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QUOTIENTS, FRACTIONS, SYMMETRIZATIONS

by Ferdinando MORA and Fulvio MORA

«Langage mod. C» was introduced by Serre [12] to deal with homological computations, and formalized by Grothendieck [5] and Gabriel [4] within abelian categories.

Since homological computations can be improved when made not in an abelian category, but in a «distributive» extension of it, a so-called CΘ-category [1], which is not abelian, and in the inverse symmetrization of the latter [6, 7, 8], it is of interest to extend «langage mod. C» to CΘ-categories and to interpret it in their inverse symmetrization.

In [9] a definition of quotient categories is given in the context of «left- (and right-) exact categories», a suitable extension of exact categories (including Groups, Topological Abelian Groups, Pointed Sets).

In this paper it is proved the existence of quotient categories for left-exact CΘ-categories; a definition is given of quotient categories for inverse categories with zero object; and it is shown that quotientation and symmetrization are commuting operations.

More in detail, Section 1 gives a construction of fraction categories for involutive regular categories, Section 2 does the same for inverse categories.

Section 3 shows that if C is a CΘ-category, and S a CΘ-subcategory of C verifying some «closure» properties, then taking fractions mod. S and symmetrizing are commuting operations.

Section 4 studies fractions and quotients within the «category» of inverse categories with decompositions. Section 5 shows that l.e. quotient categories exist for l.e. CΘ-categories and that, for such categories, taking quotients and taking inverse symmetrizations are commuting operations.

Section 6 gives an example which shows that the latter result is no more true when quaternary symmetrizations are used instead of inverse ones.

INTRODUCTION.

Let A be an abelian category. Many homological questions in A substantially consist in the study of subquotients (quotients of subobjects) in A, which may be thought of as subobjects in A^W, the category of relations (quaternary symmetrization) of A.

The concept of canonical isomorphism between subquotients is crucial if homological questions are investigated in this context: unfortunately canonical isomorphisms between subquotients are not composable in any abelian non trivial category: actually an equivalent condition is that the lattice of subobjects of any object in A be distributive.

This condition is not globally verified, but can be obtained locally: taking together the locally admissible situations one obtains the category (Mitchell-exact and not abelian) $A^\#$ whose subquotients have composable canonical isomorphisms: $A^\#$ and its symmetrizations (the quaternary one and the inverse one: in the latter canonical isomorphic subquotients are identified) are therefore the good context to do homological considerations about the original abelian category A [14].

Perhaps it is better to remark that considerations about exact sequences in $A^\#$ become semilattice considerations about subquotients in its inverse symmetrization.

A tool used in Homology is the «Langage mod. C» introduced by Serre [12], which was defined and formalized [4] (using the concept of quotient categories) in the abelian context. The remarks made above clarify the interest in extending the «Langage mod. C» to distributive exact categories and in studying the consequences on symmetrizations.

Substantially, we show that quotienting E (exact and distributive) by a given set of objects (or, which is the same, making a given set of maps isomorphisms) is possible and preserves exactness and distributivity; moreover the inverse symmetrization of the quotient of E by a given set of objects may be thought as the quotient, by the same set of objects, of the inverse symmetrization of E, and quotienting preserves decompositions (which are the analogue of exactness in the symmetrization).

These results allow to apply Serre's techniques in the context of distributive categories and their inverse symmetrizations.

Lastly, some technical notes are perhaps suitable:

1. The logical pattern of exposition is the opposite of the one used in this introduction; the reason is that the existence, in the symmetrization of E, of formal inverses of the maps which are to be made isomorphisms easily allows to explicitly construct a category of fractions of the symmetrization and then the category of fractions of E as a subcategory of the former.

2. The context of this work is more general than the one described here. This is due to the belief shared by the authors that the good concept of exactness is weaker than the one introduced by Mitchell. The only difficulty is that the subobjects are no more a lattice (for the different behaviours of the normal ones and the not normal ones); therefore the condition analogous to distributivity is more complex.

3. Lastly, the example made in Section 5 shows that our results cannot be extended to quaternary symmetrizations.

1. FRACTIONS ON INVOLUTIVE CATEGORIES.

1.1. A category is called involutive if it is endowed with a contravariant functor j, which is the identity on the objects and such that $jj = 1$; $j(a)$ is denoted $\tilde{a}$.

An involutive category is called *regular* iff for any morphism a, $a\tilde{a}a = a$ [6].

A category is called *inverse* if for any morphism a, there exists one and only one morphism $\tilde{a}$ such that $a = a\tilde{a}a$, $\tilde{a} = \tilde{a}a\tilde{a}$; such a category is regular involutive [8].

Let H be a regular involutive category and let S be a subcategory of H. Putting on H the congruence (i.e. the equivalence relation compatible with the composition) generated by

$$a\tilde{a}\,R\,1, \quad \tilde{a}a\,R\,1 \quad \text{for every } a \text{ in } S,$$

which is obviously compatible with the involution, H/R has a canonical regular involution and the projection functor $\phi: H \to H/R$ respects involution.

1.2. *H/R coincides with the category of fractions $S^{-1}H$* [3].

PROOF. Let ϕ be the projection functor. For every α in S, $\phi(\alpha)$ is an isomorphism. If $f: H \to K$ is a functor such that $f(\alpha)$ is an isomorphism whenever α is in S, the congruence of H associated to f is less fine than R as, if α is in S, $\alpha\bar{\alpha}\alpha = \alpha$, then $f(\alpha)f(\bar{\alpha})f(\alpha) = f(\alpha)$ and, $f(\alpha)$ being an isomorphism,

$$f(\alpha)f(\bar{\alpha}) = 1, \quad f(\bar{\alpha})f(\alpha) = 1.$$

So a functor $g: H/R \to K$ exists such that $g\phi = f$, and it is obviously unique.

1.3. *If H is factorizing (i.e. has unique-up-to-isomorphism epic-monic factorizations), so is $S^{-1}H$.*
In fact, all monics (resp. epics) of H are coretractions (resp. retractions) hence are preserved by ϕ; analogously for monics and epics of $S^{-1}H$, which proves the uniqueness of epic-monic factorizations in the latter.

1.4. We call *sat(S)* (the *saturated of* S) the subcategory of H of all α such that $\phi(\alpha)$ is an isomorphism in $S^{-1}H$. If $S = sat(S)$, then S is said to be *saturated*.

1.5. *If H is factorizing, α is in S and $\alpha = \mu\pi$ is an epic-monic factorization, then π and μ are in $sat(S)$.*
Actually, if α is in S, $\alpha\bar{\alpha}\ R\ 1$ and $\bar{\alpha}\alpha\ R\ 1$. So,

$$1 = \phi(\alpha\bar{\alpha}) = \phi(\mu\bar{\mu}) = \phi(\mu)\phi(\bar{\mu}), \quad 1 = \phi(\bar{\alpha}\alpha) = \phi(\bar{\pi}\pi) = \phi(\bar{\pi})\phi(\pi),$$

therefore $\phi(\mu)$ and $\phi(\pi)$ are isomorphisms.

1.6. *If H is orthodox* [8], *then:*

1.6.1. Any idempotent (resp. any projection) of $S^{-1}H$ is an image of idempotents (resp. projections) of H.

1.6.2. $S^{-1}H$ is orthodox.

1.6.3. If moreover H is quasi-inverse (resp. inverse) [8] *so is also $S^{-1}H$.*

PROOF. If α is in H and $\phi(\alpha)$ is idempotent, then $\alpha\alpha\ R\ \alpha$, $\bar{\alpha}\bar{\alpha}\ R\ \bar{\alpha}$. So,

$$\alpha = \alpha\bar{\alpha}\alpha\ R\ \alpha(\bar{\alpha}\bar{\alpha})\alpha \quad \text{and} \quad \phi(\alpha) = \phi(\alpha\bar{\alpha}\bar{\alpha}\alpha),$$

where $(\alpha\tilde{\alpha})(\tilde{\alpha}\alpha)$ is idempotent in H which is orthodox [8]. If moreover $\phi(\alpha)$ is a projection, then $\alpha\ R\ \tilde{\alpha}\ R\ \alpha\alpha$ and $\alpha\ R\ \alpha\tilde{\alpha}$.

2 and 3 are easy consequences of 1, by recalling that a regular involutive category is inverse iff its idempotents commute.

2. FRACTIONS ON INVERSE CATEGORIES.

2.1. Let H be an inverse factorizing category and S a subcategory of H verifying:

(C0) $Iso(H) \subset S$,

(C1) If $\alpha = \mu\pi$ is an epic-monic factorization and α is in S, then $\tilde{\alpha}$, μ, π are in S,

(C2) If $[\lambda, \xi, \mu, \nu]$ is a pullback of monics and μ is in S, also ξ is in S.

Let $\sim$ be the relation:

$\alpha \sim \beta$ iff there exist λ and ξ monics in S such that $\mu\lambda = \mu_1\xi$, $\nu\lambda = \nu_1\xi$, $\alpha = \mu\tilde{\nu}$, $\beta = \mu_1\tilde{\nu}_1$ being epic-monic factorizations.

2.2. If M is the subcategory of monics of H, N the subcategory of monics of S, H_1 the subcategory of H containing all the maps $\lambda\tilde{\mu}$, $\mu \in N$, then $N^{-1}M = H_1/\sim$ ([13], page 129).

2.3. *Let H, S, $\sim$ be as above. Then $\sim$ is a congruence compatible with involution. Moreover $H/\sim$ is the category of fractions $S^{-1}H$, factorizing and inverse.*

PROOF. $N^{-1}M$ is a category of monics with pullbacks and the canonical functor $\psi: M \to N^{-1}M$ preserves pullbacks. A straightforward verification shows that the inverse factorizing category canonically associated to $N^{-1}M$ is $H/\sim$ and that the functor $\phi: H \to H/\sim$ which extends ψ is associated to $\sim$. Moreover if $f: H \to K$ is a functor such that $f(\alpha)$ is an isomorphism whenever α is in S, the congruence of H associated to f is weaker than $\sim$. So $H/\sim = S^{-1}H$.

2.4. *Any saturated subcategory S verifies (C0)-(C2) and*

(C3) A monic μ is in S iff there exists a monic ν so that $\mu\nu$ is in S.

PROOF. (C0) is obvious.

(C1) If α is in S and $\alpha = \mu\pi$ is an epic-monic factorization, then

$$1 = \phi(\bar{\alpha})\phi(\alpha) = \phi(\bar{\pi})\phi(\pi) \quad \text{and} \quad 1 = \phi(\alpha)\phi(\bar{\alpha}) = \phi(\mu)\phi(\bar{\mu})\ ;$$

therefore $\phi(\bar{\alpha})$, $\phi(\mu)$, $\phi(\pi)$ are isomorphisms; $\bar{\alpha}$, μ, π are in S.

(C2) Let $[\xi,\lambda,\mu,\nu]$ be a pullback of monics, hence a bicommutative square; let μ be in S. $\phi(\mu)\phi(\bar{\mu}) = 1$ implies

$$\phi(\lambda)\phi(\bar{\lambda}) = \phi(\lambda\bar{\xi})\phi(\xi\bar{\lambda}) = \phi(\bar{\nu})\phi(\mu\bar{\mu})\phi(\nu) = 1$$

and λ is in S.

(C3) $\phi(\mu\nu)$ is an isomorphism, so $\phi(\mu)$ is such.

2.5. *If S is a subcategory verifying (C0)-(C2), then sat(S) is the subcategory of all $\alpha = \mu\bar{\nu}$ such that there exist λ, ξ monics such that $\mu\lambda$ and $\nu\xi$ are in S.*

PROOF. If α is in *sat(S)*, 2.4 states that also μ and ν are in *sat(S)*; but then $\mu\bar{\mu} - 1$ and there exist μ_1 and λ monics in S and $\mu_1 = \mu\lambda$. The same for ν. The reciprocal inclusion follows from 2.4.

2.6. *If H has a zero-object, $S^{-1}H$ has a zero and ϕ preserves zero.*

3. FRACTIONS ON CΘ-CATEGORIES.

3.1. Let $s: C \to H$ be a symmetrization of a category C; it is called an SΘ-*symmetrization* iff it verifies axioms (weaker than those of [1]):

(SΘ1) $C = (C, M, P)$ is a bicategory (obviously, an abuse of notation; moreover, it should be noticed that the axioms SΘ and CΘ concern also the bicategory structure on C).

(SΘ2) For any pair of morphisms (m,p) of C, m in M, p in P, having the same codomain, there exist morphisms m' in M, p' in P such that $mp' = pm'$.

(SΘ3) The functor s is faithful and sends morphisms of M to monics and morphisms of P to epics.

(SΘ4) H is a regular involution category.

(SΘ5) H is an inverse category.

(SΘ6) s has quaternary factorizations, i. e. each morphism a of H has a (not necessarily unique) factorization $a = n\tilde{q}p\tilde{m}$, where m and n are in M, p and q are in P.

(SΘ7) $\{ a \mid a = n\tilde{q}p\tilde{1} \}$ and $\{ a \mid a = n\tilde{1}p\tilde{m} \}$ are subcategories of H.

3.2. C is a CΘ-*category* iff it verifies the following axioms (weaker than those of [1]) :

(CΘ1) $C = (C, M, P)$ is a bicategory.

(CΘ2) identical to SΘ2.

(CΘ3) The morphisms of M have pullbacks in C.

(CΘ3*) The morphisms of P have pushouts in C.

(CΘ4) Direct images of morphisms in M preserve (finite) intersections.

(CΘ4*) Inverse imagse of morphisms in P preserve (finite) intersections.

3.3. *A category C has an SΘ-symmetrization iff it is a CΘ-category; the SΘ-symmetrization is determined up to an isomorphism of categories (cf.* [1], Theorem 3.1).

3.4. Let C be a CΘ-category. Let us consider the following properties on a subcategory S containing the isomorphisms of C :

(W1) If $f = mp$ is a P-M factorization and f is in S, then m and p are in S.

(W2) If $[m', n', n, m]$ is a pullback of morphisms of M and m is in S, then m' is in S.

(W3) Let m, n be in M, p, q in P, $pm = nq$; if m is in S, n is in S ; if p is in S, q is in S.

(W4) If $[p, q, q', p']$ is a pushout of morphisms of P and p is in S, p' is in S.

(W5) For any pair (m, p) of morphisms, m in M, p in P, m in S, with the same codomain, there exist q in P, n in M and in S such that:

$m q = p n$.

3.5. *Let C be a $C\Theta$-category, S a subcategory verifying* (W 1)-(W 5) *and containing the isomorphisms of C, H the $S\Theta$-symmetrization of C. Then S is a $C\Theta$-subcategory of C [1] and $S^{-1} H = S^{\Theta\text{-}1} H = H/\sim$ where*

$\alpha \sim \beta$ iff there exist λ, ξ in S^{Θ}, monics in H such that $\alpha = \mu \tilde{\nu}$, $\beta = \mu_1 \tilde{\nu}_1$ are epic-monic factorizations, then $\mu\lambda = \mu_1 \xi$, $\nu\lambda = \nu_1 \xi$.

Let χ be the canonical functor from H to $S^{-1} H$; let D be the subcategory of H of all morphisms having a quaternary factorization $\alpha = n \tilde{q} p \tilde{m}$ with m and q in S. Then $\chi(D) = S^{-1} C$ and, if $s: \chi(D) \to S^{-1}H$ is the canonical immersion, then $(\chi(D), S^{-1} H, s)$ is a $S\Theta$-symmetrization.

PROOF. Any functor $f: C \to C'$ such that $f(\alpha)$ is an isomorphism whenever α is in S can be extended in a unique way to a functor $g: D \to C'$, where $g(\alpha) = f(n) f(q)^{-1} f(p) f(m)^{-1}$, if $\alpha = n \tilde{q} p \tilde{m}$ is a factorization of α with m, q in S. $S^{-1} H = S^{\Theta\text{-}1} H$ because S^{Θ} is contained in $sat(S)$, taking S as a subcategory of H. $S^{\Theta\text{-}1} H = H/\sim$ follows from 2.3, as S^{Θ} verifies (C 0)-(C 2).

Now, if $f: C \to C'$ is a functor such that $f(\alpha)$ is an isomorphism whenever α is in S, and $g: D \to C'$ is the extension of f, then $g(\alpha)$ is an isomorphism whenever α is in S^{Θ}, there exists a unique functor $g': \chi(D) \to C'$ such that $g'\chi = g$; its definition is $g'(\alpha) = g(\alpha')$ if $\alpha = \chi(\alpha')$; an easy verification shows that it is well-defined; so $\chi(D) = S^{-1} C$.

$s: \chi(D) \to S^{-1} H$ is obviously a symmetrization, verifying ($S\Theta$ 4), ($S\Theta$ 5), ($S\Theta$ 6). Let M be the subcategory of $\chi(D)$ of all α, $\alpha = \chi(\alpha')$ where α' has a quaternary factorization $\alpha' = n \tilde{q} p \tilde{m}$ with m, p and q in S; let P be the subcategory of $\chi(D)$ of all α, $\alpha = \chi(\alpha')$, where $\alpha' = n \tilde{q} p \tilde{m}$, with m, q and n in S.

If α is in S and $\alpha = n \tilde{q} p \tilde{m}$ is a quaternary factorization with m and n in S, then $\chi(\alpha) = \chi(n \tilde{q}) \chi(p \tilde{m})$ is a P-M factorization of $\chi(\alpha)$ and two different quaternary factorizations of α give the same P-M factorizations of $\chi(\alpha)$. Moreover if $\alpha \sim \beta$ where $\alpha = \mu \tilde{\nu}$, $\beta = \mu_1 \tilde{\nu}_1$ are factorizations in H, then there exist λ and ξ in S^{Θ} such that

$$\mu\lambda = \mu_1 \xi, \quad \nu\lambda = \nu_1 \xi.$$

$\bar{\lambda}\tilde{\nu} = \bar{\lambda}\tilde{\nu}_1$ implies $\lambda\bar{\xi}\tilde{\nu}_1 = \tilde{\nu}$ and $\mu\lambda = \mu_1\xi$ implies $\mu_1\xi\bar{\lambda} = \mu$, where $\chi(\xi\bar{\lambda})$ is an isomorphism in $\chi(D)$; so $(\chi(D), M, P)$ is a bicategory.

(SΘ 3) is then trivial.

(SΘ 2) Let μ be in M, π be in P; we can suppose $\mu = m\tilde{p}$, $\nu = q\tilde{n}$ where n and p are in S; let $[m_1, q_1, m, q]$ be commutative (in C) by (CΘ 2); so $[nm_1, pq_1, q\tilde{n}, m\tilde{p}]$ is a commutative square in $\chi(D)$.

(SΘ 7) It is an easy consequence of the fact that, if μ is a monic in D and ν is a monic in H, and $\tilde{\nu}\mu = \xi\bar{\lambda}$ is an epic-monic factorization, then ξ is in D.

4. QUOTIENTS ON INVERSE D-CATEGORIES.

4.1. Let H be a factorizing inverse category with zero, let M be a subcategory of monics of H verifying:

(M1) $Iso(H) \subset M$.

(M2) If $[\lambda, \xi, \nu, \mu]$ is a pullback of monics and $\mu \in M$, then $\lambda \in M$.

(M3) For each μ_0,

$$\mu_0 \in M(A) = \{ \mu \in M \mid cod(\mu) = A \}$$

there exist $\mu_1, \dots \mu_n \in M(A)$ such that $\mu_i \cap \mu_j = 0$ if $i \neq j$, and if ν is a monic with A as codomain such that $\mu_i \cap \nu = 0$ whenever $i \neq i_0$ then $\nu < \mu_{i_0}$.

(H, M) is called an *inverse D-category*; a family $(\mu_0, \dots, \mu_n)$ satisfying the conditions of (M 3) is called a *decomposition of* A.

If (H, M) and (K, N) are inverse D-categories and if $\phi: H \to K$ is a zero-preserving functor such that $\phi(M) \subset N$ and preserving decompositions, $\phi: (H, M) \to (K, N)$ is called a *D-functor* [10].

4.2. Let (H, M) be an inverse D-category, S a family of morphisms in H. An inverse D-category (H_S, M_S) endowed with a D-functor

$$\phi: (H, M) \to (H_S, M_S)$$

is called a *category of D-fractions of (H, M) by S* if:

i) if s is in S, then $\phi(s)$ is an isomorphism.

ii) If α is a D-functor from (H, M) to any inverse D-category (K, N)

such that $\alpha(s)$ is an isomorphism whenever s is in S, then there exists a unique D-functor $\beta:(H_S, M_S) \to (K, N)$ such that $\alpha = \beta\phi$.

4.3. *Let (H,M) be an inverse D-category, H_1 be the inverse subcategory of H generated by M; let $S \subset H_1$ be a subcategory verifying (CO)-(C2), $S^{-1}H$ the category of fractions of H by S, $\phi: H \to S^{-1}H$ the canonical functor. We can suppose that $S^{-1}H_1$ is a subcategory of $S^{-1}H$. Let M_S be the subcategory of monics of $S^{-1}H_1$. Then the following are equivalent:*

4.3.1. $(S^{-1}H, M_S)$ endowed with ϕ is a category of D-fractions of (H,M) by S.

4.3.2. $(S^{-1}H, M_S)$ is an inverse D-category and ϕ a D-functor.

4.3.3. If B_0 is a subobject of A, it is in sat(S) iff there exists a decomposition $(A_0, \dots, A_n)$ of A such that

$$A_0 \subset B_0 \text{ and } O_{0A_i} \in S \text{ for } i > 0.$$

Any of these conditions is implied by:

4.3.4. If A_0 is a subobject of A, it is in S iff there exists a decomposition $(A_0, \dots, A_n)$ such that $O_{0A_i} \in S$ for $i > 0$.

PROOF. 4.3.2 ⇒ 4.3.1: We have only to show that the functor β such that $\alpha = \beta\phi$ (which exists and is unique, for $S^{-1}H$ is a category of fractions) is a D-functor, i.e. [10] that for any object A' of $S^{-1}H$, for any subobject A'_0 of A' in M_S, there exists a decomposition $(A'_0, \dots, A'_n)$ preserved by β. Without loss of generality, we can suppose that $A' = \phi(A)$, $A'_0 = \phi(A_0)$ and A_0 is a subobject of A in M. Let then $(A_0, \dots, A_n)$ be a decomposition of A in (H,M); $(\phi(A_0), \dots, \phi(A_n))$ is a decomposition of A' in (S^{-1}, M_S) and

$$(\beta\phi(A_0), \dots, \beta\phi(A_n)) = (\alpha(A_0), \dots, \alpha(A_n))$$

is a decompostion of $\alpha(A) = \beta(A')$ in (K,N).

4.3.3 ⇒ 4.3.2. If A'_0 is a subobject of A' in M_S, we can suppose $A' = \phi(A)$, $A'_0 = \phi(A_0)$, A_0 a subobject of A in M_S. Let $(A_1, \dots, A_n)$ be a decomposition of A in (H,M); our goal is to prove that A' has the decomposition $(\phi(A_0), \dots, \phi(A_n))$ in $(S^{-1}H, M_S)$. Let then C' be a subobject of A', $C'=\phi(C)$ (and we can suppose also that C be a subobject of A), such that $C'\cap\phi(A_i)=0$ for any $i\neq i_0$. Then

$$\phi(C\cap A_i)=0 \text{ and } O_{0,\,C\cap A_i}\in S \text{ for any } i\neq i_0 .$$

$(C\cap A_0,\dots,C\cap A_n)$ being a decomposition of C in (H,M) [10], $C\cap A_{i_0}$ is a subobject of C in $sat(S)$. So $C'=\phi(C\cap A_{i_0})\subset\phi(A_{i_0})$.

4.3.2 ⇒ 4.3.3. If B_0 is a subobject of A in $sat(S)$, then there exists a subobject A_0 of B_0 which, taken as a subobject of A, is in S. Let $(A_0,\dots,A_n)$ be a decomposition of A; $(\phi(A_0),\dots,\phi(A_n))$ is a decomposition of $\phi(A)$; but $\phi(A_0)\approx\phi(A)$, therefore

$$\phi(A_0)=0 \text{ and } O_{0A_i}\in S \text{ for } i>0.$$

Conversely, if $(A_0,\dots,A_n)$ is a decomposition of A and O_{0A_i} in S for $i>0$, then $\phi(A_0)\approx\phi(A)$, A_0 is a subobject of A in $sat(S)$ and any B_0 such that $A_0\subset B_0\subset A$ is also a subobject of A in $sat(S)$.

4.3.4 ⇒ 4.3.3 follows easily from 2.4.

4.4. Let (H,M) be an inverse D-category, S a family of morphisms of H. An inverse D-category $(H/S,M/S)$ endowed with a D-functor

$$\phi:(H,M)\to(H/S,M/S)$$

is called a *category of D-quotients of* (H,M) *by* S if

i) if s is in S, then $\phi(s)$ is a zero-morphism.

ii) if α is a D-functor from (H,M) to any inverse D-category (K,N) such that $\alpha(s)$ is zero whenever s is in S, then there exists a unique D-functor $\beta:(H/S,M/S)\to(K,N)$ such that $\alpha=\beta\phi$.

4.5. Let (H,M) be an inverse D-category, S a family of morphisms of H. Let F be the family of D-functors α from (H,M) to any inverse D-category (K,N) such that $\alpha(s)=0$ whenever s is in S, $\overline{S}$ the family of morphisms g in (H,M) such that $\alpha(g)=0$ whenever α is in F, and $th(S)$ the family of objects C in (H,M) such that $\alpha(1_C)=0$ whenever α is in F. Then $\overline{S}$ contains S and is the ideal generated by $th(S)$

4.6. A family H' of objects of (H,M) is called *thick* if $H'=th(H')$.

4.7. *If H' is thick then it verifies the following conditions:*

(DT1) If $A \in H'$ and B is a subobject of A, then $B \in H'$.

(DT2) If $(A_0, \ldots, A_n)$ is a decomposition of A and $A_i \in H'$ for all i's, then $A \in H'$.

4.8. *Let (H, M) be an inverse D-category, M containing all the zero monics, H' be a family of objects verifying (DT1)-(DT2), let $S = S(H')$ be the inverse subcategory generated by the family S_0 of monics $A_0 \rightarrowtail A$ in M such that there exists a decomposition $(A_0, \ldots, A_n)$ of A, where A_i is in H' whenever $i > 0$. Then S verifies (C0)-(C2) and $(S^{-1}H, M_S)$ endowed with its canonical functor ϕ is a category of D-quotients of (H,M) by H'.*

PROOF. S_0 is a subcategory for, if A_0 is a subobject of A in S_0 and B_0 a subobject of A_0 in S_0, then there exist a decomposition $(A_0, \ldots, A_n)$ of A and a decomposition $(B_0, \ldots, B_k)$ of A_0 such that A_i $(i > 0)$ and B_j $(j > 0)$ are in H'. But then [10] $(B_0, \ldots, B_k, A_1, \ldots, A_n)$ is a decomposition of A, where any member different from B_0 is in H', so B_0 is a subobject of A in S_0.

S_0 verifies also (C0) and (C2), therefore $S = \{\mu \tilde{\nu} \mid \mu, \nu \in S_0\}$ verifies (C0), (C1) and (C2). Moreover S verifies 4.3.4 for, if A_0 is a subobject of A in S_0 and $(A_1, \ldots, A_n)$ a decomposition of A, there exists a decomposition $(A_0, B_1, \ldots, B_k)$ of A with all the B_i's in H'. But then for all j's, $(A_j \cap B_1, \ldots, A_j \cap B_k)$ is a decomposition of A_j with all members in H', and so A_j is in H' for all j's. $(A, 0)$ being a decomposition of A, we have

$$A \in H' \text{ iff } 0_{0A} \in S_0 \text{ iff } \phi(A) = 0.$$

So $(S^{-1}H, M_S)$ endowed with ϕ is a category of D-quotients of (H,M) by H', and H' is thick.

4.9. *If (H, M) is an inverse D-category, M containing all zero monics, for any family S of morphisms in H, the category of D-quotients of (H, M) by S exists. The thick families of objects are exactly those verifying (DT1)-(DT2).*

5. QUOTIENTS ON LEFT-EXACT CATEGORIES.

5.1. Let C be a left-exact CΘ-category [9] verifying Axiom A6 of [2]:

If $nq = pm$, n, m monics, p, q conormal epics, then if m is normal, n is such,

C^{Θ} be its SΘ-symmetrization,

$$M_C = \{ m\tilde{p} \mid m \text{ subnormal monic, } p \text{ conormal epic} \}.$$

Then (C^{Θ}, M_C) is an inverse D-category [10].

5.2. *Let C' be a family of objects in C. C' verifies (DT 1)-(DT 2) in (C^{Θ}, M_C) iff it verifies in C:*

(TH 1) If $A \in C'$ and there exists a monic $B \rightarrowtail A$, then $B \in C'$.

(TH 2) If $A \in C'$ and there exists a conormal epic $A \twoheadrightarrow B$, then $B \in C'$.

(TH3) If $B \rightarrowtail A \twoheadrightarrow H$ is a short exact sequence and $B, H \in C'$, then $A \in C'$.

PROOF. (DT 1) is obviously equivalent to (TH 1) and (TH 2).
(TH3) follows from the fact that (B, C) is a decomposition of A [10].
(DT2): It is obvious for unary decompositions. So, by induction, let it be proved for n-ary decompositions and let $(A_0, \dots, A_n)$ be a decomposition of A such that $A_i \in C'$ for all i's. A_0 is in M_C, so there is a chain $H_0 \subset H_1 \subset \dots \subset H_k = A$ of subobjects of A in C', H_i normal in H_{i+1} for all i's, such that $A_0 = H_1/H_0$. Moreover

$$(H_0, H_1/H_0, \dots, H_i/H_{i-1}, \dots, H_k/H_{k-1})$$

is a decomposition of A [10]. But then $(H_0 \cap A_1, \dots, H_0 \cap A_n)$ is a decomposition of H_0, and

$$(H_i/H_{i-1} \cap A_1, \dots, H_i/H_{i-1} \cap A_n)$$

is a decomposition of H_i/H_{i-1} for $i = 2, \dots, k$. By (DT 1) all the members of these n-ary decompositions are in C', so H_0 and H_i/H_{i-1} $(i = 1, \dots, k)$ are in C'. But H_{i-1} and H_i/H_{i-1} are in C', then, by (TH 3), also H_i is in C'; so, inductively, $A = H_k$ is in C'.

5.3. *Let C be a left-exact CΘ-category verifying Axiom A6, C' a family* *of objects verifying (TH1-3), T the subcategory generated by normal monics with cokernels in C' and conormal epics with kernels in C'. Then $T^{-1}C$ is a left-exact quotient category of C by C'. It is a $C\Theta$-category and its symmetrization is the category of D-quotients of (C^{Θ}, M_C) by C'.*

PROOF. By A6, T is the subcategory of all $f = m\tilde{p}$, p a conormal epic with kernel in C', $m = m_n \dots m_2$, m_i normal monics with cokernels in C'. It moreover verifies (W1-5). Then $T^{-1}C$ is a $C\Theta$-category and its $S\Theta$-symmetrization is $T^{\Theta-1}C^{\Theta}$.

But $T^{\Theta} = S(C')$: both are contained in M_C ; any morphism $m\tilde{p}$ in M_C defines a «normal chain»

$$H_0 \xrightarrow{m_1} H_1 \xrightarrow{m_2} H_2 \dots H_{n-1} \xrightarrow{m_n} H_n$$

with

$$p = coker\, m_1, \quad m = m_n \dots m_2 ;$$

$m\tilde{p}$ is in T^{Θ} iff H_0, H_i / H_{i-1} $(i = 1, \dots, k)$ are in C', iff $m\tilde{p}$ is in S_0. So the $S\Theta$-symmetrization of $T^{-1}C$ is the category of D-quotients of C^{Θ} by C'.

$T^{-1}C$ is a left-exact category and $\phi: C \to T^{-1}C$ is a left-exact functor: let $p_1 \in P_{T^{-1}C}$; without loss of generality we can suppose that $p_1 = \phi(p)$, p a conormal epic in C ; let $n = ker\, p$ in C ; then $(n, \tilde{p})$ is a decomposition in (C^{Θ}, M_C) ; $(\phi(n), \tilde{p}_1)$ is a decomposition in the category of D-quotients of (C^{Θ}, M_C) by C' ; it is then easy to infer that $\phi(n) = ker\, p_1$.

Also, $p_1 = coker\phi(n)$: let, let q be in $P_{T^{-1}C}$ and $q\phi(n) = 0$; then $\tilde{q} < \tilde{p}_1$, there exist $m \in M_{T^{-1}C}$, $x \in P_{T^{-1}C}$ such that $q = \tilde{p}\, m\, \tilde{x}$. By ($C\Theta$ 2), there exist

$$z \in P_{T^{-1}C},\ m_1 \in M_{T^{-1}C} \quad \text{such that } mz = p_1 m,\ m_1 \tilde{z}\tilde{x} = \tilde{q}.$$

If $[q, p_1, a, b]$ is a pushout, by ($C\Theta$ 4) we have that a is an isomorphism so $q = bp_1$, $p_1 = coker\,\phi(n)$. An easy calculation shows that $T^{-1}C$ is just the left-exact quotient category of C by C'.

5.4. *If C is a left-exact $C\Theta$-category verifying A6, S any family of mor-*

phisms in C, then the left-exact quotient category of C by S exists. Thick families (in the sense of [9]) are exactly the ones verifying (TH 1-3). Moreover the left-exact quotient category of C by S is a left-exact $C\Theta$-category and its $S\Theta$-symmetrization is the category of D-quotients of C^{Θ} by S.

6. AN EXAMPLE.

6.1. Orthoquatemary categories [7] are a particular case of CΘ-categories; not only they have a SΘ-symmetrization, but also an (orthodox) quatemay one [7], of which the SΘ-symmetrization is a «quotient» by a congruence.

One could ask if the results of 3.5 apply to orthoquaternary categories and their quatemary symmetrizations, i. e.: if C is an orthoquatemary category, C^W its quatemary symmetrization, S a subcategory of C, then $S^{-1}C$ is orthoquatemary and $(S^{-1}C)^W = S^{-1}(C^W)$.

The following example shows that this is false, also when C is left-exact and $S^{-1}C$ is a left-exact quotient category.

6.2. Let C be the category with zero whose non-zero objects are

A, B, C, D, E, F, G, H, I, L,

whose non-zero and non-identity morphisms are given by:

$C(A,B) = \{m_1\}$, $C(A,C) = \{m_3\}$, $C(A,F) = \{m_9\}$,
$C(A,H) = \{m_{14}\}$, $C(B,C) = \{m_2\}$, $C(B,D) = \{m_5\}$,
$C(B,E) = \{m_6\}$, $C(B,F) = \{m_8\}$, $C(B,G) = \{m_{11}\}$,
$C(B,H) = \{m_{13}\}$, $C(B,I) = \{m_{17}\}$, $C(C,D) = \{m_4\}$,
$C(C,H) = \{m_{12}\}$, $C(C,I) = \{m_{16}\}$, $C(D,I) = \{m_{15}\}$,
$C(E,D) = \{m_7\}$, $C(E,G) = \{m_{10}\}$, $C(E,I) = \{m_{18}\}$,
$C(F,G) = \{m_{20}\}$, $C(F,H) = \{m_{19}\}$, $C(F,I) = \{m_{22}\}$,
$C(F,L) = \{m_{26}\}$, $C(G,I) = \{m_{21}\}$, $C(G,L) = \{m_{25}\}$,
$C(H,I) = \{m_{23}\}$, $C(H,L) = \{m_{27}\}$, $C(I,L) = \{m_{24}\}$.

with the following relations (plus their consequences):

$m_3 = m_2 m_1$, $m_5 = m_4 m_2 = m_7 m_6$, $m_4 m_3 = 0$, $m_6 m_1 = 0$,

$m_9 = m_8 m_1$, $m_{11} = m_{10} m_6 = m_{20} m_8$, $m_{13} = m_{12} m_2 = m_{19} m_8$,
$m_{14} = m_{12} m_3$, $m_{16} = m_{15} m_4 = m_{23} m_{12}$, $m_{17} = m_{15} m_5$,
$m_{18} = m_{15} m_7 = m_{21} m_{10}$, $m_{22} = m_{21} m_{20} = m_{23} m_{19}$,
$m_{24} m_{15} = 0$, $m_{25} = m_{24} m_{21}$, $m_{26} = m_{24} m_{22}$, $m_{27} = m_{24} m_{23}$.

6.3. C is a left-exact orthoquaternary category; $C' = \{D, E\}$ is a thick class, $S = \{m_1, m_2, m_3, m_{21}, m_{24}, m_{25}\}$. So we can take the left-exact quotient category $(C/C', f)$, where

$f(D)$, $f(E)$ are zero,

$f(m_4)$, $f(m_5)$, $f(m_6)$, $f(m_7)$, $f(m_{10})$, $f(m_{11})$, $f(m_{15})$, $f(m_{16})$, $f(m_{17})$, $f(m_{18})$ are zero,

$f(m)$ is an isomorphism iff $m \in S$.

6.4. More exactly, C/C' can be represented by the category D whose non-zero objects are

A', B', C', F', G', H', I', L',

whose non-zero and non-identity morphisms are given by:

$D(A,B) = \{n_1\}$, $D(A,C) = \{n_3\}$, $D(A,F) = \{n_9\}$,
$D(A,H) = \{n_{14}\}$, $D(B,A) = \{n_{28}\}$, $D(B,C) = \{n_2\}$,
$D(B,F) = \{n_8\}$, $D(B,H) = \{n_{13}\}$, $D(C,A) = \{n_{31}\}$,
$D(C,B) = \{n_{29}\}$, $D(C,F) = \{n_{30}\}$, $D(C,H) = \{n_{12}\}$,
$D(F,G) = \{n_{20}\}$, $D(F,H) = \{n_{19}\}$, $D(F,I) = \{n_{22}\}$,
$D(F,L) = \{n_{26}\}$, $D(G,I) = \{n_{21}\}$, $D(G,L) = \{n_{25}\}$,
$D(H,G) = \{n_{33}\}$, $D(H,I) = \{n_{23}\}$, $D(H,L) = \{n_{27}\}$,
$D(I,G) = \{n_{32}\}$, $D(I,L) = \{n_{24}\}$, $D(L,G) = \{n_{35}\}$,
$D(L,I) = \{n_{34}\}$, where

$$n_{28} = n_1^{-1},\ n_{29} = n_2^{-1},\ n_{30} = n_8 n_2^{-1},\ n_{31} = n_3^{-1},$$
$$n_{32} = n_{21}^{-1},\ n_{33} = n_{21}^{-1} n_{23},\ n_{34} = n_{24}^{-1},\ n_{35} = n_{25}^{-1}.$$

Compositions are induced by those of C through the functor f which is defined in the following way:

as to objects, if X is in C', $f(X) = 0$; otherwise $f(X) = X'$,

as to morphisms: $f(m_i) = 0$ if $i = 4, 5, 6, 7, 10, 11, 15, 16, 17, 18$, otherwise $f(m_i) = n_i$.

6.5. It is easy to see that D is an orthoquaternary category, that in C, $\{m_{19}, m_{20}, m_{23}, m_{21}\}$ is a pullback while in D, $\{n_{19}, n_{20}, n_{23}, n_{21}\}$ is not one; so f is not a $\mathbb{W}$-functor [6] and f cannot be extended to a functor $f^{W}: C^{W} \to D^{W}$ commuting with symmetrization functors; if $D^{W} = S^{-1}(C^{W})$ such a functor would necessarily exist.

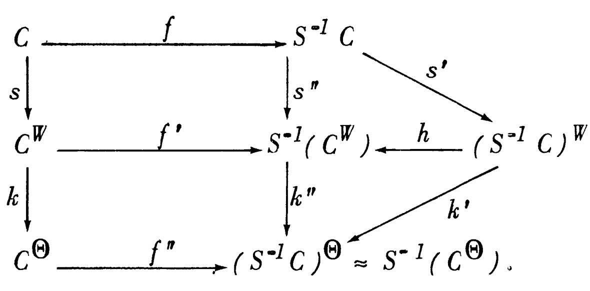
However, one can take the category of fractions $(S^{-1}(C^{W}), f')$ which turns out to be a symmetrization of $D \approx S^{-1}C$, through the functor s'' which exists for fraction properties; using the results of [6] one can prove that there exists a functor

$$h : (S^{-1}C)^{W} \to S^{-1}(C^{W})$$

such that $hs' = s''$.

Moreover, both C and D have a SΘ-symmetrization which is a quotient of the quaternary one by a congruence; to these the results of 3.5 apply.

In conclusion, we have the following diagram, where f, f' and f'' are the canonical «fraction» functors, s, s', k, k' the canonical symmetrization functors, k'' such that $k''f' = f''k$ exists for fraction properties and $k' = k''h$ for symmetrization properties.



6.6. Categories C and D have been built from their underlying graph, using the algorithm described in [11]. The numbering of the morphisms is the order by which the algorithm generates them.

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