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ON A THEOREM OF BAUMSLAG, DYER AND HELLER LINKING GROUP THEORY AND TOPOLOGY

by Aristide DELEANU

The object of this paper is to construct a direct proof for the following recent result of G. Baumslag, E. Dyer and A. Heller [1] showing that homotopy theory can be reconstructed within group theory:

THEOREM 1. *The category of pointed, connected CW-complexes and pointed homotopy classes of maps is equivalent to a category of fractions of the category of pairs* (G, P), *where* G *is a group and* P *is a perfect normal subgroup.*

The proof is direct, and somewhat elementary and self-contained, in that it uses as ingredients only the result of Kan and Thurston that every connected space has the homology of an aspherical space, the «plus construction» of Quillen, and some standard facts of algebraic topology and category theory. The original proof of the theorem, given in the ample paper [1], is based on a rather elaborate machinery, involving some general notions of realizations of abstract simplicial complexes, as well as explicit use of deep considerations of group theory, such as the fact that any group can be imbedded in an acyclic group; this machinery is also used to obtain a strengthened version of the Kan-Thurston result, which is not assumed in [1].

We first recall the details of the ingredients mentioned above. A space X is said to be *aspherical* if it is pathwise connected and $\pi_i X = 0$ for $i \geq 2$. A group G is said to be *perfect* if $G = [G, G]$ or, equivalently, if $H_1(G) = 0$.

The following result is due to D.M. Kan and W.P. Thurston [6]:

THEOREM 2. *For every pointed connected CW-complex* X *there exists a*

Serre fibration $t_X : TX \to X$ *which is natural with respect to* X *and has the following properties :*

(i) The map t_X *induces an isomorphism of singular homology and cohomology with local coefficients*

$$H_*(TX;A) \approx H_*(X;A), \quad H^*(TX;A) \approx H^*(X;A)$$

for every $\pi_1 X$*-module* A.

(ii) TX *is aspherical, the sequence*

$$1 \longrightarrow Ker\, \pi_1 t_X \longrightarrow \pi_1 TX \xrightarrow{\pi_1 t_X} \pi_1 X \longrightarrow 1$$

is exact, and the group $Ker\, \pi_1 t_X$ *is perfect.*

The following result is due to D. Quillen [8], but is stated in an explicit manner by J. B. Wagoner in [10] :

THEOREM 3. *For every pointed CW-complex* X *and for every perfect normal subgroup* P *of* $\pi_1 X$ *there exist a CW-complex* $(X, P)^+$ *and a cofibration* $i_{(X,P)} : X \to (X,P)^+$ *which have the following properties :*

(i) The sequence

$$1 \longrightarrow P \longrightarrow \pi_1 X \xrightarrow{\pi_1 i_{(X,P)}} \pi_1 (X,P)^+ \longrightarrow 1$$

is exact.

(ii) The map $i_{(X,P)}$ *induces an isomorphism of singular homology with local coefficients*

$$H_*(X;A) \approx H_*((X,P)^+;A)$$

for every $\pi_1 (X,P)^+$*-module* A.

(iii) If $f: X \to Y$ *is a map such that* $\pi_1 f(P) = 1$, *then there exists a map* $\tilde{f}$ *unique up to homotopy, which gives a homotopy commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{i_{(X,P)}} & (X,P)^+ \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \tilde{f}} & \\ & Y & \end{array}$$

We will also need the following generalization of the classical theorem of J.H.C. Whitehead, which can be proved by using universal covering spaces and the Serre spectral sequence:

THEOREM 4. *Let $f: X \to Y$ be a map of pointed connected CW-complexes such that*

(i) $\pi_1 f: \pi_1 X \xrightarrow{\approx} \pi_1 Y$, and

(ii) $H_* f: H_*(X;A) \xrightarrow{\approx} H_*(Y;A)$ for every $\pi_1 Y$-module A.

Then f is a homotopy equivalence.

Finally, we will also use the following result, whose proof follows straightforwardly from the definition of a category of fractions (see [4]; if Δ is a family of morphisms of a category $\mathfrak{A}$, then $\mathfrak{A}[\Delta^{-1}]$ denotes the category of fractions of $\mathfrak{A}$ with respect to Δ, and Q_Δ is the canonical functor from $\mathfrak{A}$ to $\mathfrak{A}[\Delta^{-1}]$).

LEMMA 5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be categories, Δ and Ω families of morphisms of $\mathfrak{A}$ and $\mathfrak{B}$, respectively, and $U, V: \mathfrak{A} \to \mathfrak{B}$ functors with the property that $U(\Delta) \subset \Omega$, $V(\Delta) \subset \Omega$, so that there exist induced functors $\bar{U}$, $\bar{V}$ rendering the following diagrams commutative:*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{U} & \mathfrak{B} \\ {\scriptstyle Q_\Delta}\downarrow & & \downarrow{\scriptstyle Q_\Omega} \\ \mathfrak{A}[\Delta^{-1}] & \xrightarrow{\bar{U}} & \mathfrak{B}[\Omega^{-1}] \end{array} \qquad \begin{array}{ccc} \mathfrak{A} & \xrightarrow{V} & \mathfrak{B} \\ {\scriptstyle Q_\Delta}\downarrow & & \downarrow{\scriptstyle Q_\Omega} \\ \mathfrak{A}[\Delta^{-1}] & \xrightarrow{\bar{V}} & \mathfrak{B}[\Omega^{-1}] \end{array}.$$

Then every natural transformation $\epsilon: U \to V$ induces a natural transformation $\bar{\epsilon}: \bar{U} \to \bar{V}$ defined by $\bar{\epsilon}_A = Q_\Omega(\epsilon_A)$ for each $A \in |\mathfrak{A}|$.

We now proceed with the

PROOF OF THEOREM 1. Consider the following diagram of categories and functors:

$$\mathcal{H}o\mathcal{CW} \underset{(\ )^+}{\overset{J}{\rightleftarrows}} \mathcal{H}o\mathcal{C} \xleftarrow{C} \mathcal{C} \underset{I}{\overset{T}{\rightleftarrows}} \mathcal{K} \xrightarrow{C} \mathcal{H}o\mathcal{K} \underset{B}{\overset{\pi}{\rightleftarrows}} \mathcal{GP},$$

in which the cast of characters is as follows:

$\mathcal{H}o\mathcal{CW}$ is the category whose objects are pointed connected CW-complexes and whose morphisms are pointed homotopy classes of maps;

$\mathcal{C}$ is the category whose objects are pairs (X, P), where X is a pointed connected CW-complex, and P is a perfect normal subgroup of $\pi_1 X$, and whose morphisms $f:(X,P)\to(X',P')$ are pointed continuous maps $f: X\to X'$ such that $\pi_1 f(P)\subset P'$;

$\mathcal{K}$ is the full subcategory of $\mathcal{C}$ whose objects are pairs (X,P) where X is an aspherical CW-complex;

$\mathcal{H}o\,\mathcal{C}$ (respectively $\mathcal{H}o\,\mathcal{K}$) is the category having the same objects as $\mathcal{C}$ (resp. $\mathcal{K}$), but whose morphisms are homotopy classes of morphisms of $\mathcal{C}$ (resp. $\mathcal{K}$);

$\mathcal{G}\mathcal{P}$ is the category whose objects are pairs (G,P), where G is a group and P is a perfect normal subgroup of G, and whose morphisms $g:(G,P)\to(G',P')$ are homomorphisms $g: G\to G'$ such that $g(P)\subset P'$;

J is the full embedding of $\mathcal{H}o\,\mathcal{C}\mathcal{W}$ into $\mathcal{H}o\,\mathcal{C}$, which sends each $X\in|\mathcal{H}o\,\mathcal{C}\mathcal{W}|$ onto $(X,1)\in|\mathcal{H}o\,\mathcal{C}|$;

$(\ )^+$ is the functor which associates with each $(X,P)\in|\mathcal{H}o\,\mathcal{C}|$ the space $(X,P)^+\in|\mathcal{H}o\,\mathcal{C}\mathcal{W}|$ provided by Theorem 3, and which is defined on morphisms by using Theorem 3 (iii) (one sees immediately that, if one identifies $\mathcal{H}o\,\mathcal{C}\mathcal{W}$ under the embedding J with a full subcategory of $\mathcal{H}o\,\mathcal{C}$, then $(\ )^+$ yields a reflection of $\mathcal{H}o\,\mathcal{C}$ onto $\mathcal{H}o\,\mathcal{C}\mathcal{W}$);

C denotes the functors which are the identities on objects and send each map to its homotopy class;

I is the full embedding of $\mathcal{K}$ into $\mathcal{C}$;

π is the functor which carries each $(X,P)\in|\mathcal{H}o\,\mathcal{K}|$ onto $(\pi_1 X, P)$ in $|\mathcal{G}\mathcal{P}|$;

B is the classifying space functor [7], which assigns to each (G,P) in $|\mathcal{G}\mathcal{P}|$ the pair $(K(G,1),P)\in|\mathcal{H}o\,\mathcal{K}|$;

T is the functor which sends each $(X,P)\in|\mathcal{C}|$ onto

$$(TX, (\pi_1 t_X)^{-1}(P))\in|\mathcal{K}|,$$

where the space TX and the map t_X are provided by Theorem 2. (TX is an aspherical space by Theorem 2 (ii), and to verify that $(\pi_1 t_X)^{-1}(P)$ is perfect, it is sufficient to consider a portion of the 5-term exact sequence [5, page 203]

$$Z\otimes_P H_1(Ker\,\pi_1 t_X) \longrightarrow H_1((\pi_1 t_X)^{-1}(P)) \xrightarrow{H_1(\pi_1 t_X)} H_1(P) \longrightarrow 0$$

associated with the short exact sequence

$$1 \longrightarrow Ker\,\pi_1 t \xrightarrow{t_X} (\pi_1 t_X)^{-1}(P) \xrightarrow{\pi_1 t_X} P \longrightarrow 1$$

and to observe that, by Theorem 2 (ii), $Ker\,\pi_1 t_X$ is perfect.)

This concludes the description of the categories and functors involved in the above diagram. We now introduce five families of morphisms as follows:

Ψ will denote the family of the morphisms $f: (X,P) \to (X',P')$ of $\mathcal{C}$ such that

$$\pi_1 f: \pi_1 X/P \xrightarrow{\approx} \pi_1 X'/P' \quad \text{and} \quad H_* f: H_*(X;A) \xrightarrow{\approx} H_*(X';A)$$

for every $\pi_1 X'/P'$-module A ;

Φ will denote the subfamily of Ψ consisting of all those elements which belong to the full subcategory $\mathcal{K}$;

Σ (respectively Λ) will denote the family of morphisms of $\mathcal{H}o\mathcal{C}$ (resp. $\mathcal{H}o\mathcal{K}$) which is the image of Ψ (resp. Φ) under the functor C ;

Γ will denote the family of the morphisms $g: (G,P) \to (G',P')$ of $\mathcal{G}\mathcal{P}$ such that

$$g: G/P \xrightarrow{\approx} G'/P' \quad \text{and} \quad H_* g: H_*(G;A) \xrightarrow{\approx} H_*(G';A)$$

for every G'/P'-module A .

Now, we observe that Σ coincides with the family of all those morphisms of $\mathcal{H}o\mathcal{C}$ which are carried into isomorphisms by the functor $(\ )^+$. This follows by utilizing Theorem 4 from the commutative diagrams induced by a map $f: (X,P) \to (X',P')$

$$\begin{array}{ccc} \pi_1 X/P & \xrightarrow{\pi_1 f} & \pi_1 X'/P' \\ {\scriptstyle \pi_1 i_{(X,P)}} \downarrow \approx & & \approx \downarrow {\scriptstyle \pi_1 i_{(X',P')}} \\ \pi_1(X,P)^+ & \xrightarrow{\pi_1 f^+} & \pi_1(X',P')^+ \end{array}$$

and

$$\begin{array}{ccc} H_*(X;A) & \xrightarrow{H_*f} & H_*(X';A) \\ {\scriptstyle H_*i_{(X,P)}}\downarrow\approx & & \approx\downarrow{\scriptstyle H_*i_{(X',P')}} \\ H_*((X,P)^+;A) & \xrightarrow{H_*f^+} & H_*((X',P')^+;A), \end{array}$$

where A is an arbitrary $\pi_1 X'/P'$-module and the vertical arrows are isomorphisms by Theorem 3 (i), (ii). Thus, according to Proposition 1.3 on page 7 of [4], the unique functor S which makes the diagram

$$\begin{array}{ccc} \mathcal{H}_o\mathcal{C}\mathcal{W} & \xleftarrow{(\ )^+} & \mathcal{H}_o\mathcal{C} \\ & {\scriptstyle S}\nwarrow & \downarrow{\scriptstyle Q_\Sigma} \\ & & \mathcal{H}_o\mathcal{C}[\Sigma^{-1}] \end{array}$$

commutative is an equivalence of categories.

Next, we note that the functor T carries elements of Ψ into elements of Φ. This follows from the commutative diagrams induced by the map $f:(X,P)\to(X',P')$

$$\begin{array}{lccc} (1) & \pi_1 TX/(\pi_1 t_X)^{-1}(P) & \xrightarrow{\pi_1 Tf} & \pi_1 TX'/(\pi_1 t_{X'})^{-1}(P') \\ & {\scriptstyle \pi_1 t_X}\downarrow\approx & & \approx\downarrow{\scriptstyle \pi_1 t_{X'}} \\ & \pi_1 X/P & \xrightarrow{\pi_1 f} & \pi_1 X'/P' \end{array}$$

and

$$\begin{array}{lccc} (2) & H_*(TX;A) & \xrightarrow{H_*Tf} & H_*(TX';A) \\ & {\scriptstyle H_*t_X}\downarrow\approx & & \approx\downarrow{\scriptstyle H_*t_{X'}} \\ & H_*(X;A) & \xrightarrow{H_*f} & H_*(X';A)\ , \end{array}$$

where A is an arbitrary $\pi_1 X'/P'$-module and the vertical arrows are isomorphisms by the «third isomorphism theorem» and by Theorem 2 (i). Thus T induces a functor $\bar{T}$ rendering commutative the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{T} & \mathcal{K} \\ {\scriptstyle Q_\Psi}\downarrow & & \downarrow{\scriptstyle Q_\Phi} \\ \mathcal{C}[\Psi^{-1}] & \xrightarrow{\bar{T}} & \mathcal{K}[\Phi^{-1}] \end{array}$$

Since obviously I carries elements of Φ into elements of Ψ, it induces a functor $\bar{I}$ rendering commutative the diagram

$$\begin{array}{ccc} \mathcal{C} & \xleftarrow{\quad I \quad} & \mathcal{K} \\ {\scriptstyle Q_\Psi}\downarrow & & \downarrow{\scriptstyle Q_\Phi} \\ \mathcal{C}[\Psi^{-1}] & \xleftarrow{\quad \bar{I} \quad} & \mathcal{K}[\Phi^{-1}] \end{array} .$$

Now, define a natural transformation $\psi : IT \to 1_{\mathcal{C}}$ by

$$\psi_{(X,P)} = t_X : (TX, (\pi_1 t_X)^{-1}(P)) \to (X, P)$$

for every $(X, P) \in |\mathcal{C}|$, and a natural transformation $\phi : TI \to 1_{\mathcal{K}}$ by

$$\phi_{(X,P)} = t_X : (TX, (\pi_1 t_X)^{-1}(P)) \to (X, P)$$

for every $(X, P) \in |\mathcal{K}|$. By Lemma 5, ψ and ϕ induce natural transformations

$$\bar{\psi} : \bar{I}\bar{T} \to 1_{\mathcal{C}[\Psi^{-1}]}, \quad \bar{\phi} : \bar{T}\bar{I} \to 1_{\mathcal{K}[\Phi^{-1}]}$$

defined by

$$\bar{\psi}_{(X,P)} = Q_\Psi(\psi_{(X,P)}) \quad \text{for every } (X, P) \in |\mathcal{C}|,$$

$$\bar{\phi}_{(X,P)} = Q_\Phi(\phi_{(X,P)}) \quad \text{for every } (X, P) \in |\mathcal{K}| .$$

Inspection of the vertical arrows in diagrams (1) and (2) shows that $\psi_{(X,P)}$ (respectively $\phi_{(X,P)}$) belongs to Ψ (resp. Φ) for every $(X, P) \in |\mathcal{C}|$ (resp. $(X, P) \in |\mathcal{K}|$). Therefore, $\bar{\psi}_{(X,P)}$ (resp. $\bar{\phi}_{(X,P)}$) is an isomorphism for every $(X, P) \in |\mathcal{C}[\Psi^{-1}]|$ (resp. $(X, P) \in |\mathcal{K}[\Phi^{-1}]|$) so that $\bar{\psi}$ and $\bar{\phi}$ are natural equivalences. In conclusion $\bar{T}$ and $\bar{I}$ are equivalences of categories.

Now by the definition of Σ, the functor C carries elements of Ψ into elements of Σ; thus, there exists a functor $\bar{C}$ making commutative the diagram

$$\begin{array}{ccc} \mathcal{H}_o\mathcal{C} & \xleftarrow{\quad C \quad} & \mathcal{C} \\ {\scriptstyle Q_\Sigma}\downarrow & & \downarrow{\scriptstyle Q_\Psi} \\ \mathcal{H}_o\mathcal{C}[\Sigma^{-1}] & \xleftarrow{\quad \bar{C} \quad} & \mathcal{C}[\Psi^{-1}] \end{array} .$$

Furthermore, since every functor which carries homotopy equivalences to isomorphisms carries homotopic maps to the same map ([2], page 244), there exists a functor F such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{H}_o\mathcal{C} & \xleftarrow{\quad C \quad} & \mathcal{C} \\ & \searrow^{F} & \downarrow{\scriptstyle Q_\Psi} \\ & & \mathcal{C}[\Psi^{-1}] \end{array} .$$

Moreover, since F carries the elements of Σ into isomorphisms, there exists a functor E such that the diagram

$$\begin{array}{ccc} \mathcal{H}_o\mathcal{C} & & \\ {\scriptstyle Q_\Sigma}\downarrow & \searrow^{F} & \\ \mathcal{H}_o\mathcal{C}[\Sigma^{-1}] & \xrightarrow{\quad E \quad} & \mathcal{C}[\Psi^{-1}] \end{array}$$

is commutative. It is easy to see that

$$E\bar{C} = 1_{\mathcal{C}[\Psi^{-1}]} \quad \text{and} \quad \bar{C}E = 1_{\mathcal{H}_o\mathcal{C}[\Sigma^{-1}]},$$

so that E and $\bar{C}$ are isomorphisms of categories.

In an entirely analogous manner, one can construct a commutative diagram

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\quad C \quad} & \mathcal{H}_o\mathcal{K} \\ {\scriptstyle Q_\Phi}\downarrow & \swarrow_{G} & \downarrow{\scriptstyle Q_\Lambda} \\ \mathcal{K}[\Phi^{-1}] & \underset{L}{\overset{\bar{C}}{\rightleftarrows}} & \mathcal{H}_o\mathcal{K}[\Lambda^{-1}] \end{array},$$

in which L and $\bar{C}$ are isomorphisms of categories.

Finally, it is evident that the functor π (respectively B) carries elements of Λ (resp. Γ) into elements of Γ (resp. Λ). Thus, there exist functors $\bar{\pi}$ and $\bar{B}$ rendering commutative the following diagrams:

$$\begin{array}{ccc} \mathcal{H}_o\mathcal{K} & \xrightarrow{\quad \pi \quad} & \mathcal{G}\mathcal{P} \\ {\scriptstyle Q_\Lambda}\downarrow & & \downarrow{\scriptstyle Q_\Gamma} \\ \mathcal{H}_o\mathcal{K}[\Lambda^{-1}] & \xrightarrow{\quad \bar{\pi} \quad} & \mathcal{G}\mathcal{P}[\Gamma^{-1}] \end{array} \qquad \begin{array}{ccc} \mathcal{H}_o\mathcal{K} & \xleftarrow{\quad B \quad} & \mathcal{G}\mathcal{P} \\ {\scriptstyle Q_\Lambda}\downarrow & & \downarrow{\scriptstyle Q_\Gamma} \\ \mathcal{H}_o\mathcal{K}[\Lambda^{-1}] & \xleftarrow{\quad \bar{B} \quad} & \mathcal{G}\mathcal{P}[\Gamma^{-1}] \end{array} .$$

There clearly exists a natural equivalence $\eta : \pi B \xrightarrow{\approx} 1_{\mathcal{GP}}$; there also exists a natural equivalence $\theta : 1_{\mathcal{H}o\mathcal{K}} \xrightarrow{\approx} B\pi$, as can be seen by using the natural bijection ([9], page 427)

$$\mathcal{H}o\mathcal{CW}(X, Y) \xrightarrow{\approx} Hom(\pi_1 X, \pi_1 Y), \quad X, Y \in |\mathcal{H}o\mathcal{CW}|, \quad Y \text{ aspherical},$$

and defining $\theta_{(X,P)}$, for each $(X, P) \in |\mathcal{H}o\mathcal{K}|$, to be the homotopy class corresponding under this bijection to $1_{\pi_1 X}$ when $(Y, P) = B\pi(X, P)$. According to Lemma 5, η and θ induce natural equivalences

$$\bar{\eta} : \bar{\pi}\bar{B} \xrightarrow{\approx} 1_{\mathcal{GP}[\Gamma^{-1}]}, \quad \bar{\theta} : 1_{\mathcal{H}o\mathcal{K}[\Lambda^{-1}]} \xrightarrow{\approx} \bar{B}\bar{\pi}$$

defined by

$$\bar{\eta}_{(G,P)} = Q_\Gamma(\eta_{(G,P)}) \text{ for every } (G, P) \in |\mathcal{GP}|,$$

$$\bar{\theta}_{(X,P)} = Q_\Lambda(\theta_{(X,P)}) \text{ for every } (X, P) \in |\mathcal{H}o\mathcal{K}| .$$

Thus $\bar{\pi}$ and $\bar{B}$ are equivalences of categories.

To complete the proof of Theorem 1 it is sufficient to observe that the composition $S\bar{C}\bar{I}L\bar{B}$ establishes an equivalence between the categories $\mathcal{H}o\mathcal{CW}$ and $\mathcal{GP}[\Gamma^{-1}]$, and that the composition $\bar{\pi}\bar{C}\bar{T}EQ_\Sigma J$ is its equivalence-inverse.

As a corollary of the above proof, we obtain the following theorem, which is a more precise form of a result stated without proof in [6]:

THEOREM 6. *For every pointed connected CW-complex X there exists a natural homotopy class*

$$\tau_X : (TX, Ker\,\pi_1 t_X)^+ \to X$$

which is an isomorphism in $\mathcal{H}o\mathcal{CW}$ and has the property that the following diagram is commutative in $\mathcal{H}o\mathcal{CW}$:

$$\begin{array}{ccc} TX & \xrightarrow{[i_{(TX,\, Ker\pi_1 t_X)}]} & (TX, Ker\pi_1 t_X)^+ \\ {\scriptstyle [t_X]}\downarrow & \swarrow {\scriptstyle \tau_X} & \\ X & & \end{array}$$

PROOF. The morphism of $\mathcal{C}$

$$t_X : (TX,\ Ker\,\pi_1\, t_X) \to (X, 1)$$

gives rise to the following commutative diagram in $\mathcal{H}o\,\mathcal{C}\mathcal{W}$:

$$\begin{array}{ccc} TX & \xrightarrow{[\,i_{(TX,\,Ker\pi_1 t_X)}]} & (TX,\ Ker\,\pi_1\, t_X)^+ \\ {\scriptstyle [\,t_X]}\downarrow & & \downarrow{\scriptstyle (C t_X)^+} \\ X & \xrightarrow{[\,i_{(X,1)}]} & (X, 1)^+ \end{array} .$$

$i_{(X,1)}$ is clearly the identity map of X, and $(C t_X)^+$ is an isomorphism, since $(C t_X)^+ = SQ_\Sigma\, C(t_X)$ and we have seen above that $C\, t_X$ belongs to Σ. Thus it is sufficient to take $\tau_X = (C\, t_X)^+$. The verification of the naturality of τ_X is a straightforward manipulation of diagrams.

REMARK. One can say that, in a certain sense, the Kan-Thurston construction inverts the Quillen «plus construction». This assertion is justified by Theorem 6 and by the fact that, for each object (X, P) of $\mathcal{H}o\,\mathcal{C}$, there exists a natural isomorphism *in* $\mathcal{H}o\,\mathcal{C}\,[\Sigma^{-1}]$

$$\sigma_{(X,P)} : (T(X,P)^+,\ Ker\,\pi_1\, t_{(X,P)^+}) \approx (X, P).$$

This follows from the diagram in $\mathcal{C}$

$$(X, P) \xrightarrow{i_{(X,P)}} ((X, P)^+, 1) \xleftarrow{t_{(X,P)^+}} (T(X, P)^+,\ Ker\pi_1\, t_{(X,P)^+})$$

and the observation that both $i_{(X,P)}$ and $t_{(X,P)^+}$ belong to Ψ for every $(X, P) \in |\mathcal{C}|$, so that we can take

$$\sigma_{(X,P)} = (Q_\Sigma\, C\, i_{(X,P)})^{-1} \circ (Q_\Sigma\, C\, t_{(X,P)^+}).$$

Another consequence of the proof of Theorem 1 is the following description of the Quillen «plus construction» as a generalized Adams completion in the sense of [3] :

THEOREM 7. *The generalized Adams Σ-completion of every $(X, P) \in |\mathcal{H}o\,\mathcal{C}|$ is $((X, P)^+, 1)$.*

PROOF. We have to show that there exists for every $(X, P) \in |\mathcal{H}o\,\mathcal{C}|$ a natural equivalence of functors from $\mathcal{H}o\,\mathcal{C}$ to sets:

$$\mathcal{H}o\,\mathcal{C}[\Sigma^{-1}](Q_\Sigma -, Q_\Sigma(X, P)) \approx \mathcal{H}o\,\mathcal{C}(-, ((X, P)^+, 1)).$$

But we have seen above that S is an equivalence of categories, and that J is a right-adjoint of $(\ )^+$, so that

$$\mathcal{H}o\,\mathcal{C}[\Sigma^{-1}](Q_\Sigma -, Q_\Sigma(X,P)) \approx \mathcal{H}o\,\mathcal{C}\mathcal{W}(SQ_\Sigma -, SQ_\Sigma(X,P)) =$$
$$= \mathcal{H}o\,\mathcal{C}\mathcal{W}((-)^+, (X,P)^+) \approx \mathcal{H}o\,\mathcal{C}(-, J(X,P)^+) =$$
$$= \mathcal{H}o\,\mathcal{C}(-, ((X,P)^+, 1)).$$

REMARK. By Theorem 3 (i) the map $i_{(X,P)}$ defines a morphism of $\mathcal{C}$

$$i_{(X,P)} : (X,P) \to ((X,P)^+, 1).$$

Then, for every $(X,P) \in |\mathcal{H}o\,\mathcal{C}|$, the canonical morphism from (X,P) to its Σ-completion is $C\,i_{(X,P)}$.

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