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SYNTACTICAL THEORY OF FUNCTORS

by Axel MÖBUS

A logical theory is regarded as a category. This motivates a syntactical definition of a functor between theories. The definition given here covers only those functors in the usual sense that preserve monomorphisms. As a compensation we need less data to describe a functor.

1. FUNCTORS.

If L is a language and Γ is a theory in L, then (L,Γ) is a category in the following sense [MR, Ch. 8]:

The objects of (L,Γ) are equivalence classes of formulae $\phi(\vec{x})$, the morphisms are equivalence classes of functional formulae, i.e. formulae

$$\alpha(\vec{x},\vec{y}): \phi(\vec{x}) \to \psi(\vec{y}),$$

such that

$$\Gamma \vdash \alpha(\vec{x},\vec{y}) \Rightarrow \phi(\vec{x}) \wedge \psi(\vec{y}) \qquad (F1)$$

$$\Gamma \vdash \alpha(\vec{x},\vec{y}) \wedge \alpha(\vec{x},\vec{y}') \Rightarrow \vec{y} = \vec{y}' \qquad (F2)$$

$$\Gamma \vdash \phi(\vec{x}) \Rightarrow \exists \vec{y}: \alpha(\vec{x},\vec{y}) \qquad (F3)$$

(If the ranges of the sequences $\vec{x}$, $\vec{y}$ are not disjoint, the definition has to be modified accordingly.)

A *functor between theories* $(L,\Gamma) \to (L',\Gamma')$ will be given by its operation on formulae and the values it takes at «projections». So a functor $(T,\pi): (L,\Gamma) \to (L',\Gamma')$ consists of:

i) an assignment: to each formula $\psi(\vec{x})$ in L we assign a formula $\{T(\psi(\vec{x}))\}(t_{\vec{x}})$ in one variable in L'. If $type(\vec{x}) = type(\vec{x}')$, then $type(t_{\vec{x}}) = type(t_{\vec{x}'})$ is required. (The sequence $\vec{x}$ may be the empty sequence. In this case $\vec{x} = \vec{x} \equiv true$.)

ii) a family of formulae $\pi^{\vec{x}\vec{y}}_{\vec{x}}(t_{\vec{x}\vec{y}}, t_{\vec{x}})$ in L'.

To increase legibility the arrow indicating a sequence of variables is omitted in the sequel.

$$\{T(\psi(x_1, \dots, x_n, y))\}<t_{x_1}, \dots, t_{x_n}>$$

is to be an abbreviation for

$$\exists t'_{x_1 \dots x_n y}: \{T(\psi(x_1, \dots, x_n, y))\}(t'_{x_1 \dots x_n y}) \wedge \bigwedge_{i=1\dots n} \pi^{x_1 \dots x_n y}_{x_i}(t'_{x_1 \dots x_n y}, t_{x_i}).$$

DEFINITION 1. $(T,\pi):(L,\Gamma) \to (L',\Gamma')$ is a functor if (A1)-(A4) are derivable in Γ' and the deduction rules (R1)-(R3) are valid (i.e., whenever the hypothesis is derivable in Γ, then the conclusion is derivable in Γ').

(A1) $\pi^{xy}_x(t_{xy}, t_x)$ is functional with domain $\{T(x=x \wedge y=y)\}(t_{xy})$ and codomain $\{T(x=x)\}(t_x)$.

(A2) $\{T(x=x)\}(t_x) \Rightarrow \pi^x_x(t_x, t_x)$.

(A3) $\pi^{xyz}_{xy}(t_{xyz}, t_{xy}) \wedge \pi^{xy}_x(t_{xy}, t_x) \Rightarrow \pi^{xyz}_x(t_{xyz}, t_x)$.

(A4) $\{T(\psi(x))\}(t_x) \wedge t_x = t_{x'} \Rightarrow \{T(\psi(x) \wedge x=x')\}<t_x, t_{x'}>$.

(R1) $$\frac{\beta(x,y) \Rightarrow \psi(y)}{\{T(\beta(x,y))\}<t_y> \Rightarrow \{T(\psi(y))\}(t_y)}.$$

(R2) $$\frac{\beta(x,y) \wedge \beta(x,y') \Rightarrow y=y'}{\exists t_x: \{T(\beta(x,y))\}<t_x, t_y> \wedge \{T(\beta(x,y))\}<t_x, t'_y> \Rightarrow t_y = t'_y}.$$

(R3) $$\frac{\text{same hypothesis as in (R2)}}{\{T(\exists y: \beta(x,y))\}(t_x) \Rightarrow \exists t_y: \{T(\beta(x,y))\}<t_x, t_y>}.$$

EXAMPLES.

1) (Q,π) «object representing partial morphisms» [JTT, 1.26]:

$$\{Q(\phi(x))\}(q_x) \equiv (x \in q_x \wedge x' \in q_x) \Rightarrow (\phi(x) \wedge x=x'),$$

$$\pi^{xy}_x(q_{xy}, q_x) \equiv x \in q_x \Leftrightarrow \exists y: (x,y) \in q_{xy}.$$

2) (F,ρ) «filter functor»:

$$\{F(\phi(x))\}(f_x) \equiv \begin{cases} [(x \in A \wedge A \in f_x) \Rightarrow \phi(x)] \\ \wedge [(A \in f_x \wedge B \in f_x) \Rightarrow A \cap B \in f_x] \\ \wedge [A \in f_x \wedge A \subset B \wedge B \subset \{x \mid \phi(x)\} \Rightarrow B \in f_x]. \end{cases}$$

3) Let $\lambda(y)$ be a formula of L. The «comma theory» $(L,\Gamma)/\lambda(c)$ is formed by adding a constant «c» to L and $\lambda(c)$ to the axioms of Γ. If $\Gamma \vdash \alpha(y,z) \Rightarrow \lambda(y) \wedge \kappa(z)$, then α induces a functor

$$\alpha_* = (A, \underline{\alpha}): (L,\Gamma)/\lambda(c) \to (L,\Gamma)/\kappa(d)$$

where

$$\{A(\phi(x,c))\}(a_x) \equiv \exists x \exists y: a_x = (x,y) \wedge \phi(x,y) \wedge \alpha(y,d)$$

and

$$\underline{\alpha}_x^{xv}(a_{xv}, a_x) \equiv \exists x \exists y \exists v: a_{xv} = (x,v,y) \wedge a_x = (x,y) \wedge \alpha(y,d).$$

Another functor

$$\alpha^* = (A^\#, \bar{\alpha}): (L,\Gamma)/\kappa(d) \to (L,\Gamma)/\lambda(c)$$

is defined in the same way with y and z exchanged.

PROPOSITION 1. *Let (T,π) be a functor.*

i) If $\alpha(x,y): \phi(x) \to \psi(y)$ is functional, then

$$\{T(\alpha(x,y))\}\langle t_x, t_y\rangle : \{T(\phi(x))\}(t_x) \to \{T(\psi(y))\}(t_y)$$

is functional.

ii) T preserves composition of functional formulae.

iii) T preserves identities.

iv) T preserves monos (i. e., 1-1 functional formulae).

PROOF.

i) (F1): $\{T(\alpha(x,y))\}\langle t_x, t_y\rangle \Rightarrow \{T(\alpha(x,y))\}\langle t_x\rangle$

$\Rightarrow \{T(\phi(x))\}(t_x)$ (R1).

(F2): this is just an instance of (R2).

(F3): $\phi(x) \Rightarrow \exists y: \alpha(x,y)$. Hence

$$\{T(\phi(x))\}(t_x) \Rightarrow \{T(\exists y: \alpha(x,y))\}(t_x)$$

by (R1). Now apply (R3).

ii) Let $\alpha(x,y)$ and $\beta(x,y)$ be functional. Then

$$\{T(\exists y: \alpha(x,y) \wedge \beta(y,z))\}(t_{xz})$$
$$\Rightarrow \exists t_y: \{T(\alpha(x,y) \wedge \beta(y,z))\} < t_{xz}, t_y > ,$$

i.e., $\exists t_{xyz} \exists t_y \; \{T(a(x,y) \wedge \beta(y,z))\}(t_{xyz}) \wedge \pi(t_{xyz}, t_{xz}) \wedge \pi(t_{xyz}, t_y)$

$$\Rightarrow \exists t_{xyz} \exists t_{xy} \exists t_{yz} \exists t_y \{T(\alpha(x,y) \wedge \beta(y,z))\}(t_{xyz})$$
$$\wedge \pi(t_{xyz}, t_{xz}) \wedge \pi(t_{xyz}, t_y) \wedge \pi(t_{xyz}, t_{xy}) \wedge \pi(t_{xyz}, t_{yz}) \text{ (A 1)}$$
$$\Rightarrow \exists t_{xyz} \exists t_{xy} \exists t_{yz} \exists t_y \{T(\alpha(x,y))\}(t_{xy}) \wedge \{T(\beta(y,z))\}(t_{yz})$$
$$\wedge \pi(t_{xyz}, t_{xz}) \wedge \pi(t_{xyz}, t_y) \wedge \pi(t_{xyz}, t_{xy}) \wedge \pi(t_{xyz}, t_{yz}) \text{ (R 1)}.$$

Therefore

$$\{T(\exists y: \alpha(x,y) \wedge \beta(y,z))\} < t_x, t_z > \qquad (*)$$
$$\Rightarrow \exists t_{xz} \exists t_{xyz} \exists t_{xy} \exists t_{yz} \exists t_y:$$
$$\{T(\alpha(x,y))\}(t_{xy}) \wedge \{T(\beta(y,z))\}(t_{yz}) \wedge \pi(t_{xyz}, t_{xz}) \wedge \pi(t_{xyz}, t_y)$$
$$\wedge \pi(t_{xyz}, t_{xy}) \wedge \pi(t_{xyz}, t_{yz}) \wedge \pi(t_{xz}, t_x) \wedge \pi(t_{xz}, t_z).$$

One can add

$$\pi(t_{xy}, t_x), \quad \pi(t_{xy}, t_y), \quad \pi(t_{yz}, t_y) \text{ and } \pi(t_{yz}, t_z)$$

by (A1) and (A3). Thus

$$(*) \Rightarrow \exists t_y: \{T(\alpha(x,y))\} < t_x, t_y > \wedge \{T(\beta(y,z))\} < t_y, t_z > \quad (**)$$

The converse implication $(**) \Rightarrow (*)$ need not be shown, since both $(*)$ and $(**)$ are functional.

iii) Since $\phi(x) \wedge x = x' : \phi(x) \to \phi(x')$ is functional, so is

$$\{T(\phi(x) \wedge x = x')\} < t_x, t_{x'} > .$$

The statement is then a consequence of (A 4).

iv) This follows from (R 2).

PROPOSITION 2. *If* U *is a monomorphism preserving functor in the usual sense between the categories associated to* (L, Γ) *and* (L', Γ'), *there is a functor* (T, π) *in the sense of Definition 1 whose interpretation according to Proposition 1 (i) is equivalent to* U.

PROOF. For each type X choose an appropriate sequence x. Then, if $U(x=x) \equiv \xi(y)$ set

$$\{T(x=x)\}(t_x) \equiv \exists y : \xi(y) \wedge t_x = y$$

(t_x of suitable type). If

$$U(\phi(x') \xrightarrow[\phi(x') \wedge x'=x]{} x=x) \equiv \kappa(w,y),$$

set

$$\{T(\phi(x'))\}(t_{x'}) \equiv \exists w \exists y : \kappa(w,y) \wedge t_{x'} = y.$$

And if

$$U(x=x \wedge z=z) \equiv \lambda(v), \quad U(x=x \wedge z=z \xrightarrow[pr_0]{} x=x) \equiv \mu(v,y),$$

set

$$\pi_x^{xz}(t_{xz}, t_x) \equiv \exists v \exists y : \mu(v,y) \wedge t_{xz} = v \wedge t_x = y.$$

Then (A1)-(A3) are obviously fulfilled by (T, π).

Now apply U to the following diagrams:

(A4)
$$\begin{array}{ccc} \phi(x) & \longrightarrow & x=x \\ \downarrow & & pr_0 \uparrow\uparrow pr_1 \\ \phi(x) \wedge x=x' & \longrightarrow & x=x \wedge x'=x' \end{array}$$

(R1)
$$\begin{array}{ccc} \phi(x,y) & \longrightarrow & x=x \wedge y=y \\ \downarrow & & \downarrow \\ \psi(y) & \longrightarrow & y=y \end{array}$$

(R2)
$$\begin{array}{ccc} \beta(x,y) & \longrightarrow & x=x \wedge y=y \\ & \searrow^{m} & \downarrow \\ & & y=y \end{array}$$

using Um mono,

(R3)
$$\begin{array}{ccc} \exists y : \beta(x,y) \wedge \beta(x',y) & \longrightarrow & x=x \wedge x'=x' \\ j \downarrow & & \uparrow \\ \beta(x,y) \wedge \beta(x',y) & \longrightarrow & x=x \wedge x'=x' \wedge y=y \end{array}$$

where j sends (x, x') to the unique corresponding y.

If application of U to the first diagram yields

$$\begin{array}{ccc}
\alpha(a) & \xrightarrow{\kappa(a,c)} & \gamma(c) \\
\Big\downarrow \lambda(a,b) & & \mu(d,c) \Big\uparrow \Big\uparrow \nu(d,c) \\
\beta(b) & \xrightarrow[\rho(b,d)]{} & \delta(d)
\end{array}$$

then

$$\{T(\phi(x))\}(t_x) \equiv \exists a\, \exists c:\, \kappa(a,c) \wedge c = t_x \qquad (*)$$

$$\{T(\phi(x) \wedge x = x')\}(t_{xx'}) \equiv \exists b\, \exists d:\, \rho(b,d) \wedge d = t_{xx'},$$

$$\{T(\phi(x) \wedge x = x')\}<t_x, t_{x'}> \equiv$$

$$\equiv \exists b\, \exists d\, \exists c\, \exists c':\, \rho(b,d) \wedge \mu(d,c) \wedge \nu(d,c') \wedge c = t_x \wedge c' = t_{x'}.$$

But (*) implies

$$\exists b\, \exists d\, \exists c:\, \rho(b,d) \wedge \mu(d,c) \wedge \nu(d,c) \wedge c = t_x$$

by functionality of $\lambda(a,b)$ and commutativity of the diagram.
The verification of (R1)-(R3) proceeds in a similar way.

DEFINITION 2. If

$$(T,\pi):(L,\Gamma) \to (L',\Gamma') \quad \text{and} \quad (S,\sigma):(L',\Gamma') \to (L'',\Gamma'')$$

are functors, the *composite* $(ST, \sigma.\pi)$ is defined as follows:

$$\{ST(\psi(x))\}(st_x) \equiv \{S(\{T(\psi(x))\}(t_x))\}(st_x),$$

$$\sigma.\pi_x^{xy}(st_{xy}, st_x) \equiv \{S(\pi_x^{xy}(t_{xy}, t_x))\}<st_{xy}, st_x>.$$

The *identity* functor is given by

$$\{I(\psi(x))\}(i_x) \equiv \exists x:\, i_x = x \wedge \psi(x),$$

$$\iota_x^{xy}(i_{xy}, i_x) \equiv \exists x\, \exists y:\, i_{xy} = (x,y) \wedge i_x = x.$$

The identity functor is a neutral element with respect to composition, and composition is associative up to provable equivalence and exchange of variables.

2. NATURAL TRANSFORMATIONS.

DEFINITION 3. If $(T,\pi), (S,\sigma):(L,\Gamma) \to (L',\Gamma')$ are functors, a *nat-*

ural transformation $\eta : S \to T$ is a family of Γ'-functional formulae

$$\eta_x(s_x, t_x) : \{S(x=x)\}(s_x) \to \{T(x=x)\}(t_x)$$

such that

$$\Gamma' \vdash \exists s_y : \{S(\alpha(x,y))\}<s_x, s_y> \wedge \eta_y(s_y, t_y)$$

$$\Rightarrow \exists t_x : \eta_x(s_x, t_x) \wedge \{T(\alpha(x,y))\}<t_x, t_y>$$

for any formula α and any partition $(\vec{x};\vec{y})$ of its sequence of free variables.

EXAMPLES.

4) $\kappa : (I, \iota) \to (Q, \pi)$ (Example 1),

$$\kappa_x(i_x, q_x) \equiv \exists x : i_x = x \wedge q_x = \{x\} \ ;$$

$\mu : (QQ, \pi.\pi) \to (Q, \pi)$,

$$\mu_x(qq_x, q_x) \equiv q_x = \{x \mid qq_x = \{\{x\}\}\}$$

((Q, κ, μ) is a monad, the definition of monad being straightforward).

5) Let α_* and α^* be defined as in Example 3; then

$$(*) \quad \{AA^\#(\phi(x,d))\}(aa^\#_x)$$

$$\equiv \exists x \exists y \exists z : aa^\#_x = (x,z,y) \wedge \phi(x,z) \wedge \alpha(y,z) \wedge \alpha(y,d).$$

If α is a partial function, $(*) \Rightarrow \phi(x,d)$, and so

$$\lambda_x(aa^\#_x, i_x) \equiv \exists x \exists y \exists x : aa^\#_x = (x,z,y) \wedge i_x = x$$

is a natural transformation $\lambda : \alpha_*\alpha^* \to Id$.

If α is functional, there is an analogous $\mu : Id \to \alpha^*\alpha_*$ and λ and μ fulfill the usual conditions of a counit and unit.

LITERATURE.

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