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BARRY MITCHELL

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MONOIDAL STRUCTURES ON GRADED CATEGORIES *)

by Barry MITCHELL

Two monoidal structures on a category $\underline{V}$ are *equivalent* if there is a bimonoidal (that is, strict monoidal) structure on the identity functor $1_{\underline{V}}$, using one monoidal structure in the domain and the other in the range. Also if $\underline{V}$ and $\hat{\underline{V}}$ are monoidal categories, then two bimonoidal structures on a functor $T: \underline{V} \to \hat{\underline{V}}$ are *equivalent* if there is a monoidal isomorphism $T \approx T$ using one bimonoidal structure in the domain and the other in the range.

Let G be a group acting on a monoidal category $\underline{V}$. This means that for each $x \in G$, there is a bimonoidal equivalence $T_x : \underline{V} \to \underline{V}$ and monoidal isomorphisms

$$(1) \qquad T_x T_y \approx T_{xy}, \quad T_1 \approx 1_{\underline{V}}$$

making a couple of obvious diagrams commute. Such an action induces an action of G on the abelian group $Z^*\underline{V}$ of automorphisms of $1_{\underline{V}}$. Let $G\underline{V}$ denote the category of G-graded objects of $\underline{V}$ (that is, the direct product of G copies of $\underline{V}$). If $\underline{V}$ has coproducts, we can define a tensor product in $G\underline{V}$ by the rule

$$(2) \qquad (A \otimes B)_z = \bigoplus_{xy=z} A_x \otimes T_x B_y .$$

Under the assumption that the tensor product of $\underline{V}$ preserve coproducts and epimorphisms and that the unit of this tensor product be a generator for $\underline{V}$, we show:

THEOREM 1. *The equivalence classes of monoidal structures on $G\underline{V}$ using the tensor product (2) are in 1-1 correspondence with the elements of $H^3(G, Z^*\underline{V})$. Moreover, the equivalence classes of bimonoidal structures on $1_{G\underline{V}}$, using any one of the above monoidal structures in both domain and*

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range, are in 1-1 correspondence with the elements of $H^2(G, Z^*\underline{V})$.

Two symmetric monoidal structures on a category $\underline{V}$ are *equivalent* if there is a symmetric bimonoidal structure on $1_{\underline{V}}$ making the monoidal structures equivalent. Now for $G\underline{V}$ with the tensor product (2) to have a symmetric monoidal structure at all, one must assume that $\underline{V}$ is symmetric monoidal and that G is abelian and acts trivially on $\underline{V}$ (that is, $T_x = 1_{\underline{V}}$ for all $x \in G$ with the isomorphisms (1) identities). In this case the monoidal structure on $G\underline{V}$ using the tensor product (2) and the trivial 3-cocycle will be referred to as the *trivial* monoidal structure. Then again with the above blanket assumptions on $\underline{V}$, we show:

THEOREM 2. *If G is abelian and $\underline{V}$ is symmetric monoidal, then the equivalence classes of symmetric structures on the trivial monoidal structure on $G\underline{V}$ are in 1-1 correspondence with the equivalence classes of bilinear antisymmetric maps $f: G \times G \to Z^*\underline{V}$, where two such maps f, f' are equivalent if there is a 2-dimensional cocycle h such that*

$$f'(x,y) - f(x,y) = h(x,y) - h(y,x)$$

for all $x, y \in G$.

An immediate consequence of the above theorems, using the fact that the group of integers has cohomological dimension one, is that if K is a commutative ring, then up to equivalence there is precisely one monoidal structure on the category of Z-graded K-modules (with the usual graded tensor product), and the symmetries for this structure are in 1-1 correspondence with the elements $k \in K$ such that $k^2 = 1$. In particular, if K is a domain, we find that the only symmetries are given by

$$a_p \otimes b_q \mapsto b_q \otimes a_p \quad \text{and} \quad a_p \otimes b_q \mapsto (-1)^{pq} b_q \otimes a_p .$$

Finally, if we start with an abelian group K on which a group G acts, then we can take $\underline{V}$ to be $Sets^K$ (so that $Z^*\underline{V} \approx K$ as G-modules), in which case Theorem 1 gives new interpretations of the cohomology groups $H^3(G, K)$ and $H^2(G, K)$.

Details of this work will be appearing in reference [5].

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Department of Mathematics
Rutgers University
NEW BRUNSWICK, N. J. 08903
U. S. A.