. Any $\mathcal{V}$-functor $T: \mathfrak{A} \Rightarrow \mathfrak{B}$ gives rise to evident $\mathcal{V}$-modules $T_*: \mathfrak{A} \to \mathfrak{B}$ and $T^*: \mathfrak{B} \to \mathfrak{A}$, where

$$(19) \qquad T_*(A,B) = \mathfrak{B}(TA,B), \quad T^*(B,A) = \mathfrak{B}(B,TA);$$

and a $\mathcal{V}$-natural $\alpha: T \to S$ gives rise to evident 2-cells

$$\alpha_*: S_* \to T_* \quad \text{and} \quad \alpha^*: T^* \to S^*.$$

Since $\alpha \mapsto \alpha_*$ and $\alpha \mapsto \alpha^*$ are both fully faithful by the Yoneda lemma, any one of α, α_*, α^* determines the others.

For $T: \mathfrak{A} \Rightarrow \mathfrak{B}$ and $\psi: \mathfrak{B} \to \mathcal{C}$ we get from (6) and (7) natural isomorphisms

$$(20) \qquad (\psi T_*)(A,C) \approx \psi(TA,C) \approx [T^*,\psi](A,C);$$

and dually, for $\phi: \mathfrak{A} \to \mathfrak{B}$ and $S: \mathcal{C} \Rightarrow \mathfrak{B}$,

$$(21) \qquad (S^*\phi)(A,C) \approx \phi(A,SC) \approx \{S_*,\phi\}(A,C).$$

In particular, for $T: \mathfrak{A} \Rightarrow \mathfrak{B}$ and $Q: \mathfrak{B} \Rightarrow \mathcal{C}$., we have natural isomorphisms

$$(QT)_* \approx Q_* T_* \quad \text{and} \quad (QT)^* \approx T^* Q^*.$$

Thus we have locally-fully-faithful embeddings

$$(\ )_*: (\mathcal{V}\text{-}CAT)^{co} \to MOD \quad \text{and} \quad (\ )^*: (\mathcal{V}\text{-}CAT)^{op} \to MOD$$

of «partial bicategories».

The isomorphisms

$$(22) \qquad \psi T_* \approx [T^*,\psi], \quad \text{resp.} \quad S^*\phi \approx \{S_*,\phi\},$$

resulting from (20) and (21) admit of further analysis when $\mathfrak{A}$ (resp. $\mathcal{C}$) is small. We consider the first of these, and observe that, when $\mathfrak{A}$ is small, θT^* exists for any $\theta: \mathfrak{A} \to \mathcal{C}$. In particular $T_* T^*$ exists, and we have a canonical $\epsilon: T_* T^* \to 1_{\mathfrak{B}}$ induced by the composition

$$\mathfrak{B}(TA,B') \otimes \mathfrak{B}(B,TA) \to \mathfrak{B}(B,B').$$

On the other hand, T^*T_* always exists by (20), with

$$(T^*T_*)(A,A') = \mathfrak{B}(TA,TA');$$

and we have a canonical $\eta: 1_{\mathfrak{A}} \to T^*T_*$ with components

$$T_{AA'} : \mathfrak{A}(A, A') \to \mathfrak{B}(TA, TA').$$

An easy Yoneda-lemma calculation shows that η, ϵ are the unit and counit of an adjunction $T_* \dashv T^*$ in MOD. It is further easy to check that the evaluation $[T^*, \psi]\, T^* \to \psi$ corresponds under (22) to $\psi \epsilon : \psi T_* T^* \to \psi$. Now the conclusion from (22), that there is a natural bijection between 2-cells $\theta \to \psi T_*$ and 2-cells $\theta T^* \to \psi$, is just a classical consequence (see [6]) of this adjunction. The other classical consequence - the bijection between 2-cells $T_* \chi \to \psi$ and 2-cells $\chi \to T^* \psi$ - corresponds to the second isomorphism in (22).

Note that *the unit* $\eta : 1_{\mathfrak{A}} \to T^* T_*$ *of the adjunction above is an isomorphism exactly when* T *is fully faithful.* In this case we have a somewhat less evident result:

PROPOSITION 1. *Let* $\mathfrak{A}$ *be small, let* $T : \mathfrak{A} \Rightarrow \mathfrak{B}$ *be fully faithful, and let* $\theta : \mathfrak{A} \to \mathfrak{C}$. *Then* $[T_*, \theta]$ *exists, and the evaluation* $[T_*, \theta]\, T_* \to \theta$ *is an isomorphism.*

PROOF. Since $\mathfrak{A}$ is small, $[T_*, \theta]$ exists by (11) and is given by

$$[T_*, \theta](B, C) = \int_A [\mathfrak{B}(TA, B), \theta(A, C)].$$

By (20) we have

$$([T_*, \theta] T_*)(A', C) = \int_A [\mathfrak{B}(TA, TA'), \theta(A, C)].$$

Because T is fully faithful, the right side here is isomorphic to

$$\int_A [\mathfrak{A}(A, A'), \theta(A, C)],$$

which is itself isomorphic by (7) to $\theta(A', C)$. It is an easy verification that this isomorphism is in fact the evaluation. □

REMARK 2. If we were prepared to use the isomorphism

$$[T^*, [T_*, \theta]] \approx [T^* T_*, \theta]$$

of 2.6, which we have preferred not to discuss, we could have expressed the evaluation here directly as the composite isomorphism

$$[T_*, \theta] T_* \approx [T^*, [T_*, \theta]] \approx [T^* T_*, \theta] \xrightarrow[{[\eta, 1]}]{} [1_{\mathfrak{A}}, \theta] \approx \theta .$$

REMARK 3. When $\mathcal{C}$ too is small, and $\theta : \mathcal{A} \to \mathcal{C}$ corresponds as in 2.7 to $H : \mathcal{A} \Rightarrow P'\mathcal{C}$, Proposition 1 reduces to the classical result (see [5] Proposition 4.23) that $(Lan_T H)T = H$ for a fully faithful T ; at least for symmetric $\mathcal{V}$, but in fact for any $\mathcal{V}$ if the (pointwise) left Kan extension is appropriately defined. Similarly for the dual result $S^*\{S^*, \theta\} \approx \theta$ when S is fully faithful, interpreting θ now as a $\mathcal{V}$-functor $\mathcal{C} \Rightarrow P\mathcal{A}$.

REMARK 4. In so far as composition of modules is defined only to within isomorphism, it does no harm to express Proposition 1 as $[T_*, \theta] T_* = \theta$, with the identity as the evaluation.

PROPOSITION 5. *Let $\mathcal{A}$ be small, let the module $\theta : \mathcal{A} \Rightarrow \mathcal{C}$ correspond as in 2.7 to the $\mathcal{V}$-functor $G : \mathcal{C} \Rightarrow P\mathcal{A}$, and let $Y : \mathcal{A} \Rightarrow P\mathcal{A}$ be the Yoneda embedding. Then we have a natural isomorphism*

$$[Y_*, \theta] \approx G^* ; \tag{23}$$

and taking this, as we may, to be an equality, we have the decomposition

$$\theta = G^* Y_* . \tag{24}$$

PROOF. Since Y is fully faithful by (18), we have by Proposition 1 and Remark 4 only to verify (23). By (19) and (17) we have

$$Y_*(A, \zeta) = (P\mathcal{A})(YA, \zeta) \approx \zeta A ;$$

so that (11) gives

$$[Y_*, \theta](\zeta, C) \approx \int_A [\zeta A, \theta(A, C)].$$

By (16) this is $\int_A [\zeta A, (GC)A]$, which by (15) is $(P\mathcal{A})(\zeta, GC)$; that is, $G^*(\zeta, C)$, as desired. □

REMARK 6. We regard (24) as the *canonical decomposition* of any $\theta : \mathcal{A} \to \mathcal{C}$ with $\mathcal{A}$ small. There is a dual decomposition when $\mathcal{C}$ is small, and θ corresponds to $H : \mathcal{A} \Rightarrow P'\mathcal{C}$; namely $\theta = Y'^* H_*$, where $Y' : \mathcal{C} \Rightarrow P'\mathcal{C}$ is the dual Yoneda embedding ; and in place of (23) we now have

$$\{Y'^*, \theta\} \approx H_* .$$

REMARK 7. We can generalize Proposition 5 as follows. Let $Q\mathcal{A}$ be a full subcategory of $P\mathcal{A}$ containing the representables, and let $Z : \mathcal{A} \Rightarrow Q\mathcal{A}$ be the Yoneda embedding seen as landing in $Q\mathcal{A}$. Let $\theta : \mathcal{A} \to \mathcal{C}$ *be such that the corresponding* $G : \mathcal{A} \Rightarrow P\mathcal{C}$ *lands in* $Q\mathcal{C}$, giving $K : \mathcal{A} \Rightarrow Q\mathcal{C}$. Then a trivial modification of the proof of Proposition 5 gives $[Z_*, \theta] = K^*$, so that $\theta = K^* Z_*$.

3. COFIBRATIONS.

3.1. For any objects $\mathcal{A}$, $\mathcal{C}$ of any bicategory K with finite bilimits, Street [9] defines the bicategory $Fib(\mathcal{A}, \mathcal{C})$ of *fibrations* $\mathcal{A} \to \mathcal{C}$ in K as that of the algebras for a certain bimonad on the bicategory of spans from $\mathcal{A}$ to $\mathcal{C}$. One can define «discrete object» in any bicategory; and the full sub-bicategory of $Fib(\mathcal{A}, \mathcal{C})$ given by its discrete objects forms the category $DFib(\mathcal{A}, \mathcal{C})$ *of discrete fibrations* $\mathcal{A} \to \mathcal{C}$.

When K $= Cat$ there is a biequivalence between $Fib(\mathcal{A}, \mathcal{C})$ and the bicategory of bifunctors* $\mathcal{A}^{op} \times \mathcal{C} \Rightarrow Cat$, and an equivalence between $DFib(\mathcal{A}, \mathcal{C})$ and the category of functors $\mathcal{A}^{op} \times \mathcal{C} \Rightarrow Set$ (or modules $\mathcal{A} \to \mathcal{C}$) ; so that in this case fibrations are, in effect, two-sided versions of those first introduced by Grothendieck. However the equivalence between $DFib(\mathcal{A}, \mathcal{C})$ and $Mod(\mathcal{A}, \mathcal{C})$ fails totally for K $= \mathcal{V}$-Cat, even for good $\mathcal{V}$.

Nevertheless, as Street observes, we can still recover $\mathcal{V}$-Mod directly from the 2-category $\mathcal{V}$-Cat, by looking instead at fibrations in $(\mathcal{V}\text{-}Cat)^{op}$, which he calls *cofibrations in* $\mathcal{V}$-Cat. He analyzes the nature of these in elementary terms, and shows that the *codiscrete cofibrations* $\mathcal{A} \to \mathcal{C}$ are (to within equivalence) the modules $\mathcal{A} \to \mathcal{C}$.

3.2. Since we do not want to restrict ourselves to *small* $\mathcal{V}$-categories, we first give the results of Street's analysis in a form which makes sense without smallness. We also, by abuse of language, use the name «cofibrations» for the concrete objects to which the actual cofibrations are only biequivalent.

* We use «bifunctor» for what some call a «pseudofunctor» or a «homomorphism of bicategories».

Then a cofibration $\mathfrak{A} \to \mathcal{C}$ in $\mathcal{V}$-*Cat* may be identified with a cospan $\mathfrak{A} \Rightarrow \mathfrak{D} \Leftarrow \mathcal{C}$ of the following special kind: the $\mathcal{V}$-functors $\mathfrak{A} \Rightarrow \mathfrak{D}$ and $\mathcal{C} \Rightarrow \mathfrak{D}$ are fully faithful, and may as well be taken to be the inclusions of full subcategories; these subcategories $\mathfrak{A}$ and $\mathcal{C}$ are disjoint; and if $\mathfrak{B}$ is the full subcategory of $\mathfrak{D}$ given by the objects neither in $\mathfrak{A}$ nor in $\mathcal{C}$, we have

$$\mathfrak{D}(B, A) = \mathfrak{D}(C, B) = \mathfrak{D}(C, A) = 0$$

(the initial object of $\mathcal{V}$) whenever $A \in \mathfrak{A}$, $B \in \mathfrak{B}$, and $C \in \mathcal{C}$. A morphism of cofibrations from $\mathfrak{A} \Rightarrow \mathfrak{D} \Leftarrow \mathcal{C}$ to $\mathfrak{A} \Rightarrow \mathfrak{D}' \Leftarrow \mathcal{C}$ is a $\mathcal{V}$-functor $T : \mathfrak{D} \Rightarrow \mathfrak{D}'$ which is the identity on $\mathfrak{A}$ and on $\mathcal{C}$ and which maps $\mathfrak{B}$ into the corresponding $\mathfrak{B}'$. A 2-cell $\sigma : T \to \bar{T}$ is a $\mathcal{V}$-natural transformation whose components σ_D are identities when $D \in \mathfrak{A}$ or $D \in \mathcal{C}$. The bicategory $Cofib(\mathfrak{A}, \mathcal{C})$ so described is in fact a 2-category. The codiscrete cofibrations are those for which $\mathfrak{B}$ is the empty $\mathcal{V}$-category 0.

3.3. Street carries the analysis further in terms of modules. To give the cofibration $\mathfrak{A} \Rightarrow \mathfrak{D} \Leftarrow \mathcal{C}$ is to give the $\mathcal{V}$-category $\mathfrak{B}$, the objects $\mathfrak{D}(A, B)$, $\mathfrak{D}(B, C)$, and $\mathfrak{D}(A, C)$ of $\mathcal{V}$, and the law of composition in $\mathfrak{D}$. If we write $\phi(A, B)$, $\psi(B, C)$, and $\theta(A, C)$ for $\mathfrak{D}(A, B)$, $\mathfrak{D}(B, C)$, and $\mathfrak{D}(A, C)$, the composition

$$\mathfrak{D}(B, B') \otimes \mathfrak{D}(A, B) \to \mathfrak{D}(A, B')$$

is just a left action $\mathfrak{B}(B, B') \otimes \phi(A, B) \to \phi(A, B')$, and so on; in short, we are to give *modules* $\phi : \mathfrak{A} \to \mathfrak{B}$, $\psi : \mathfrak{B} \to \mathcal{C}$, and $\theta : \mathfrak{A} \to \mathcal{C}$. The composition is now determined except for its components

$$\mathfrak{D}(B, C) \otimes \mathfrak{D}(A, B) \to \mathfrak{D}(A, C) ;$$

and the giving of the cofibration is complete when we give this as a family

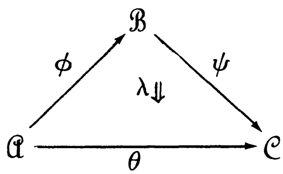
$$\lambda_{ABC} : \psi(B, C) \otimes \phi(A, B) \to \theta(A, C)$$

that is $\mathcal{V}$-natural in all variables. When the composite $\psi\phi$ exists this is just to give a 2-cell $\lambda : \psi\phi \to \theta$ of modules.

Since we need below the case where $\mathfrak{B}$ is as large as $P\mathfrak{A}$ for a small $\mathfrak{A}$, we do not automatically have $\psi\phi$ as a $\mathcal{V}$-module; and yet it would be intolerably tedious to work with λ_{ABC} instead of λ. Accordingly we suppose $\mathcal{V}$ embedded, as in [5] Section 3.12 (which does not use the symmetry of $\mathcal{V}$), into a $\mathcal{V}'$ corresponding to a higher universe *Set'* that sees *Set* and $\mathcal{V}$ as small, in such a way that the inclusion of $\mathcal{V}$ in $\mathcal{V}'$ preserves limits and *Set'*-small colimits. Then we have $\psi\phi$ as a $\mathcal{V}'$-module, which coincides with the module $\psi\phi$ when the latter exists; and similarly for such other module-composites as we need below.

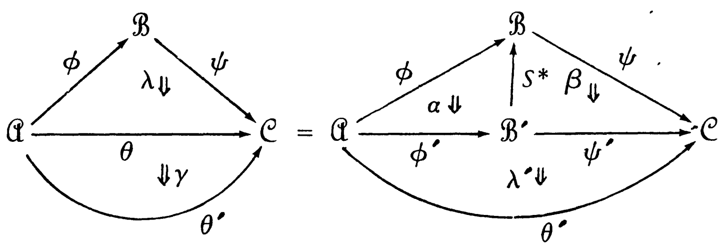
With this understanding, then, a cofibration $\mathfrak{A} \Rightarrow \mathfrak{D} \Leftarrow \mathfrak{C}$ is identified with a pentad $(\mathfrak{B};\phi,\psi,\theta;\lambda)$ where $\mathfrak{B}$ is a $\mathcal{V}$-category, ϕ,ψ,θ are modules, and λ is a 2-cell, as in

(25)



The corresponding analysis of a morphism $T: \mathfrak{D} \Rightarrow \mathfrak{D}'$ in $Cofib(\mathfrak{A},\mathfrak{C})$ identifies it with a tetrad $(S;\alpha,\beta,\gamma)$ where $S: \mathfrak{B} \Rightarrow \mathfrak{B}'$ is a $\mathcal{V}$-functor and where α, β, γ are 2-cells of modules satisfying the equation

(26)



Here S is the restriction of T to $\mathfrak{B}$, the component

$$\alpha_{AB}: \phi(A,B) \to \phi'(A,SB) \quad \text{of} \quad \alpha: \phi \to S^*\phi'$$

is $T_{AB}: \mathfrak{D}(A,B) \to \mathfrak{D}(TA,TB)$, and the component γ_{AC} of $\gamma: \theta \to \theta'$ is $T_{AC}: \mathfrak{D}(A,C) \to \mathfrak{D}(TA,TC)$. On the other hand T_{BC} is the component, not of $\beta: \psi S^* \to \psi'$, but of the $\dot{\beta}: \psi \to \psi' S_*$ which corresponds it under the adjunction $S_* \dashv S^*$ of 2.8. The equation (26) is what is needed to ensure that the T so defined is a $\mathcal{V}$-functor.

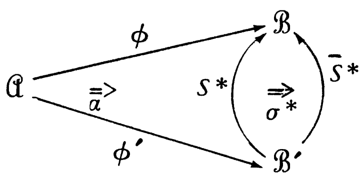
When

$$T, \bar{T}: (\mathcal{B}; \phi, \psi, \theta; \lambda) \to (\mathcal{B}'; \phi', \psi', \theta'; \lambda')$$

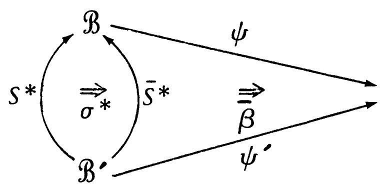
are given by $(S; \alpha, \beta, \gamma)$ and $(\bar{S}; \bar{\alpha}, \bar{\beta}, \bar{\gamma})$, it is easy to see that there are no 2-cells $T \to \bar{T}$ in $Cofib(\mathcal{A}, \mathcal{C})$ unless

(27) $$\gamma = \bar{\gamma} .$$

When this does hold, a 2-cell $\sigma: T \to \bar{T}$ is at once seen to be a $\mathcal{V}$-natural $\sigma: S \to \bar{S}$ satisfying the equations

(28) 

$= \bar{\alpha}$,

(29) 

$= \beta$.

The full sub-2-category of codiscrete cofibrations is obtained by setting $\mathcal{B} = 0$ throughout. An object is just a module $\theta : \mathcal{A} \to \mathcal{C}$, and a morphism just a 2-cell $\gamma: \theta \to \theta'$; there are no non-identity 2-cells here, so that we are dealing with a mere category. We need no new name for it; it is just $MOD(\mathcal{A}, \mathcal{C})$.

3.4. We define the *behaviour* 2-functor

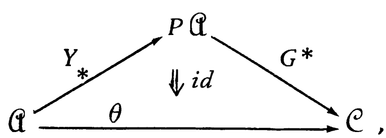
$$B: Cofib(\mathcal{A}, \mathcal{C}) \to MOD(\mathcal{A}, \mathcal{C})$$

to be that sending $(\mathcal{B}; \phi, \psi, \theta; \lambda)$ to $\theta: \mathcal{A} \to \mathcal{C}$, sending $(S; \alpha, \beta, \gamma)$ to $\gamma: \theta \to \theta'$, and sending a 2-cell σ to the identity.

For small $\mathcal{A}$, we define a functor

$$R: MOD(\mathcal{A}, \mathcal{C}) \to Cofib(\mathcal{A}, \mathcal{C})$$

that we may call *(canonical) realization.* It sends $\theta : \mathcal{A} \to \mathcal{C}$ to the cofibration $R\theta$ given by



,

using the canonical decomposition (24) of θ. As for its effect on a 2-cell $\gamma : \theta \to \theta'$, we can by 2.7 identify γ with a $\mathcal{O}$-natural $\gamma : G \to G'$, and hence by 2.8 with a 2-cell $\gamma^* : G^* \to G'^*$; and we set $R\gamma = (1_{P\mathfrak{A}}; id, \gamma^*, \gamma)$. This is indeed a morphism in $Cofib(\mathfrak{A}, \mathcal{C})$, for the condition (26) reduces here to $\gamma^* Y_* = \gamma$, which follows from the naturality of the evaluation in Proposition 1 and of the isomorphism (23). Clearly $BR = 1$; in this sense $R\theta$ is a «realization» of θ, since its «behaviour» is θ.

If X is an arbitrary cofibration $(\mathfrak{B}; \phi, \psi, \theta; \lambda)$ and $\theta' : \mathfrak{A} \to \mathcal{C}$ an arbitrary module, the category $Cofib(\mathfrak{A}, \mathcal{C})(X, R\theta')$ is neither isomorphic nor equivalent to the discrete category given by the set $MOD(\mathfrak{A}, \mathcal{C})(BX, \theta')$; so that R is neither a right adjoint nor even a right biadjoint to the 2-functor B. We have only the following still weaker result:

PROPOSITION 8. *The functor*

$$(30) \quad B_{X, R\theta'} : Cofib(\mathfrak{A}, \mathcal{C})(X, R\theta') \to Mod(\mathfrak{A}, \mathcal{C})(BX, BR\theta') = Mod(\mathfrak{A}, \mathcal{C})(BX, \theta')$$

has a left adjoint.

Since the codomain of (30) is only a set, this has a very simple meaning in elementary terms. The functor (30) sends a typical morphism $(S; \alpha, \beta, \gamma) : X \to R\theta'$ of cofibrations to $\gamma : \theta \to \theta'$, and a 2-cell between such morphisms to the identity. To say that it has the left adjoint L is to assert a natural bijection between 2-cells $\sigma : L\delta \to (S; \alpha, \beta, \gamma)$ and maps $\delta \to \gamma$ in the *discrete* category $MOD(\mathfrak{A}, \mathcal{C})(\theta, \theta')$. Thus there is to be exactly one 2-cell $\sigma : L\delta \to (S; \alpha, \beta, \gamma)$ if $\delta = \gamma$, and none otherwise. Since the equality (27) is a necessary condition for the existence of a 2-cell, Proposition 8 in fact asserts the following:

THEOREM 9. *Let $\mathfrak{A}$ be small, let $X = (\mathfrak{B}; \phi, \psi, \theta; \lambda)$ be a cofibration $\mathfrak{A} \to \mathcal{C}$ and let $\theta' : \mathfrak{A} \to \mathcal{C}$ be a module. Then for each $\gamma : \theta \to \theta'$, the full subcategory of $Cofib(\mathfrak{A}, \mathcal{C})(X, R\theta')$, given by those $(S; \alpha, \beta, \gamma) : X \to R\theta'$ with this particular γ, has an initial object.*

PROOF. To give a morphism $(S; \alpha, \beta, \gamma)$ from $X = (\mathfrak{B}; \phi, \psi, \theta; \lambda)$ to $R\theta' = (P\mathfrak{A}; Y_*, G'^*, \theta'; id)$, where $G' : \mathcal{C} \Rightarrow P\mathfrak{A}$ corresponds as in 2.7

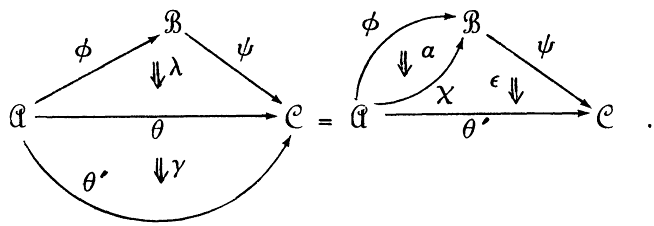
to θ': $\mathfrak{A} \to \mathfrak{C}$, is by 3.3 to give

$$S : \mathfrak{B} \Rightarrow P\mathfrak{A}, \quad \alpha : \phi \to S^* Y_*, \quad \beta : \psi S^* \to G'^* \quad \text{and } \gamma : \theta \to \theta',$$

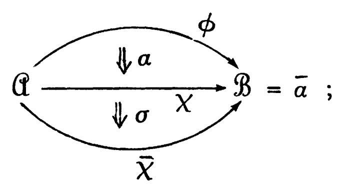
satisfying the appropriate instance of (26). To give $S : \mathfrak{B} \Rightarrow P\mathfrak{A}$ is equally to give the module $\chi : \mathfrak{A} \to \mathfrak{B}$ which corresponds to it as in 2.7. Then $S^* Y_* = \chi$ by (24), so that α is just a 2-cell $\phi \to \chi$. Since $G'^* = [Y_*, \theta']$ by (23), and since the evaluation $[Y_*, \theta'] Y_* \to \theta'$ is the identity by Remark 4, there is a bijection between 2-cells $\beta : \psi S^* \to G'^*$ and 2-cells $\epsilon : \psi S^* Y_* \to \theta'$, where $\epsilon = \beta Y_*$. Using $S^* Y_* = \chi$ once more, we conclude that to give a morphism $(S; \alpha, \beta, \gamma) : X \to R\theta'$ is to give

$$\chi : \mathfrak{A} \to \mathfrak{B}, \quad \alpha : \phi \to \chi, \quad \epsilon : \psi\chi \to \theta', \quad \text{and } \gamma : \theta \to \theta',$$

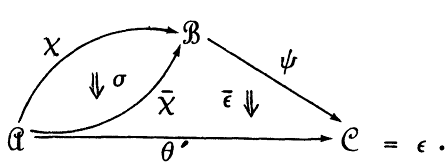
satisfying the instance of (26) which now becomes

(31) 

Consider now a 2-cell $\sigma : (S; \alpha, \beta, \gamma) \to (\bar{S}; \bar{\alpha}, \bar{\beta}, \gamma)$, where γ remains fixed. To give the $\mathcal{V}$-natural $\sigma : S \to \bar{S}$ is equally by 2.7 to give a 2-cell $\sigma : \chi \to \bar{\chi}$ of modules. The condition (28), since $\sigma^* Y_* = \sigma$ by the naturality in Proposition 1 and that of (23), becomes

(32) 

and the condition (29) becomes

(33) 

We get the initial object by setting $\chi = \phi$, $a = id$, and ϵ equal to the composite $\gamma\lambda$ on the left side of (31). For then, by (32), a 2-cell σ is forced to by $\bar{a}$ itself; and this satisfies (33) by the $(\bar{\chi};\bar{a},\bar{\epsilon},\gamma)$-case of (31), whose left side is still $\gamma\lambda = \epsilon$. □

REMARK 10. There is a dual result, when $\mathcal{C}$ is small, corresponding to the dual decomposition of θ in Remark 6.

REMARK 11. Clearly the results of Proposition 8 and Theorem 9 remain unchanged if we restrict ourselves to those cofibrations $(\mathcal{B};\phi,\psi,\theta;\lambda)$ in which λ is the identity.

4. NON-DETERMINISTIC AUTOMATA.

4.1. We recall the description of non-deterministic automata in terms of enriched categories, given by Betti [2].

The dynamics of a deterministic automaton with Q for its set of states and L for its alphabet of inputs is given by a function $L \times Q \to Q$, which at once extends to an *action* $M \times Q \to Q$ where M is the free monoid on the set L. Because there are also practical applications where the monoid M is not free, it is usual to generalize at once to the case of any monoid M. Thus a deterministic dynamics is a monoid M together with an M-set Q.

To obtain the notion of a non-deterministic dynamics, we merely replace the function $M \times Q \to Q$ by a relation $\rightsquigarrow$ between $M \times Q$ and Q, satisfying the following generalizations of the associativity and unit laws for an action:

if $(m,q) \rightsquigarrow r$ and $(n,r) \rightsquigarrow s$ then $(nm,q) \rightsquigarrow s$;

and $(e,q) \rightsquigarrow q$ for all q, where e is the identity of M.

A relation between $M \times Q$ and Q is equally a relation between $Q \times Q$ and M, which is in turn the same thing as a function $\mathcal{Q} : Q \times Q \to \mathcal{P}M$ into the set $\mathcal{P}M$ of subsets of M.

The ordered set $\mathcal{P}M$, seen as a category, is in fact a biclosed monoidal category $\mathcal{V}$. The tensor product $X \otimes Y$ is just the set

$$\{ xy \mid x \in X \text{ and } y \in Y \},$$

while the unit object I is $\{ e \}$. One internal-hom $[Y, Z]$ is

$$\{ x \mid xy \in Z \quad \text{for all } y \in Y \},$$

and the other one $\{ X, Z \}$ is

$$\{ y \mid xy \in Z \quad \text{for all } x \in X \}.$$

The transforms of the generalized associativity and unit laws are the inequalities

$$\mathcal{Q}(r, s) \otimes \mathcal{Q}(q, r) \leq \mathcal{Q}(q, s) \quad \text{and} \quad I \leq \mathcal{Q}(q, q),$$

which inform us precisely that we have a $\mathcal{V}$-category $\mathcal{Q}$ with Q for its set of objects. Thus a non-deterministic dynamics with monoid M is nothing but a $\mathcal{V}$-category $\mathcal{Q}$, where $\mathcal{V} = \mathcal{P}M$.

Accordingly, if we now fix once for all the monoid M, the possible dynamics form the objects of a 2-category, namely the 2-category $\mathcal{V}$-*Cat* of small $\mathcal{V}$-categories. Thus a morphism $S: \mathcal{Q} \Rightarrow \mathcal{Q}'$ of dynamics is just a $\mathcal{V}$-functor; and since $\mathcal{P}M$ is only an ordered set, this is just a function $S: Q \to Q'$ such that $\mathcal{Q}(q, r) \leq \mathcal{Q}'(Sq, Sr)$; or equivalently, such that

$$(m, q) \leadsto r \quad \text{implies} \quad (m, Sq) \leadsto Sr.$$

Again, there is at most one 2-cell $\sigma: S \to \bar{S}$, this existing when

$$e \in \mathcal{Q}'(Sq, \bar{S}q) \quad \text{for all } q.$$

4.2. A *non-deterministic automaton* is a (non-deterministic) dynamics $\mathcal{Q}$ together with a subset $J \subset Q$ of *initial states* and some notion of *output.* We restrict ourselves to the simplest case where the automaton is a mere *recognizer*, with possible outputs 0 and 1 ; so that to give the output function $Q \to \{0, 1\}$ is just to give a subset $T \subset Q$ of *terminal states.* Then the *behaviour* $b(\mathcal{Q}; J, T)$ of the automaton is the subset of M given by those m such that, for some $j \in J$ and some $t \in T$, we have $(m, j) \leadsto t$. In other words,

(34) $$b(\mathcal{Q}; J, T) = \Sigma_{j \in J,\, t \in T} \mathcal{Q}(j, t),$$

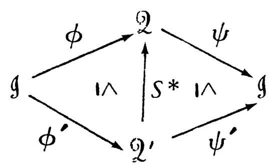
where the coproduct Σ in $\mathcal{V}$ is in fact the union in $\mathcal{P}M$.

Following Betti [2], we define a morphism $(\mathcal{Q}; J, T) \to (\mathcal{Q}'; J', T')$ of automata (still for the fixed M) to be a morphism $S: \mathcal{Q} \Rightarrow \mathcal{Q}'$ of dynamics such that the function $S: Q \to Q'$ satisfies

$$(35) \qquad S(J) \leq J' \quad \text{and} \quad S(T) \leq T' ;$$

and we define a 2-cell to be just any 2-cell $\sigma: S \to \bar{S}$ of dynamics. Thus automata are the objects of a 2-category *Aut*. Since (35) clearly implies that $b(\mathcal{Q}; J, T) \leq b(\mathcal{Q}'; J', T')$, behaviour becomes a 2-functor b from *Aut* to the underlying category $\mathcal{V}_0 = \mathcal{P}M$ of $\mathcal{V}$; of course b sends any 2-cell to an identity.

The condition (35) is not, on the face of it, a «categorical» one. Still following Betti [2] (although not in terminology), we define a 2-category *Gen* of *generalized automata* (for the given monoid M). A generalized automaton is a dynamics $\mathcal{Q}$ along with modules $\phi: \mathcal{I} \to \mathcal{Q}$ (the *input*) and $\psi: \mathcal{Q} \to \mathcal{I}$ (the *output*). A morphism $(\mathcal{Q}; \phi, \psi) \to (\mathcal{Q}'; \phi', \psi')$ is a morphism $S: \mathcal{Q} \Rightarrow \mathcal{Q}'$ of dynamics satisfying the inequalities (that is, 2-cells of modules)

(36)  ;

and a 2-cell is any $\sigma: S \to \bar{S}$. There is a *behaviour* 2-functor

$$B: Gen \to Mod(\mathcal{I}, \mathcal{I}) = \mathcal{V}_0 ,$$

sending $(\mathcal{Q}; \phi, \psi)$ to the composite $\psi\phi: \mathcal{I} \to \mathcal{I}$; for clearly (36) implies that $\psi\phi \leq \psi'\phi'$.

We have a 2-functor $F: Aut \to Gen$, sending $(\mathcal{Q}; J, T)$ to $(\mathcal{Q}; \phi, \psi)$ where

$$(37) \qquad \phi(q) = \Sigma_{j \in J} \mathcal{Q}(j, q), \quad \psi(q) = \Sigma_{t \in T} \mathcal{Q}(q, t),$$

and sending S, σ to themselves; for it is easy to see that (35) implies (36) when ϕ, ψ are defined by (37). It is further clear from (34) that F

preserves behaviour: we have $BF = b$. Betti seems to us to be mistaken, however, in asserting that F is fully faithful.

As he correctly observes, F has a right adjoint H, sending $(\mathcal{Q}; \phi, \psi)$ to $(\mathcal{Q}; J, T)$ where

(38) $$J = \{ q \mid e \in \phi q \}, \quad T = \{ q \mid e \in \psi q \},$$

and sending S, σ to themselves. Indeed, an easy calculation shows that, if

$$F(\mathcal{Q}; J, T) = (\mathcal{Q}; \phi, \psi) \quad \text{and} \quad H(\mathcal{Q}'; \phi', \psi') = (\mathcal{Q}'; J', T'),$$

the inequalities (36) become exactly $S(J) \leq J'$ and $S(T) \leq T'$. However the unit $1 \to HF$ of the adjunction is not an isomorphism; for

$$HF(\mathcal{Q}; J, T) = (\mathcal{Q}; J^{\dagger}, T^{\dagger})$$

where

(39) $$J^{\dagger} = \{ q \mid e \in \mathcal{Q}(j, q) \text{ for some } j \in J \},$$

(40) $$T^{\dagger} = \{ q \mid e \in \mathcal{Q}(q, t) \text{ for some } t \in T \};$$

and the inequalities $J \leq J^{\dagger}$ and $T \leq T^{\dagger}$ are in general strict.

What we do have at once from the triangular adjunction equations, since both F and H are the identity at the level of the dynamics, is that

(41) $$FHF = F \quad \text{and} \quad HFH = H.$$

Let us call the idempotent 2-functor $(\;)^{\dagger} = HF : Aut \to Aut$ the *normalization* functor, and call an automaton $A = (\mathcal{Q}; J, T)$ *normal* if $J^{\dagger} = J$ and $T^{\dagger} = T$; which is equivalently to say that A is in the image of H. Note that (41) gives

(42) $$FA^{\dagger} = FA,$$

whence, applying B and using $BF = b$, we have

(43) $$bA^{\dagger} = bA,$$

so that *normalization preserves behaviour*. Observe that, although F is not fully faithful, we do have:

PROPOSITION 12. *The functor* $F_{A\,A'} : Aut(A, A') \to Gen(FA, FA')$ *is an isomorphism if the automaton* A' *is normal.* □

4.3. We may identify a generalized automaton $(\mathcal{Q};\ \phi, \psi)$ with the cofibration $(\mathcal{Q};\phi,\psi,\theta;\lambda) : \mathcal{I} \to \mathcal{I}$, where $\theta = \psi\phi$ and $\lambda = id$. Then a morphism S of generalized automata coincides with a morphism $(S;\alpha,\beta,\gamma)$ of cofibrations; α and β are the inequalities in (36), and γ is the induced inequality $\psi\phi \leq \psi'\phi'$. A 2-cell $\sigma: S \to \bar{S}$ in *Gen* is also a 2-cell

$$\sigma: (S;\alpha,\beta,\gamma) \to (\bar{S};\bar{\alpha},\bar{\beta},\bar{\gamma})$$

in $Cofib(\mathcal{I},\mathcal{I})$; for the equations (27), (28), and (29) are automatic when $\mathcal{V}_0$ is merely an ordered set. Thus *Gen* is identified with the full subcategory of $Cofib(\mathcal{I},\mathcal{I})$ given by those cofibrations with $\lambda = id$. Moreover the behaviour 2-functor $B: Gen \to MOD(\mathcal{I},\mathcal{I}) = \mathcal{V}_0$ of 4.2 is just the restriction of the behaviour 2-functor $B: Cofib(\mathcal{I},\mathcal{I}) \to MOD(\mathcal{I},\mathcal{I})$ of 3.4.

The 2-functor $R: MOD(\mathcal{I},\mathcal{I}) \to Cofib(\mathcal{I},\mathcal{I})$ of 3.4 may be seen as a 2-functor $R: \mathcal{V}_0 \to Gen$, sending $\theta \in \mathcal{P}M$ to $R\theta = (P\mathcal{I};\ Y_*, G^*)$. Since $P\mathcal{I} = \mathcal{V}$, since $Y_*: \mathcal{I} \to \mathcal{V}$ is given by

$$Y_*X = [I, X] = X,$$

and since $G^*: \mathcal{V} \to \mathcal{I}$ is given by $G^*X = [X,\theta]$, we in fact have $R\theta = F(\mathcal{V};\{I\},\{\theta\})$. The automaton $(\mathcal{V};\{I\},\{\theta\})$, however, is not normal; its normalization is $N\theta = (\mathcal{V};\pi,\hat{\theta})$ where

$$(44) \qquad \pi = \{X \subset M \mid e \in X\}, \quad \hat{\theta} = \{X \subset M \mid X \subset \theta\},$$

and $R\theta$ is equally $FN\theta$ by (42).

Theorem 9 now gives, when we use Proposition 12:

THEOREM 13. *Let $A = (\mathcal{Q};J,T)$ be any automaton, let its behaviour be θ, and let $\theta' \geq \theta$ in $\mathcal{P}M$. Then the category of morphisms*

$$(\mathcal{Q};\ J, T) \to (\mathcal{V};\pi,\hat{\theta}'),$$

and 2-cells between them, has an initial object. Equivalently, the functor

$$b_{A,N\theta'}: Aut(A, N\theta') \to \mathcal{V}_0(bA, bN\theta') = \mathcal{V}_0(bA,\theta')$$

has a left adjoint. □

This is the essence of the theorem of Betti and Kasangian in [3]. It is true that they consider a richer notion of morphism for automata, which involves a *re-coding* given by a monoidal functor from $\mathcal{V}$ to itself; but their result in fact follows from Theorem 13 without any significant further calculation.

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KASANGIAN: Istituto Matematico « F. Enriques»,
Via Saldini 50, 20123 MILANO, ITALY

KELLY: Department of Pure Mathematics
University of Sydney, N.S.W. 2006. AUSTRALIA

ROSSI: Istituto di Matematico,
Universitá degli Studi di Trieste
P.le Europa 1, 34100 TRIESTE, ITALY