GEORG GREVE

JENÖ SZIGETI

WALTER THOLEN

Lifting tensorproducts along non-adjoint functors

LIFTING TENSORPRODUCTS A LONG NON-ADJOINT FUNCTORS

by Georg GREVE, Jenö SZIGETI and Walter THOLEN

This note contains an attempt to axiomatize structure properties of non-adjoint functors which still allow canonical liftings of tensorproducts. The type of functors which is adequate for this purpose is described by the notion of a *presemitopological functor*. These functors are, roughly spoken, non-adjoint semitopological functors [12], more precisely: a functor is semitopological iff it is presemitopological and right adjoint (see the Proposition below). It turns out that the major results for semitopological functors hold in a generalized form for presemitopological functors: they are faithful, they can be completely described by a factorization structure (Theorem 1), and they are characterized as full reflexive restrictions of so called *pretopological functors* which can be correspondingly considered as «non-adjoint topological functors» (Theorem 2).

Although presemitopologicity is an extremely weak property one is able to lift tensorproducts along presemitopological functors (Theorem 3). Even more: within the context of closed categories these tensorproducts are left adjoint to inner homfunctors, provided the lifting functor is also co-semitopological. This holds in particular for every topological functor and the inclusion functor of every full coreflexive subcategory. Therefore one has plenty of examples only a few of them could be mentioned in this paper.

The definition of a presemitopological functor was given before by the first author [5] who already mentioned the useful lifting properties of presemitopological functors. Theorem 1 was first announced (in a slightly different form) by the second author [11].

1. PRESEMITOPOLOGICAL FUNCTORS.

Throughout this paper let $P: \underline{A} \to \underline{X}$ be a functor. For a diagram

(= functor) $D : \underline{D} \to \underline{A}$ ($\underline{D}$ may be empty or a proper class), an object A of $\underline{A}$, and a cocone $\xi : P D \to \Delta P A$, we call an $\underline{A}$-morphism $e : A \to B$ a P-lifter of (D, ξ, A) iff the following holds:

There exists a cocone $\beta : D \to \Delta B$ in $\underline{A}$ with $P\beta = \Delta P e . \xi$ such that for all $\underline{A}$-morphisms $f : A \to C$ and all cocones $\gamma : D \to \Delta C$ in $\underline{A}$ with $P\gamma = \Delta P f . \xi$ there is a unique $\underline{A}$-morphism $t : B \to C$ with

$$te = f \text{ and } \Delta t . \beta = \gamma .$$
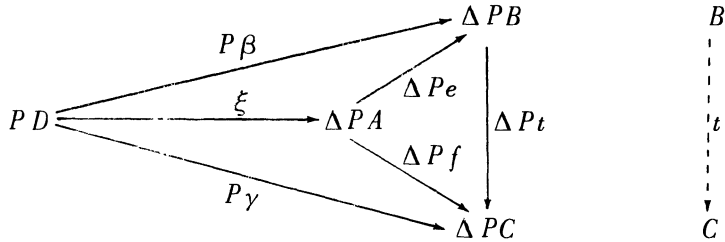
P is called presemitopological (pretopological resp.) iff each triple (D, ξ, A) admits a P-lifter (a P-lifter whose P-image is an $\underline{X}$-isomorphism resp.). Dual notions: *P-colifter, copresemitopological, copretopological.*

REMARKS. 1. If P is faithful, every P-lifter is an $\underline{A}$-epimorphism; this can be directly seen from the definition.

2. *If P is presemitopological, then P is faithful.* This can be shown as in Lemma 3.2 of [12] using the principle developed in [1] (see also [11]). Since only P-lifters to discrete data, i. e. sinks $(x_i : P A_i \to P A)_{i \in I}$ of $\underline{X}$-morphisms, are needed, we therefore get from Remark 1:

3. *P is presemitopological iff every sink $(x_i : P A_i \to P A)_{i \in I}$ admits a P-lifter.*

PROPOSITION. *1. Every full and faithful functor is pretopological and, therefore, presemitopological.*

2. P is semitopological [12] iff P is presemitopological and has a left adjoint.

3. P is topological [12] iff P is pretopological and has a full and faithful left adjoint.

PROOF. 1. All P-lifters can be taken to be identities.

2. Every semitopological functor P is known to be right adjoint. Now, let (D, ξ, A) be given as above. We form $\underline{D}_0 = \underline{D} \cup \underline{1}$ with $\underline{1}$ being the discrete category with one object O, $D_0 : \underline{D}_0 \to \underline{A}$ with

$$D_0 O = A, \quad D_0 / \underline{D} = D,$$

and $\xi_0 : PD_0 \to \Delta PA$ with

$$\xi_0 O = 1_{PA}, \quad \xi_0 / \underline{D} = \xi.$$

A P-semifinal prolongation $q : PA \to PB$ of ξ_0 (cf. [12]) lifts to an $\underline{A}$-morphism $e : A \to B$ which easily turns out to be a P-lifter of (D, ξ, A).

Vice versa, let P be presemitopological and right adjoint, and let $\zeta : PD \to \Delta X$ be a P-cocone. Its P-semifinal prolongation can be constructed as $(Pf)(\eta X) : X \to PB$ where $\eta X : X \to PFX$ is the unit of the left adjoint F of P at X and where $f : FX \to B$ is a P-lifter of the triple $(D, \Delta \eta X. \zeta, FX)$.

3. A left adjoint of P is full and faithful iff the units are isomorphisms and a topological functor has a full and faithful left adjoint (cf. [12]). Thus the assertion follows from the constructions given in 2. $\|$

Assertion 3 of the Proposition is formally similar to a result due to Fay, Brümmer and Hardie [4] : P is topological iff P is semitopological and has a full and faithful left adjoint.

By the Proposition one has immediately a whole host of examples which are, however, not typical because they are described by stronger notions like topologicity or semitopologicity. The following examples are not covered by these stronger notions.

EXAMPLES. 1. Let $(X, \leq)$ be a partially ordered set considered as a category $\underline{X}$ with $Ob \underline{X} = X$, $|\underline{X}(x, y)| \leq 1$ for all $x, y \in X$, and

$$|\underline{X}(x, y)| = 1 \quad \text{iff} \quad x \leq y.$$

The unique faithful functor $P : \underline{X} \to \underline{1}$ (see above) is pretopological iff every non-void subset of X has a supremum in X. From this example one sees that «pretopological» is *not* selfdual whereas «topological» is (cf. [12]).

2. Let Grp_{fin} be the category of finite groups. The underlying functor $P: Grp_{fin} \rightarrow Ens_{fin}$ is presemitopological. The same holds if Grp_{fin} is replaced by $\underline{A}_{fin}$ with $\underline{A}$ being any monadic category over Ens.

3. Many categories in algebra which are not semitopological over Ens are still presemitopological over Ens, for instance: finitely generated groups, rings or algebras; solvable groups or algebras; p-groups; torsion groups; simple modules or algebras; semisimple modules or algebras; Noetherian modules or rings, etc.

4. Let Met_1 be the category of metric spaces whose metric is bounded by 1; morphisms $f: (X, d) \rightarrow (X', d')$ are non-expanding maps, i.e.,

$$d'(f(x), f(y)) \le d(x, y) \quad \text{for all} \quad x, y \in X.$$

The underlying set functor $P: Met_1 \rightarrow Ens$ is known to be semitopological but not cosemitopological because of the missing right adjoint (there is no couniversal solution for a two elements set). Nevertheless, P is copretopological: given spaces (X, d) and (X_i, d_i), $i \in I$, and set mappings $f_i: X \rightarrow X_i$, we can provide X with the new metric

$$d^*(x, y) = sup \{ d(x, y), d_i(f_i(x), f_i(y)), i \in I \}.$$

Then $id_X: (X, d^*) \rightarrow (X, d)$ is a P-colifter of $((X_i, d_i)_I, (f_i)_I, (X, d))$.

Since P has a full and faithful left adjoint this example shows that Assertion 3 of the Proposition becomes wrong if «pretopological» is replaced by «copretopological».

5. Take $P: Conn_* \rightarrow Top$ to be the base point forgetting functor from pointed connected spaces to topological spaces. P is not semitopological, but presemitopological: given objects (X, x), (X_i, x_i), $i \in I$, in $Conn_*$ and a sink $(f_i: X_i \rightarrow X)_{i \in I}$ of continuous mappings one gets a P-lifter $\pi: X \rightarrow X/\sim$ from the smallest equivalence relation $\sim$ identifying all $f_i(x_i)$ with x. The same result can be obtained for other notions of connectedness. Similarly one proves presemitopologicity of the forgetful functor from connected groups to Top.

6. Take (P_0, η, μ) to be the covariant power set monad with

$$P_0 X = \{ Y \subset X \mid Y \ne \emptyset \}.$$

The Kleisli category of this monad is the category Ens_{Rel} of sets, with morphisms $R : X \to Y$ being relations $R \subset X \times Y$ with $R^{-1} Y = X$ (cf. [10] 1.3). The inclusion functor $J : Ens \to Ens_{Rel}$ turns out to be presemitopological; given a sink $(R_i : X_i \to X)_I$ of relations one gets a J-lifter $\pi : X \to X/\sim$ from the smallest equivalence relation $\sim$ such that

$$x \sim y \quad \text{if there is an } x_i \in X_i \text{ with } x_i R_i x \text{ and } x_i R_i y.$$

The power set monad can be lifted to the so called Vietoris monad on the category of compact Hausdorff spaces (cf. [14], [10] Ex. 1.5.23). Correspondingly to the result above one can show that the canonical left adjoint functor into the Kleisli category of this monad is presemitopological.

2. THE CHARACTERIZATION THEOREM.

In this section we will characterize pretopological and presemitopological functors by factorization structures. From [12] we recall some phrases: Let $\underline{E} \subset Mor \underline{A}$ be a subclass which is closed under composition with isomorphisms. An $(\underline{E}\text{-})$ factorization of a cone $\alpha : \Delta A \to D$ in $\underline{A}$ consists of an $\underline{A}$-morphism $e : A \to B$ (in $\underline{E}$) and a cone $\mu : \Delta B \to D$ in $\underline{A}$ with $\alpha = \mu . \Delta e$. This factorization is called:

- *rigid* iff every endomorphism t of B with $te = e$ and $\mu . \Delta t = \mu$ is the identity morphism,

- *P-semiinitial* iff for every cone $\gamma : \Delta C \to D$ in $\underline{A}$ and every $\underline{X}$-morphism $x : PC \to PA$ with $P\alpha . \Delta x = P\gamma$ there is a unique morphism

$$b : C \to B \quad \text{with} \quad Pb = (Pe)x \text{ and } \mu . \Delta b = \gamma,$$

- *locally orthogonal* (with respect to $\underline{E}$) iff for every $p : K \to L$ in $\underline{E}$, every $k : K \to A$ in $\underline{A}$, and every cone $\lambda : \Delta L \to D$ in $\underline{A}$ with $\lambda . \Delta p = \alpha . \Delta k$ there is a unique morphism $t : L \to B$ with $tp = ek$ and $\mu . \Delta t = \lambda$.

As in [12], Lemma 2.9, one can show:

LEMMA. *Let $\alpha = \mu . \Delta e$ be a P-semiinitial factorization. Then for every P-lifter $p : K \to L$, every $x : PK \to PA$ in $\underline{X}$, and every cone $\lambda : \Delta L \to D$ in A with $P\lambda . \Delta Pp = Pa . \Delta x$ there is a unique morphism $t : L \to B$ with*

$$(P t)(P p) = (P e)x \quad \text{and} \quad \mu . \Delta t = \lambda.$$

Therefore, if e is a P-lifter and P is faithful, the factorization is locally orthogonal with respect to the class of all P-lifters. ||

Now we can state the main result in this section:

THEOREM 1. *The following assertions are equivalent:*

   (i) P is presemitopological.

   (ii) There is a class $\underline{E}$ of $\underline{A}$-(epi)morphisms such that every cone in $\underline{A}$ admits a locally orthogonal and P-semiinitial $\underline{E}$-factorization.

   (iii) Every cone in $\underline{A}$ admits a rigid and P-semiinitial factorization.

PROOF. (i) $\Rightarrow$ (ii) : Given $\alpha : \Delta A \to D$ with $D: \underline{D} \to \underline{A}$ we form the full subcategory $\tilde{D}$ of the comma-category $(P \downarrow P A)$ consisting of all objects (C, x) with $C \in Ob \underline{A}$ and $x: PC \to PA$ such that there is a cone

$$\gamma : \Delta C \to D \quad \text{with} \quad P \gamma = P \alpha . \Delta x$$

(γ is uniquely determined as P is faithful). There is a canonical projection functor $\tilde{D}: \tilde{\underline{D}} \to \underline{A}$, $(C, x) \mapsto C$, and a cocone

$$\xi : P \tilde{D} \to \Delta P A \quad \text{with} \quad \xi (C, x) = x.$$

This cocone admits a P-lifter $e : A \to B$, and by the universal property of e one gets, for every $d \in Ob \underline{D}$ a unique morphism

$$\mu d : B \to D d \quad \text{with} \quad (\mu d) e = \alpha d.$$

Therefore we have an $\underline{E}$-factorization $\alpha = \mu . \Delta e$ with $\underline{E}$ being the class of all P-lifters. By construction of $\tilde{D}$, this factorization is obviously P-semiinitial. Therefore, by the above Lemma, the factorization is locally orthogonal.

   (ii) $\Rightarrow$ (iii) : If every cone in $\underline{A}$ admits a locally orthogonal $\underline{E}$-factorization, then $\underline{E}$ necessarily consists of epimorphisms only (cf. [12], Corollary 6.4, [2], Lemma 1). Therefore, every $\underline{E}$-factorization is in particular rigid.

   (iii) $\Rightarrow$ (i) : As in [12] one first shows P to be faithful. Thus, given (D, ξ, A) with $D : \underline{D} \to \underline{A}$, we form the full subcategory $\tilde{D}$ of the comma-category $(A \downarrow \underline{A})$ consisting of all objects (f, C) with $f : A \to C$ in $\underline{A}$ such that there is a cocone $\gamma : D \to \Delta C$ with $P \gamma = \Delta P f . \xi$. One gets a cone

$a: \Delta A \to \tilde{D}$ with $a(f, C) = f$ and $\tilde{D}: \underline{\tilde{D}} \to \underline{A}$ being the projection functor. We now consider a rigid and P-semiinitial factorization $a = \mu . \Delta e$ and have to show that $e: A \to B$ is a P-lifter of (D, ξ, A). But this can be done analogously to the proof of [12], Theorem 3.1. ||

Analyzing the equivalence (i) $\Leftrightarrow$ (iii) one obtains immediately:

COROLLARY. *P is pretopological iff for every cone $a: \Delta A \to D$ in $\underline{A}$, there is a P-initial cone $\mu: \Delta B \to D$ and an $\underline{A}$-morphism $e: A \to B$ such that $a = \mu . \Delta e$ and $P e$ is an $\underline{X}$-isomorphism.*

REMARKS. 1. From Theorem 1 it follows that $\underline{A}$ is $\underline{E}$-cocomplete for $\underline{E}$ the class of all P-lifters, provided P is presemitopological. This means that pushouts of $\underline{E}$-morphisms along arbitrary $\underline{A}$-morphisms exist and belong to $\underline{E}$ and that multiple pushouts of (class-indexed) families of $\underline{E}$-morphisms exist and belong to $\underline{E}$ (cf. [12]).

2. It is interesting to look at the meaning of Theorem 1 and the Corollary for the mentioned examples. For instance, by application of the dual of the Corollary one obtains: Given spaces (X_i, d_i), $i \in I$, in Met_1 and mappings $f_i: X_i \to X$, $i \in I$, then there is a largest metric on X making all f_i Met_1-morphisms, provided there is at least one metric on X with this property.

3. As in case of semitopological functors one has also external characterizations (cf. [13]) for presemitopological functors. There is one external characterization arising from the characterization (iii) in Theorem 1, and there is another one which can be obtained directly from the definition and which we mention here without proof:

*P is presemitopological (pretopological) iff for all functors*

$$S: \underline{B} \to \underline{A}, \quad L: \underline{C} \to \underline{A}, \quad K: \underline{C} \to \underline{B}$$

and for every natural transformation $\phi: PL \to PSK$ there exist a functor $F: \underline{B} \to \underline{A}$ and natural transformations $\rho: L \to FK$, $\sigma: S \to F$ (with σ being an isomorphism) such that $P\rho = P\sigma K . \phi$ and the following universal property holds:

For all functors $G : \underline{B} \to \underline{A}$ *and natural transformations*

$$a : L \to GK, \quad \beta : S \to G \quad \text{with} \quad Pa = P\beta K . \phi$$

there exists a unique $\delta : F \to G$ *with* $a = \delta K . \rho$ *and* $\beta = \delta . \sigma$.
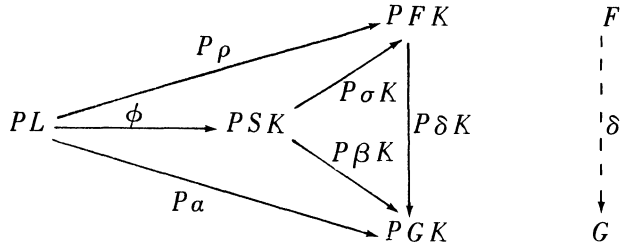
3. THE REPRESENTATION THEOREM.

Every semitopological functor is the composition of a topological functor and a preceding full reflexive embedding. This has been shown in [12], and the construction given there turned out to be the *MacNeille completion* (cf. [6]) of the given semitopological functor. We now show the non-adjoint analogue of this theorem :

THEOREM 2. *The following assertions are equivalent:*

(i) P is presemitopological.

(ii) There is a pretopological functor $T : \underline{B} \to \underline{X}$ *and a full reflexive embedding* $E : \underline{A} \to \underline{B}$ *with* $P = TE$.

(iii) Same as (ii), but, in addition, with E being initially dense (cf. [6]).

PROOF. (ii) $\Rightarrow$ (i) can be easily checked, and (iii) $\Rightarrow$ (ii) is trivial. Hence, (i) $\Rightarrow$ (iii) remains to be shown. The category $\underline{B}$ is constructed as follows: objects are all P-lifters, and a $\underline{B}$-morphism from $e : A \to B$ to $e' : A' \to B'$ is given by an $\underline{X}$-morphism x and an $\underline{A}$-morphism $g : B \to B'$ such that

commutes, composition is horizontal. The functor $T : \underline{B} \to \underline{X}$ sends such a square to x. The functor $R : \underline{B} \to \underline{A}$ which sends this square to g is the reflector of the embedding $E : \underline{A} \to \underline{B}$ which assigns to every $f : A \to A'$ the square

$$
\begin{array}{ccc}
P A & \xrightarrow{\;\; P f \;\;} & P A' \\
\Big\downarrow{\scriptstyle P 1_A} & & \Big\downarrow{\scriptstyle P 1_{A'}} \\
P A & \xrightarrow{\;\; P f \;\;} & P A'
\end{array}
$$

We apply the criterion given in the Corollary of Theorem 1 in order to show that T is pretopological. For this purpose we consider a class-indexed source of $\underline{B}$-morphisms given by squares

$$
\begin{array}{ccc}
P A & \xrightarrow{\;\; x_i \;\;} & P A_i \\
\Big\downarrow{\scriptstyle P e} & & \Big\downarrow{\scriptstyle P e_i} \\
P B & \xrightarrow{\;\; P g_i \;\;} & P B_i
\end{array} \qquad (*)
$$

By Theorem 1, $(\, g_i e : A \to B_i \,)_I$ admits a locally orthogonal and P-semi-initial factorization $g_i\, e = m_i\, \tilde{e}$, $i \in I$. There is a unique morphism t with $t e = \tilde{e}$ and $m_i\, t = g_i$ giving the following factorization of $(*)$ in $\underline{B}$:

$$
\begin{array}{ccccc}
P A & \xrightarrow{\;\; 1_{PA} \;\;} & P A & \xrightarrow{\;\; x_i \;\;} & P A_i \\
\Big\downarrow{\scriptstyle P e} & & \Big\downarrow{\scriptstyle P \tilde{e}} & & \Big\downarrow{\scriptstyle P e_i} \\
P B & \xrightarrow{\;\; P t \;\;} & P C & \xrightarrow{\;\; P m_i \;\;} & P B_i
\end{array}
$$

T maps the left factor on 1_{PA} (which is an isomorphism). The family consisting of the right factors is T-initial; this can be easily proved by application of the Lemma preceding Theorem 1. Finally, E is initially dense, since the reflection morphism

$$
\begin{array}{ccc}
P A & \xrightarrow{\;\; P e \;\;} & P B \\
\Big\downarrow{\scriptstyle P e} & & \Big\downarrow{\scriptstyle P 1_B} \\
P B & \xrightarrow{\;\; P 1_B \;\;} & P B
\end{array}
$$

is T-initial. $\;\|$

REMARK. Like semitopologicity is generalized to presemitopologicity one can analogously generalize the notion of a topologically algebraic functor (cf. $[8, 2, 7]$) : P is called *pretopologically algebraic* iff every cone $a : \Delta A \to D$ in A admits a factorization $a = \mu . \Delta e$ with $e : A \to B$ being an $\underline{A}$-epimorphism and $\mu : \Delta B \to D$ being P-initial. Without giving any details we just mention that there is a characterization of pretopologically algebraic functors which corresponds to a result due to Herrlich and Strecker $[7]$: Topologically algebraic functors are just those functors admitting a «reflexive universal initial completion».

4. LIFTINGS OF HOMFUNCTORS AND TENSORPRODUCTS.

In this section we shall describe how to use the methods developed above to lift homfunctors and tensorproducts.

Let $\underline{V}$ be a symmetric monoidal closed category with inner homfunctor H and tensorproduct $\otimes$. Take $\underline{A}$ to be a tensored $\underline{V}$-category, and let $P : \underline{A} \to \underline{V}$ be a $\underline{V}$-functor. Thus we have a $\underline{V}$-natural transformation

$$\hat{P}_{AB} : \underline{A}(A, B) \to H(PA, PB)$$

and V-adjunctions

$$- \oplus A \xrightarrow[\epsilon^A]{\eta^A} \underline{A}(A, -), \quad - \otimes X \xrightarrow[\beta^X]{a^X} H(X, -)$$

for all $A \epsilon Ob\underline{A}$ and $X \epsilon Ob\underline{V}$; here $\underline{A}(-,-) : \underline{A}^{op} \otimes \underline{A} \to \underline{V}$ denotes the external homfunctor (cf. $[3, 9]$). In the following we are mainly interested in the underlying situation, i. e. we consider the underlying functors

$$P : \underline{A}_0 \to \underline{V}_0, \quad \underline{A}(-,-) : \underline{A}_0^{op} \times \underline{A}_0 \to \underline{V}_0 \quad \text{and} \quad - \oplus - : \underline{V}_0 \times \underline{A}_0 \to \underline{A}_0$$

(there will be no confusion by using the same notations for the $\underline{V}$-data and the underlying functors). Furthermore there is a natural transformation $\tilde{P}_{XA} : X \otimes PA \to P(X \oplus A)$ defined by the following diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;a_X^{PA}\;\;} & H(PA, X \otimes PA) \\
{\scriptstyle \eta_X^A} \downarrow & & \downarrow {\scriptstyle H(PA, \tilde{P}_{XA})} \\
\underline{A}(A, X \oplus A) & \xrightarrow[\hat{P}_{A(X \oplus A)}]{} & H(PA, P(X \oplus A))
\end{array}
$$

Obviously, $\tilde{P}_{XA}$ is a P-epimorphism (cf. [12]) if P is $\underline{V}$-faithful, i.e. if $\hat{P}_{BC}$ is a monomorphism for all B, $C \in Ob\underline{A}$.

For the rest of the paper, we assume P to be $\underline{V}$-faithful and fix a full subcategory $\underline{S}_0$ of $\underline{A}_0$ and functors

$$-\boxtimes-: \underline{A}_0 \times \underline{S}_0 \to \underline{A}_0, \quad h: \underline{S}_0^{OP} \times \underline{A}_0 \to \underline{A}_0$$

such that, for every $S \in Ob\underline{S}_0$, there is an adjunction

$$-\boxtimes S \xrightarrow[\delta^S]{\gamma^S} h(S,-)$$

satisfying the following compatibility conditions (naturally for all $S \in Ob\underline{S}_0$, $A \in Ob\underline{A}_0$):

$$Ph(S,A) = \underline{A}(S,A), \quad P(A \boxtimes S) = PA \otimes PS,$$

$$a_{PA}^{PS} = \hat{P}_{S(A \boxtimes S)} \cdot P\gamma_A^S, \quad P\epsilon_A^S \cdot \tilde{P}_{\underline{A}(S,A)S} = P\delta_A^S.$$

The following Theorem describes how to extend the «partial functors» $-\boxtimes-$, h to functors with domain $\underline{A}_0 \times \underline{A}_0$, $\underline{A}_0^{OP} \times \underline{A}_0$ resp.:

THEOREM 3. *(1) Assume that, for all $A, B \in Ob\underline{A}_0$, the source*

$$(\underline{A}(A,B) \xrightarrow{\underline{A}(f,B)} Ph(S,B))_{S \in Ob\underline{S}_0, f \in \underline{A}(S,A)}$$

has a P-cosemifinal lifting (= dual of P-semifinal lifting; cf. [12]), i.e. there is an object $h(A,B)$ in $\underline{A}_0$, a morphism $e_{AB}: Ph(A,B) \to \underline{A}(A,B)$ in $\underline{V}_0$, and a source

$$(h(A,B) \xrightarrow{h(f,B)} h(S,B))_{S \in Ob\underline{S}_0, f \in \underline{A}(S,A)}$$

in $\underline{A}_0$ such that the diagram

commutes and fulfills the obvious couniversal property. Then these data can be chosen in such a way that h can be uniquely extended to a functor $h: \underline{A}_0^{OP} \times \underline{A}_0 \to \underline{A}_0$ making the morphisms e_{AB} a natural transformation $e: Ph(-,-) \to \underline{A}(-,-)$.

(2) Assume that, for all A, $B \in Ob\underline{A}_0$, *the sink*

$$(P(A \boxtimes S) \xrightarrow{PA \otimes Pf} PA \otimes PB \xrightarrow{\tilde{P}_{(PA)B}} P(PA \oplus B))_{S \in Ob\underline{S}_0, f \in \underline{A}(S,B)}$$

admits a P-lifter, i.e. there is an object $A \boxtimes B$ *in* $\underline{A}_0$, *a morphism* d_{AB}:
$PA \oplus B \to A \boxtimes B$ *in* $\underline{A}_0$, *and a sink*

$$(A \boxtimes S \xrightarrow{A \boxtimes f} A \boxtimes B)_{S \in Ob\underline{S}_0, f \in \underline{A}(S,B)}$$

in $\underline{A}_0$, *such that the diagram*

commutes and fulfills the obvious universal property. Then these data can be chosen in such a way that $\boxtimes$ *can be uniquely extended to a functor* $\boxtimes: \underline{A}_0 \times \underline{A}_0 \to \underline{A}_0$ *making the morphisms* d_{AB} *a natural transformation* $d: ((P-)\oplus -) \to (-\boxtimes -)$.

(3) Under the assumptions of (1), (2), $-\boxtimes-$ *is left adjoint to* h.
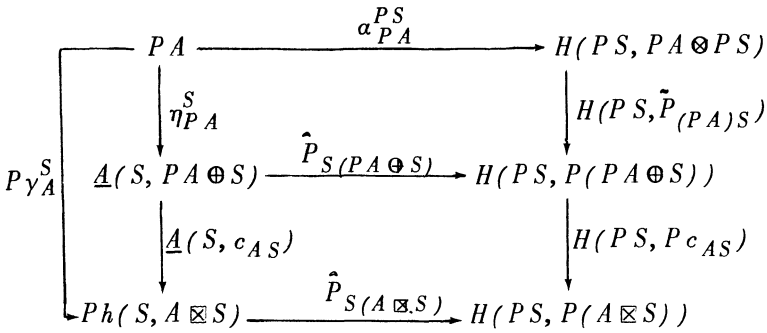
REMARK. If P is cosemitopological and presemitopological, then the assumptions of assertions (1) and (2) of Theorem 3 are fulfilled. This holds in particular if P is topological (cf. [12]) or if P is the inclusion functor of a full coreflective subcategory; in both cases, d is a natural equivalence. For P being topological, e is a natural equivalence, too, whereas e consists of coreflection maps, if P is a full coreflective embedding.

PROOF (of Theorem 3). (1) is straightforward.

(2) The only remarkable point is to show that the P-lifters d_{AB} can be chosen in a way that the construction described in the theorem really leads to an extension of the given functor $\boxtimes: \underline{A}_0 \times \underline{S}_0 \to \underline{A}_0$. To see this, for $S \in Ob\underline{S}_0$, $A \in Ob\underline{A}_0$ consider the unique $\underline{A}_0$-morphism

$$c_{AS}: PA \oplus S \to A \boxtimes S \text{ with } \underline{A}(S, c_{AS})\eta_{PA}^S = P\gamma_A^S.$$

By the commutativity of the following diagram one gets $Pc_{AS} \cdot \tilde{P}_{(PA)S} = 1$:

$$\begin{array}{ccc}
PA & \xrightarrow{\ \alpha^{PS}_{PA}\ } & H(PS, PA \otimes PS) \\
\Big\downarrow{\eta^S_{PA}} & & \Big\downarrow{H(PS,\tilde{P}_{(PA)S})} \\
\underline{A}(S, PA \oplus S) & \xrightarrow{\ \hat{P}_{S(PA \oplus S)}\ } & H(PS, P(PA \oplus S)) \\
\Big\downarrow{\underline{A}(S, c_{AS})} & & \Big\downarrow{H(PS, Pc_{AS})} \\
Ph(S, A \boxtimes S) & \xrightarrow{\ \hat{P}_{S(A \boxtimes S)}\ } & H(PS, P(A \boxtimes S))
\end{array}$$

with $P\gamma^S_A$ on the left side.

Now one can see that c_{AS} is a P-lifter of the sink

$$(P(A \boxtimes S') \xrightarrow{\ \tilde{P}_{(PA)S} \cdot (PA \otimes Pf)\ } P(PA \oplus S))_{S' \epsilon\, Ob\,\underline{S}_0,\, f \epsilon\, \underline{A}(S', S)} .$$

Hence one can choose d_{AS} to be c_{AS}, and this shows what we wanted to show.

(3) The construction of the unit κ^A_B for the adjunction $\boxtimes \dashv h$ is done by the following diagram using the couniversal property of a P-cosemifinal lifting:

$$\text{(commutative diagram)}$$
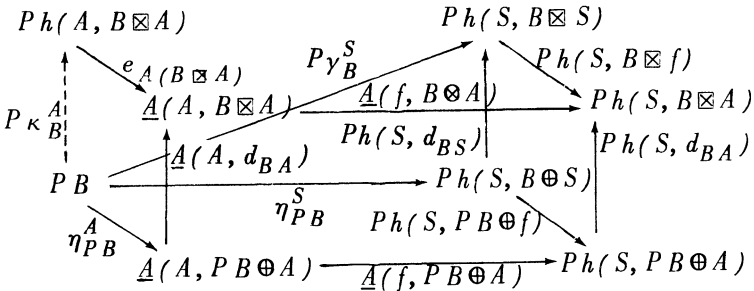
The counit λ^A_B is similarly obtained from the following diagram:

$$\text{(commutative diagram)}$$
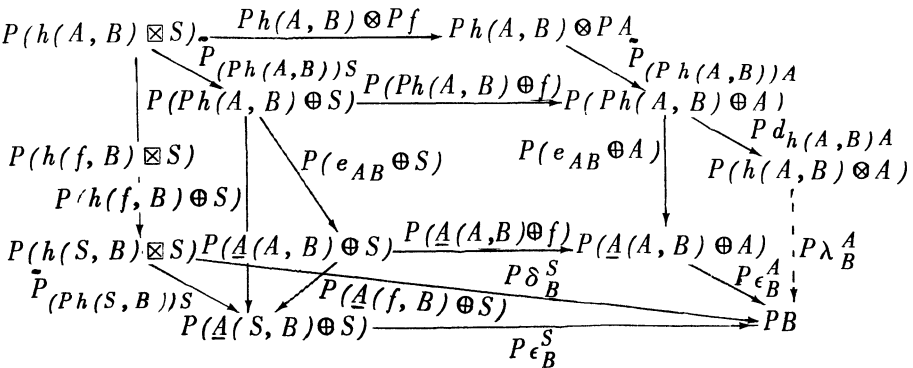
By complicated, but straigthforward diagram chasing it can be shown that κ_B^A and λ_B^A are natural in B and extraordinary natural in A. Moreover, one has the equations

$$h(A, \lambda_B^A) \kappa_{h(A,B)}^A = 1 \quad \text{and} \quad \lambda_{B \boxtimes A}^A (\kappa_B^A \boxtimes A) = 1,$$

hence h is right adjoint to $-\boxtimes-$. ∥

EXAMPLES. 1. We first mention one trivial application of Theorem 3: Take $\underline{V}$ to be symmetric monoidal closed, and let $\underline{A}$ be a full reflective subcategory of $\underline{V}$ such that $H(A, B) \in \underline{A}$ for all $A, B \in Ob \underline{A}$. Taking $\underline{S} = \emptyset$ we obtain $R(-\otimes-)$ to be left adjoint to $H: \underline{A}^{op} \times \underline{A} \to \underline{A}$, R being the reflector.

2. Let $P: Top \to Ens$ be the forgetful functor from topological spaces to sets; P is topological, and Top is a tensored Ens-category. Let $\underline{S}$ be a class of compact Hausdorff spaces (considered as a full subcategory of Top). For all $X, Y \in Top$ and $S \in \underline{S}$ one has the exponential law

$$Top(X \pi S, Y) \approx Top(X, C_{co}(S, Y))$$

where «co» stands for «compact open». Therefore,

$$C_{co}(-,-): \underline{S}^{op} \times Top \to Top$$

is left adjoint to $-\pi-: Top \times \underline{S} \to Top$. All compatibility conditions needed for Theorem 3 are fulfilled. Hence we get a function space functor

$$h: Top^{op} \times Top \to Top$$

which is right adjoint to a tensorproduct $\boxtimes: Top \times Top \to Top$, and

$$h(S, X) = C_{co}(S, X), \quad X \boxtimes S = X \pi S$$

hold for all $X \in Top$ and $S \in \underline{S}$.

3. Theorem 3 can be modified by requiring

$$Ph(S, A) = H(PS, PA), \quad \alpha_{PA}^{PS} = P\gamma_A^S \quad \text{and} \quad \beta_{PA}^{PS} = P\delta_A^S.$$

A P-cosemifinal lifting of the source

$$(A(A, B) \xrightarrow{\hat{P}_{AB}} H(PA, PB) \xrightarrow{H(f, PB)} Ph(S, B))_{S \in Ob \underline{S}_o, f \in \underline{V}_o(PS, PA)}$$

resp. a P-lifter of the sink

$$(PA \otimes PS \xrightarrow{PA \otimes f} PA \otimes PB \xrightarrow{\tilde{P}_{(PA)B}} P(PA \oplus B))_{S \in Ob \underline{S}_o, f \in \underline{V}_o(PS, PB)}$$

again yields a pair of adjoint functors

$$h: \underline{A}_0^{op} \times \underline{A}_0 \to \underline{A}_0 \quad \text{and} \quad \boxtimes: \underline{A}_0 \times \underline{A}_0 \to \underline{A}_0.$$

The following example is an application of this situation (up to canonical bijections).

Consider the forgetful functor $P: Conn_* \to Ens$ with $Conn_*$ being the category of pointed connected spaces. $Conn_*$ is a tensored Ens-category. We take $\underline{S}$ to be the subcategory containing only the one-element space. The functors

$$\boxtimes: Conn_* \times \underline{S} \to Conn_* \quad \text{and} \quad h: \underline{S}^{op} \times Conn_* \to Conn_*$$

are (isomorphic) projections. For (X, x_0), $(Y, y_0) \in Ob\,Conn_*$ we provide $Conn_*((X, x_0), (Y, y_0))$ with the initial topology with respect to all mappings

$$e_x: Conn_*((X, x_0), (Y, y_0)) \to P(Y, y_0), \quad f \mapsto f(x).$$

Let $h((X, x_0), (Y, y_0))$ be the component of the constant morphism in $Conn_*((X, x_0), (Y, y_0))$; then the inclusion mapping yields the cosemifinal factorization needed in the theorem.

Next we have to construct a P-lifter of the sink of all mappings

$$j_y: P(X, x_0) \to P(\amalg_X(Y, y_0)), \quad x \mapsto i_x(y),$$

where i_x denotes the canonical injection of the copower taken in $Conn_*$. For this, on $X \times Y$, we consider the topology of separate continuity and identify the points x_0 and y_0. So we get a connected pointed space $(X, x_0) \boxtimes (Y, y_0)$ and the desired P-lifter

$$d_{X,Y}: \amalg_X(Y, y_0) \to (X, x_0) \boxtimes (Y, y_0).$$

In $Conn_*$ we therefore have a structure of pointwise convergence and separate continuity.

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Georg GREVE & Walter THOLEN :
Fernuniversität, Fachbereich Mathematik
Postfach 940
D-5800 HAGEN. R.F.A.

Jenö SZIGETI:
Mathematical Institute of the Hungarian Academy of Sciences
Realtanoda u. 13-15
H-1053 BUDAPEST, V.   HONGRIE