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G. M. KELLY

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STRUCTURES DEFINED BY FINITE LIMITS IN THE ENRICHED CONTEXT, I

by G. M. KELLY

Carolo Ehresmann amico mentorique: in memoriam

INTRODUCTION

Ehresmann's notion of the category of algebras defined by a (projective) ***sketch***, along with the elementary properties of such categories and the basic relations between such categories, are generalized to the case of enriched categories in Chapter 6 of the author's forthcoming book [*]. That generalization, more precisely, is to the case of $\mathcal{V}$-categories where $\mathcal{V}$ is a symmetric monoidal closed category whose underlying ordinary category $\mathcal{V}_0$ is locally small, complete, and cocomplete, and where, moreover, $\mathcal{V}$ is *locally bounded* in the sense of [*] - as most base-categories of interest seem to be. In the present article we set about giving more precise results when the sketch, or the corresponding *theory*, is *finitary*: generalizing the results of Gabriel-Ulmer [7] on *locally finitely presentable* (ordinary) categories.

It turns out that finitariness has a good definition in the enriched case, leading to results analogous to the classical ones, only when the base category $\mathcal{V}$ itself is suitably special. The analogy works most perfectly when the ordinary category $\mathcal{V}_0$ is locally finitely presentable in the classical sense, and when the finitely-presentable objects of $\mathcal{V}_0$ are closed under the monoidal structure of $\mathcal{V}$; that is, when the unit-object I of $\mathcal{V}$ is finitely presentable in $\mathcal{V}_0$, and when the object $x \otimes y$ is so whenever x and y are; in which case we say that $\mathcal{V}$ is *locally finitely presentable as a closed category*.

For such a $\mathcal{V}$, a $\mathcal{V}$-category $\mathcal{A}$ is the category of algebras for a finitary $\mathcal{V}$-sketch precisely when $\mathcal{A}$ is *locally finitely presentable as a $\mathcal{V}$-category*; and the corresponding finitary $\mathcal{V}$-theory $\mathcal{T}$ is $(\mathcal{A}_f)^{op}$, where $\mathcal{A}_f$ is the full subcategory of $\mathcal{A}$ determined by the finitely-presentable objects - whereupon $\mathcal{A} = \mathcal{T}\text{-}Alg$. A functor $T: \mathcal{T}'\text{-}Alg \to \mathcal{T}\text{-}Alg$ is induced by a morphism $M: \mathcal{T} \to \mathcal{T}'$ of theories precisely when T is finitary and admits a left adjoint S; whereupon M^{op} is the restriction of S to the finitely-presentable objects. Here a *finitary $\mathcal{V}$-theory* is a small $\mathcal{V}$-category $\mathcal{T}$ that is finitely-complete in the appropriate $\mathcal{V}$-enriched sense; a morphism of theories $\mathcal{T} \to \mathcal{T}'$ and a $\mathcal{T}$-algebra $\mathcal{T} \to \mathcal{V}$ are $\mathcal{V}$-functors which are *left exact*, in that they preserve the appropriate finite $\mathcal{V}$-limits; and a $\mathcal{V}$-functor is *finitary* if it preserves (classical, conical) filtered colimits.

The underlying ordinary category $\mathcal{T}_0$ of a finitary $\mathcal{V}$-theory $\mathcal{T}$ is itself finitely complete in the classical sense; and $\mathcal{T}_0\text{-}Alg$ is precisely the underlying category $(\mathcal{T}\text{-}Alg)_0$ of $\mathcal{T}\text{-}Alg$. A given classical finitary theory may or may not be of the form $\mathcal{T}_0$ for some finitary $\mathcal{V}$-theory $\mathcal{T}$; and when it is so, $\mathcal{T}$ need not be unique.

Examples of such base-categories $\mathcal{V}$, other than $\mathcal{S}et$, include R-modules, graded R-modules, and differential graded R-modules, for any commutative ring R; and the categories $\mathcal{C}at$, $\mathcal{G}pd$, $\mathcal{O}rd$ of (small) categories, groupoids, and ordered sets.

Since many of the arguments below are direct generalizations of what is true when $\mathcal{V} = \mathcal{S}et$, it was very little extra trouble to make the article self-contained, assuming no prior knowledge of that classical case. We have therefore done this, but kept these references to the classical case brief, since so much detail is available in [7] and [4].

A second part of this article, to appear later, will study the special case of those finitary $\mathcal{V}$-theories which arise from finitary $\mathcal{V}$-monads; and will use this study to give a syntactic description of finitary $\mathcal{V}$-theories, as is done for the classical case in the thesis [3] of M. Coste or the book [13] of Makkai-Reyes.

Categories with essentially-algebraic extra structure are algebras for a finitary $\mathcal{V}$-theory $\mathcal{T}$, where $\mathcal{V}$ is usually $\mathcal{G}pd$ but sometimes $\mathcal{C}at$. The morphisms in $\mathcal{T}$-*Alg* are those that preserve the structure *strictly*; and it is more natural to study a bigger category $\mathcal{T}$-*Alg**, with the same objects, but with morphisms that preserve the structure only to within isomorphism. In a forthcoming paper [10], the author will examine the relation between $\mathcal{T}$-*Alg* and $\mathcal{T}$-*Alg**, and combine this with the results of the present article, on the left adjoint of the $\mathcal{V}$-functor $\mathcal{T}'$-*Alg* $\to$ $\mathcal{T}$-*Alg* induced by a map $\mathcal{T} \to \mathcal{T}'$ of theories, to describe the left *bi-adjoint* of the corresponding functor $\mathcal{T}'$-*Alg** $\to$ $\mathcal{T}$-*Alg**.

0. REVISION OF NOMENCLATURE

Our general reference for enriched category theory is [*]. We suppose that our chosen base-category $\mathcal{V}$ is a (symmetric monoidal) closed category whose underlying category $\mathcal{V}_0$ is locally small, complete, and cocomplete. An important special case is that where $\mathcal{V}$ is the cartesian-closed category $\mathcal{S}et$ of small sets; note that a $\mathcal{S}et$-category is a *locally-small* ordinary category. The tensor product, unit object, and internal hom of $\mathcal{V}$ are $X \otimes Y$, I, and $[X, Y]$; and we use $V: \mathcal{V}_0 \to \mathcal{S}et$ for the canonical representable functor $\mathcal{V}_0(I, -)$.

Recall from Section 1.3 of [*] the careful distinction we make between a $\mathcal{V}$-category $\mathcal{A}$ and the underlying ordinary category $\mathcal{A}_0$ - which has the same objects but has $\mathcal{A}_0(A, B) = V\mathcal{A}(A, B)$; and between a $\mathcal{V}$-functor $T: \mathcal{A} \to \mathcal{B}$ and its underlying functor $T_0 : \mathcal{A}_0 \to \mathcal{B}_0$. It is precisely this notational distinction that allows us the simplification of writing «category» for «$\mathcal{V}$-category» and «functor» for «$\mathcal{V}$-functor» when the context makes clear which base-category is meant.

Recall too that a $\mathcal{V}$-category is *small* if the set of *isomorphism classes* of its objects is small. Then an ordinary category is *small* if it is a small $\mathcal{S}et$-category. We often use lower-case letters for the objects of small categories.

Finally, as is appropriate for enriched categories, we use the unqualified word *limit* to mean *indexed limit* in the sense of [*] Ch. 3, and similarly for *colimit* - except where $\mathcal{V} = \mathcal{S}et$ and the context makes clear that we are referring to classical conical limits.

1. FILTERED COLIMITS AND FINITARY FUNCTORS

(1.1) By a *filtered colimit* in the $\mathcal{V}$-category $\mathcal{A}$ we mean the conical colimit in $\mathcal{A}$ (see [*] Section 3.8) of an ordinary functor $P: \mathcal{L} \to \mathcal{A}_0$, where the ordinary category $\mathcal{L}$ is small and filtered. Since $colim\ P$, if it exists in $\mathcal{A}$, is *a fortiori* the colimit of P in $\mathcal{A}_0$, it follows that *if* $\mathcal{A}$ *admits all filtered colimits, so does* $\mathcal{A}_0$.

(1.2) The notion of a filtered category can be found in [*] Section 4.6, where it is shown that *filtered colimits in* $\mathcal{S}et$ *commute with finite (conical) limits.*

(1.3) We call a $\mathcal{V}$-functor $T: \mathcal{A} \to \mathcal{B}$ *finitary* if $\mathcal{A}$ admits filtered colimits and T preserves them. Clearly T *is finitary if and only if* T_0 *is so, provided that* $\mathcal{A}$ *and* $\mathcal{B}$ *admit filtered colimits.* Of course *a composite* TS *is finitary if* T *and* S *are*; while, since a conservative (= isomorphism-reflecting) functor reflects such colimits as it preserves ([*] Section 3.6), S *is finitary if* TS *is finitary and* T *is finitary and conservative.*

(1.4) *Let the fully-faithful* $T: \mathcal{A} \to \mathcal{B}$ *have the left adjoint* S, *where* $\mathcal{B}$ *(and hence* $\mathcal{A}$*) admits filtered colimits. Then* T *is finitary if and only if* TS *is so.*

PROOF. For the non-trivial part let $colim\ P$ be a filtered colimit in $\mathcal{A}$. We have isomorphisms

$$colim\ TP \cong colim\ TSTP \cong TS\,colim\ TP \cong T\,colim\ STP \cong T\,colim\ P,$$

the first because $ST \cong 1$, the second because TS is finitary, the third because S is left adjoint, and the fourth because $ST \cong 1$; and the composite isomorphism is easily verified to be the canonical map $colim\ TP \to T\,colim\ P$.

2. FINITELY-PRESENTABLE OBJECTS

(2.1) We shall call the object G of the $\mathcal{V}$-category $\mathcal{A}$ *finitely-presentable* (or *f.p.*) if the representable $\mathcal{V}$-functor $\mathcal{A}(G,-): \mathcal{A} \to \mathcal{V}$ is finitary. We write $\mathcal{A}_f$ for the full subcategory of $\mathcal{A}$ given by the f.p. objects.

(2.2) EXAMPLES. The unit object I is f.p. in the $\mathcal{V}$-category $\mathcal{V}$, since

$$\mathcal{V}(I,-) = [I,-] \cong 1 : \mathcal{V} \to \mathcal{V}.$$

If x and y are f.p. objects in $\mathcal{V}$, so is $x \otimes y$, since

$$[x \otimes y, -] \cong [x, [y, -]].$$

More generally, if $x \in \mathcal{V}_f$ and $G \in \mathcal{A}_f$, the tensor product $x \otimes G$ is f.p. if it exists; for $\mathcal{A}(x \otimes G, -) \cong [x, \mathcal{A}(G,-)]$. For any small $\mathcal{V}$-category $\mathcal{T}$, the representable $Yt = \mathcal{T}(t,-)$ is f.p. in $[\mathcal{T}, \mathcal{V}]$, since $[\mathcal{T}, \mathcal{V}](Yt, -)$ is isomorphic by Yoneda to the evaluation $E_t : [\mathcal{T}, \mathcal{V}] \to \mathcal{V}$, which preserves all small colimits by [*] Section 3.6. More generally, if $G \in \mathcal{A}_f$ and if the tensor product $\mathcal{T}(t,s) \otimes G$ exists in $\mathcal{A}$ for all $t, s \in \mathcal{T}$, the object $Yt \otimes G = \mathcal{T}(t,-) \otimes G$ is f.p. in $[\mathcal{T}, \mathcal{A}]$; for now Yoneda gives

$$[\mathcal{T}, \mathcal{A}](Yt \otimes G, -) \cong \mathcal{A}(G, E_t -). \tag{2.3}$$

In fact the finite presentability of $Yt \otimes G$ could also be deduced from that of Yt and

(2.4) PROPOSITION. *If $S \dashv T : \mathcal{A} \to \mathcal{B}$ where T is finitary, we have $S(\mathcal{B}_f) \subset \mathcal{A}_f$.*

PROOF. For $G \in \mathcal{B}_f$ we have $\mathcal{A}(SG,-) \cong \mathcal{B}(G, T-)$, the composite of the finitary functors T and $\mathcal{B}(G,-)$.

(2.5) It follows from (1.2) that, *when $\mathcal{V} = \mathcal{S}et$ and $\mathcal{A}$ admits filtered colimits, the category $\mathcal{A}_f$ is closed in $\mathcal{A}$ under finite (conical) limits.*

Before we can exhibit other «suitable» $\mathcal{V}$ for which an analogue of this is true, we need more examples of finite presentability in the case $\mathcal{V} = \mathcal{S}et$: to which we now restrict ourselves up to the end of (2.9) below.

(2.6) *$\mathcal{S}et_f$ is the category of finite sets.*

PROOF. $1 \in \mathcal{S}et$ being f.p. by (2.2), so is every finite set n by (2.5). For the converse, since every set X is the filtered colimit of the diagram $n \to X$ of its finite subsets, we have $\mathcal{S}et(X,X) \cong colim\, \mathcal{S}et(X,n)$, if X is f.p.; but then $1 : X \to X$ factorizes as $X \to n \to X$ for some finite n, whence X is finite.

(2.7) Let $\mathcal{A}$ be the category of algebras for a one-sorted finitary algebraic theory in the sense of Lawvere [11], and let $U: \mathcal{A} \to \mathcal{S}et$ be the underlying-set functor. It was observed in [*] Section 4.6 that a filtered colimit in $\mathcal{A}$ is formed by taking the colimit of the underlying sets and giving this the algebra structure which it inherits by (1.2). Thus U is finitary: so that, if $F \dashv U$, the free algebra Fn on a finite set n is f.p. by (2.6) and (2.4). By (2.5), an algebra A is f.p. if it is the coequalizer of two maps $Fm \rightrightarrows Fn$ with m and n finite. {It will follow from (8.12) below that every f.p. algebra has this form: so that our definition of finite presentability agrees here with the classical one of universal algebra.}

(2.8) The results are similar when $\mathcal{A}$ is the category of algebras for a *many*-sorted finitary algebraic theory in the sense of Bénabou [2]. If X is the set of sorts, the forgetful functor $U: \mathcal{A} \to \mathcal{S}et^X$ is finitary, since its component $U_x: \mathcal{A} \to \mathcal{S}et$ is so for each $x \in X$. Hence $F_x n$ is f.p. for finite n, where $F_x \dashv U_x$. A two-sorted example is the category of (small) graphs, where by a *graph* A we mean a diagram $A_1 \rightrightarrows A_0$. The graph A is called *finite* if both A_0 and A_1 are finite; and it follows much as in (2.6) - essentially because the theory of graphs has no axioms - that the f.p. graphs are exactly the finite ones. Two $\aleph_0$- sorted examples are the categories of graded and of differential graded R-modules; it follows from the above that $A = (A_i)_{i \in \mathbb{Z}}$ is f.p. if each A_i is f.p. and if $A_i = 0$ for all but a finite number of i. {It will follow from (8.12) below that these are the only f.p. objects.}

(2.9) The structure of a category or of a groupoid, not being given by operations defined on a finite *product*, is not algebraic in the sense of (2.8); it is however what Freyd [6] called «essentially algebraic», in that it can be described in terms of finite *limits*; more precisely, as was first pointed out by Ehresmann ([5]; see also [1]), both categories and groupoids are algebras for a *finitary essentially-algebraic theory* in the sense of [*] Section 6.3 - which theories are the case $\mathcal{V} = \mathcal{S}et$ of our present object of study in this article. It follows easily using (1.2) that the forgetful functor from the category of (small) categories [resp. groupoids] to that of graphs is finitary; and hence that a category [resp. a groupoid] is f.p. if it is the coequalizer of a diagram $Fg \rightrightarrows Fh$, where Fg and Fh are the free categories [resp. groupoids] on the finite graphs g and h. {Once again it will follow from (8.12) below that these are the only f.p. objects.} In particular the category $2 = (0 \to 1)$ and the groupoid $\mathrm{I} = (0 \rightleftarrows 1)$ are f.p.; the set-valued functors they represent send a category or a groupoid to its set of morphisms, and are not only finitary but also conservative.

(2.10) We now return to the case of a general $\mathcal{V}$, and consider a $\mathcal{V}$-category $\mathcal{A}$ admitting filtered colimits. For an object G of $\mathcal{A}$, we must carefully distinguish between its finite presentability in $\mathcal{A}$ - the finitariness of $\mathcal{A}(G,-): \mathcal{A} \to \mathcal{V}$, or equally of $\mathcal{A}(G,-)_0: \mathcal{A}_0 \to \mathcal{V}_0$ - and its finite presentability in $\mathcal{A}_0$ - the finitariness of $\mathcal{A}_0(G,-): \mathcal{A}_0 \to \mathcal{S}et$, which is the composite of $\mathcal{A}(G,-)_0$ with $V: \mathcal{V}_0 \to \mathcal{S}et$. Neither implies the other in general, so that neither of $\mathcal{A}_{f0}$ and $\mathcal{A}_{0f}$ need contain the other; and this even for $\mathcal{A} = \mathcal{V}$.

(2.11) As an example of this, let M be a group, and let $\mathcal{V}$ be the category $\mathcal{V}_0 = [M, \mathcal{S}et]$ of M-sets with its cartesian closed structure. It follows easily from (2.7) that $G \in \mathcal{V}$ is f.p. in $\mathcal{V}_0$ if and only if (i) the set of orbits of G is finite, and (ii) the stabilizer of each $g \in G$ is a finitely-generated group. On the other hand, since the finitary and conservative forgetful functor $U: \mathcal{V}_0 \to \mathcal{S}et$ is represented by the M-set M, the object G is f.p. in $\mathcal{V}$ precisely when $\mathcal{V}_0(M, [G,-])$ is finitary; that is, when $\mathcal{V}_0(M \times G, -)$ is finitary, or when $M \times G$ is f.p. in $\mathcal{V}_0$; which by (i) and (ii) above is the case exactly when G is finite. Thus the unit object $I = 1$ is f.p. in $\mathcal{V}$, as it must be by (2.2); but it is f.p. in $\mathcal{V}_0$ only when M is finitely generated as a group. On the other hand the M-set M, since it represents U, is always f.p. in $\mathcal{V}_0$; but it is f.p. in $\mathcal{V}$ only when M is finite.

(2.12) *In the case $\mathcal{V} = \mathcal{S}et$, let $q_l: Pl \to C$ be the colimit of $P: \mathcal{L} \to \mathcal{A}$, where $\mathcal{L}$ is small and filtered. Then if C is f.p., some q_l is a retraction. If, moreover, every Pl is f.p., and if $P\phi$ is epimorphic for every $\phi: l \to m$*

in $\mathcal{L}$, *then some* q_l *is an isomorphism.*

PROOF. Since C is f.p. the $\mathcal{A}(C, q_l): \mathcal{A}(C, Pl) \to \mathcal{A}(C, C)$ constitute a colimit cone in $\mathcal{S}et$, so that $1_C = q_l i$ for some l and for some $i: C \to Pl$. For such an i we have

$$q_l i q_l = q_l 1 : Pl \to C ;$$

and since the $\mathcal{A}(Pl, q_m): \mathcal{A}(Pl, Pm) \to \mathcal{A}(Pl, C)$ also constitute a colimit in $\mathcal{S}et$ if Pl is f.p., it follows (see [*] Thm. 4.72) that there is some $\phi: l \to m$ in $\mathcal{L}$ with $P\phi . iq_l = P\phi$. Since $q_l = q_m . P\phi$, this gives

$$P\phi . iq_m . P\phi = P\phi, \text{ whence } P\phi . iq_m = 1$$

since $P\phi$ is epimorphic. But $q_m . P\phi . i = q_l i = 1$; so that q_m is an isomorphism.

3. LOCALLY FINITELY PRESENTABLE CATEGORIES

(3.1) PROPOSITION. *For a cocomplete* $\mathcal{V}$*-category* $\mathcal{A}$, *the following are equivalent:*

(i) $\mathcal{A}$ *has a (small) strong generator* $\mathcal{G} \subset \mathcal{A}_f$;

(ii) there is a small $\mathcal{G}$ *and a strongly-generating* $K: \mathcal{G} \to \mathcal{A}$ *with* $K(\mathcal{G}) \subset \mathcal{A}_f$;

(iii) there is a small $\mathcal{G}$ *and a right-adjoint, finitary, conservative functor* $T: \mathcal{A} \to [\mathcal{G}^{op}, \mathcal{V}]$.

PROOF. (i) and (ii) are equivalent since, by [*] Section 3.6, the full image of a strongly generating functor is a strong generator. By [*] Thm 4.51, there is an equivalence between functors $K: \mathcal{G} \to \mathcal{A}$ and right adjoints $T: \mathcal{A} \to [\mathcal{G}^{op}, \mathcal{V}]$, given by $T = \tilde{K}$ where $\tilde{K} A = \mathcal{A}(K-, A)$. By definition, K is strongly generating exactly when $\tilde{K}$ is conservative; moreover, since small colimits in $[\mathcal{G}^{op}, \mathcal{V}]$ are formed pointwise, $\tilde{K}$ is finitary exactly when $K(\mathcal{G}) \subset \mathcal{A}_f$.

(3.2) We shall call a $\mathcal{V}$-category $\mathcal{A}$ *locally finitely presentable* (or *l.f.p.*) when it is cocomplete and satisfies the equivalent conditions of (3.1).

(3.3) It follows from (3.1) (iii) that, *if* $J: \mathcal{B} \to \mathcal{A}$ *is right-adjoint, finitary and conservative, and if* $\mathcal{A}$ *is l.f.p., then* $\mathcal{B}$ *is l.f.p., provided that* $\mathcal{B}$ *is* *cocomplete*. Since the cocompleteness of $\mathfrak{B}$ follows from that of $\mathfrak{A}$ if the right-adjoint J is fully faithful, we conclude that *a reflective full subcategory $\mathfrak{B}$ of an l.f.p. $\mathfrak{A}$ is l.f.p. if the inclusion $J: \mathfrak{B} \to \mathfrak{A}$ is finitary.*

(3.4) EXAMPLES. The $\mathcal{V}$-category $\mathcal{V}$ itself is always l.f.p., since the object I, which is f.p. by (2.2), is (not merely a strong generator but) dense, by (5.17) of [*]. More generally the functor category $[\mathcal{T}, \mathcal{V}]$ is l.f.p. for a small $\mathcal{T}$, since the representables $\mathcal{T}(t, -)$ are f.p. by (2.2) and dense by Proposition 5.16 of [*]. Still more generally, if $\mathfrak{A}$ is l.f.p. with strong generator $\mathcal{G} \subset \mathfrak{A}_f$, then $[\mathcal{T}, \mathfrak{A}]$ is l.f.p.; for the $\{\mathcal{T}(t, -) \otimes G\}_{t \in \mathcal{T}, G \in \mathcal{G}}$, which are f.p. by (2.2), form a strong generator by (2.3), the $\mathfrak{A}(G, E_t -)$ jointly reflecting isomorphisms.

(3.5) EXAMPLES OF L.F.P. CATEGORIES WHEN $\mathcal{V} = \mathcal{S}et$. The category of algebras for a finitary algebraic theory, one-sorted or many-sorted, is l.f.p.; the objects $\{F_x 1\}_{x \in X}$ of (2.8) are f.p., and constitute a strong generator - indeed, a regular one. The category of small categories is l.f.p. by (2.9), the f.p. object 2 being a strong generator; and the same argument applies to the full subcategories of preordered sets and of ordered sets. The category of small groupoids is again l.f.p. by (2.9), the f.p. object I being a strong generator; and the same argument applies to the full subcategory of sets-with-an-equivalence-relation.

(3.6) By applying the last remark of (1.3) to the finitary and conservative $\mathfrak{A} \to [\mathcal{G}^{op}, \mathcal{V}]$ of (3.1), we deduce that *if $\mathfrak{A}$ is l.f.p. with the strong generator $\mathcal{G} \subset \mathfrak{A}_f$, the functors $\mathfrak{A}(G, -): \mathfrak{A} \to \mathcal{V}$ for $G \in \mathcal{G}$ jointly reflect filtered colimits; so that $S: \mathfrak{B} \to \mathfrak{A}$ is finitary if each $\mathfrak{A}(G, S-): \mathfrak{B} \to \mathcal{V}$ is so.*

(3.7) *In the classical case $\mathcal{V} = \mathcal{S}et$, let $\mathfrak{A}$ be an l.f.p. category, and suppose that $\mathfrak{A}$ admits finite (conical) limits - which is in fact automatically true by (7.2) below. Then finite limits commute with filtered colimits in $\mathfrak{A}$.*

PROOF. We are asserting that the canonical map

$$\rho: colim_l \, lim_p F(p, l) \to lim_p \, colim_l F(p, l)$$

is an isomorphism, where $F: \mathcal{P} \times \mathcal{L} \to \mathfrak{A}$ with $\mathcal{P}$ finite and $\mathcal{L}$ filtered. If $\mathcal{G} \subset \mathfrak{A}_f$ is a strong generator for $\mathfrak{A}$, it suffices to show that $\mathfrak{A}(G, \rho)$ is an isomorphism for each $G \in \mathcal{G}$. Since $\mathcal{A}(G,-): \mathcal{A} \to \mathcal{S}et$ preserves finite limits and filtered colimits, this follows from (1.2).

(3.8) *In the case $\mathcal{V} = \mathcal{S}et$, let $\mathcal{A}$ be a category with finite limits and filtered colimits. Then the n-th power functor $(\ )^n : \mathcal{A} \to \mathcal{A}$ is finitary for each finite n, if either $\mathcal{A}$ is l.f.p. or $\mathcal{A}$ is cartesian closed.*

PROOF. Since the diagonal $\Delta : \mathcal{A} \to \mathcal{A}^n$ is left adjoint and hence finitary, it suffices to show that the product functor $\Pi : \mathcal{A}^n \to \mathcal{A}$ is finitary. If $\mathcal{A}$ is l.f.p., we have this by (3.7). If $\mathcal{A}$ is cartesian closed, let $P : \mathcal{L} \to \mathcal{A}^n$ have components $P_i : \mathcal{L} \to \mathcal{A}$, where $\mathcal{L}$ is filtered. Since the diagonal $\Delta : \mathcal{L} \to \mathcal{L}^n$ is final by [*] Thm 4.10, we have

$$colim\, \Pi P \cong colim_{l \in \mathcal{L}^n} (P_1 l_1 \times \ldots \times P_n l_n) ;$$

and since $- \times A : \mathcal{A} \to \mathcal{A}$ preserves all colimits (or more trivially when $n = 1$ or 0), this is isomorphic to $colim\, P_1 \times \ldots \times colim\, P_n = \Pi\, colim\, P$.

(Note that, by the example in Section 3.3 of [9], $(\ )^2 : \mathcal{A} \to \mathcal{A}$ is not finitary when $\mathcal{A}$ is the category of topological spaces.)

4. FINITE INDEXED LIMITS.

(4.1) An indexing-type $H : \mathcal{K} \to \mathcal{V}$ (cf. [*] Section 3.1) shall be called *finite* if

(i) the set of isomorphism classes of $ob\mathcal{K}$ is finite,

(ii) for each $k, k' \in \mathcal{K}$, we have $\mathcal{K}(k,k') \in \mathcal{V}_f$,

(iii) H factorizes through the inclusion $\mathcal{V}_f \subset \mathcal{V}$.

A *finite limit* (or *colimit*) is one whose indexing-type is finite.

(4.2) It follows from (2.2) that, if $H : \mathcal{K} \to \mathcal{V}$ and $H' : \mathcal{K}' \to \mathcal{V}$ are finite, so is the functor $H \otimes H' : \mathcal{K} \otimes \mathcal{K}' \to \mathcal{V}$ sending (k,k') to $Hk \otimes H'k'$. It then follows from (3.18) of [*] that a repeated finite limit $\{H?, \{H'-, P(?,-)\}$ is a finite limit $\{H \otimes H', P\}$.

(4.3) PROPOSITION. *A $\mathcal{V}$-category $\mathcal{A}$ admits all finite (indexed) limits if it admits all conical limits indexed by finite ordinary categories and all cotensor products $x \pitchfork A$ with $x \in \mathcal{V}_f$. The converse is true if $(\ )^n : \mathcal{V}_0 \to \mathcal{V}_0$ is finitary for all finite n, and hence by (3.8) if $\mathcal{V}_0$ is l.f.p. or if $\mathcal{V}$ is* *cartesian closed.*

PROOF. Let $H : \mathcal{K} \to \mathcal{V}$ be a finite indexing-type and let $T : \mathcal{K} \to \mathcal{A}$. Replacing $\mathcal{K}$ if necessary by an equivalent category, we may suppose $ob\,\mathcal{K}$ to be in fact finite. Since each $Hk \in \mathcal{V}_f$, as does each $\mathcal{K}(k, k')$, we have the cotensor products $Hk \pitchfork Tk'$ and $\mathcal{K}(k, k') \pitchfork (Hk \pitchfork Tk')$. Since $\mathcal{A}$ admits finite products and equalizers, it admits the equalizer

$$\int_{k \in \mathcal{K}} Hk \pitchfork Tk \to \Pi_{k \in \mathcal{K}} Hk \pitchfork Tk \rightrightarrows \Pi_{k, k' \in \mathcal{K}} \mathcal{K}(k, k') \pitchfork (Hk \pitchfork Tk'),$$

which by (3.68) of [∗] is the indexed limit $\{H, T\}$.

For the converse, $x \pitchfork A$ is by [∗] Section 3.7 the indexed limit $\{x, A\}$, where x and A are identified with functors $x : \mathcal{I} \to \mathcal{V}$ and $A : \mathcal{I} \to \mathcal{A}$. Since $ob\,\mathcal{I} = \{0\}$ and since $\mathcal{I}(0, 0) = I$ is f.p. in $\mathcal{V}$ by (2.2), $x : \mathcal{I} \to \mathcal{V}$ is a finite indexing-type if $x \in \mathcal{V}_f$; thus $x \pitchfork A$ exists. Next, by [∗] Section 3.8, the conical limit in $\mathcal{A}$ of an ordinary functor $S : \mathcal{P} \to \mathcal{A}$ is the indexed limit $\{\overline{\Delta I}, \bar{S}\}$, where $\bar{S} : \mathcal{P}_{\mathcal{V}} \to \mathcal{A}$ is the $\mathcal{V}$-functor corresponding to S, its domain $\mathcal{P}_{\mathcal{V}}$ being the free $\mathcal{V}$-category on the ordinary category $\mathcal{P}$, and where $\overline{\Delta I} : \mathcal{P}_{\mathcal{V}} \to \mathcal{V}$ corresponds similarly to the ordinary functor $\Delta I : \mathcal{P} \to \mathcal{V}_0$ constant at I. When $\mathcal{P}$ is finite, $ob\,\mathcal{P}_{\mathcal{V}} = ob\,\mathcal{P}$ is finite, and $\overline{\Delta I}$ factorizes through $\mathcal{V}_f \subset \mathcal{V}$ since $I \in \mathcal{V}_f$ by (2.2). To show that $\overline{\Delta I}$ is a finite indexing-type, it remains to show that each $\mathcal{P}_{\mathcal{V}}(k, k')$ is f.p. in $\mathcal{V}$. But $\mathcal{P}_{\mathcal{V}}(k, k') = \mathcal{P}(k, k').I$, the coproduct of $\mathcal{P}(k, k')$ copies of I ; and $\mathcal{P}(k, k')$ is a finite set n. Since $\mathcal{V}(n.I, -) \cong (\ )^n : \mathcal{V} \to \mathcal{V}$, to show that $n.I \in \mathcal{V}_f$ is to show that $(\ )^n : \mathcal{V} \to \mathcal{V}$ is finitary. By (1.3), this is the same as the finitariness of $(\ )^n : \mathcal{V}_0 \to \mathcal{V}_0$.

(4.4) HYPOTHESIS. From now on we suppose at all times that *the ordinary category* $\mathcal{V}_0$ *is l.f.p.*

(4.5) A $\mathcal{V}$-category $\mathcal{A}$ satisfying the equivalent conditions of (4.3) shall be called *finitely complete*, or *f.c.*. A $\mathcal{V}$-functor $T : \mathcal{A} \to \mathcal{B}$ shall be called *left exact*, or *lex*, if $\mathcal{A}$ is finitely complete and T preserves all finite limits. (The duals are *finitely cocomplete*, and *right exact* or *rex*.) It is clear from the proof of (4.3) that:

(4.6) *For a finitely complete* $\mathcal{A}$, *the functor* $T : \mathcal{A} \to \mathcal{B}$ *is left exact if and* *only if it preserves finite conical limits and the cotensor products* $x \pitchfork A$ *with* $x \in \mathcal{V}_f$.

Now (2.6) gives:

(4.7) *In the classical case* $\mathcal{V} = \mathcal{S}et$, *a category* $\mathcal{A}$ *is finitely complete precisely when it admits all finite conical limits, and then* $T: \mathcal{A} \to \mathcal{B}$ *is left exact precisely when it preserves these. For a general* $\mathcal{V}$, *if the* $\mathcal{V}$-*category* $\mathcal{A}$ *is f.c., so is* $\mathcal{A}_0$; *and if* $T: \mathcal{A} \to \mathcal{B}$ *is lex, so is* $T_0: \mathcal{A}_0 \to \mathcal{B}_0$.

(4.8) Under further hypotheses that we shall later impose on $\mathcal{V}$ we shall show in (7.2) below that every l.f.p. category $\mathcal{A}$ is complete. Hence the hypothesis of finite completeness in such propositions as the next -which generalizes (3.7) - is in fact otiose in practice.

(4.9) PROPOSITION. *Let* $\mathcal{A}$ *be l.f.p. and f.c., let* $H: \mathcal{K} \to \mathcal{V}$ *be a finite indexing type, and let* $P: \mathcal{L} \to [\mathcal{K}, \mathcal{A}]_0$ *be an ordinary functor with* $\mathcal{L}$ *small and filtered. Then the evident canonical map*

$$\sigma: colim\,\{H, P\} \to \{H, colim\,P\}$$

is an isomorphism. In other words, finite limits commute with filtered colimits in such an l.f.p. $\mathcal{A}$.

PROOF. If $\mathcal{G} \subset \mathcal{A}_f$ is a strong generator for $\mathcal{A}$, it suffices to show that $\mathcal{A}(G, \sigma)$ is an isomorphism for all $G \in \mathcal{G}$. Since $\mathcal{A}(G, -): \mathcal{A} \to \mathcal{V}$ then preserves filtered colimits and all limits, we are reduced to the special case $\mathcal{A} = \mathcal{V}$. Then the functor $x \pitchfork -: \mathcal{V} \to \mathcal{V}$ is simply $[x, -] = \mathcal{V}(x, -)$, which commutes (by definition) with filtered colimits when $x \in \mathcal{V}_f$. So, by the proof of (4.3), it suffices to show that filtered colimits commute with finite conical limits in $\mathcal{V}$; which is to say that they do so in $\mathcal{V}_0$. But this is so by (3.7), given our hypothesis (4.4) that $\mathcal{V}_0$ is l.f.p.

(4.10) This result may also be expressed by saying that $\{H, -\}: [\mathcal{K}, \mathcal{A}] \to \mathcal{A}$ *is finitary for an l.f.p. and f.c.* $\mathcal{A}$ *when* $H: \mathcal{K} \to \mathcal{V}$ *is a finite indexing type; or equally by saying that* $colim: [\mathcal{L}_{\mathcal{V}}, \mathcal{A}] \to \mathcal{A}$ *is left exact for an l.f.p. and f.c.* $\mathcal{A}$ *when* $\mathcal{L}$ *is a small filtered ordinary category.* (Recall from [*] Section 2.5 that the $\mathcal{V}$-category $[\mathcal{L}_{\mathcal{V}}, \mathcal{A}]$ has $[\mathcal{L}, \mathcal{A}_0]$ as underlying category.)

(4.11) Taking $\mathcal{A} = \mathcal{V}$ in the first statement of (4.10), we conclude that *if* $H: \mathcal{K} \to \mathcal{V}$ *is a finite indexing type, then* $H \in [\mathcal{K}, \mathcal{V}]_f$.

(4.12) *Let* $\mathcal{A}$ *be l.f.p. and f.c. Then, in the sense made precise in the proof, a filtered colimit of left exact functors into* $\mathcal{A}$ *is left exact.*

PROOF. We consider an f.c. category $\mathcal{B}$, not necessarily small, and an ordinary functor $Q: \mathcal{L} \to [\mathcal{B}, \mathcal{A}]_0$ where $\mathcal{L}$ is small and filtered and each $Ql: \mathcal{B} \to \mathcal{A}$ is left exact; here $[\mathcal{B}, \mathcal{A}]_0$ is the ordinary category of $\mathcal{V}$-functors $\mathcal{B} \to \mathcal{A}$ and $\mathcal{V}$-natural transformations between them, which exists even when $[\mathcal{B}, \mathcal{A}]$ is too big to exist as a $\mathcal{V}$-category; and what we assert is that $colim Q: \mathcal{B} \to \mathcal{A}$ is left exact. To give Q, however, is equivalently to give a left exact $R: \mathcal{B} \to [\mathcal{L}_{\mathcal{V}}, \mathcal{A}]$; and $colim Q$, being the composite of R with $colim: [\mathcal{L}_{\mathcal{V}}, \mathcal{A}] \to \mathcal{A}$, is left exact by the second assertion of (4.10).

(4.13) *Let* $\mathcal{A}$ *be l.f.p. and f.c. Then, in the sense made precise in the proof, a finite limit of finitary functors into* $\mathcal{A}$ *is finitary.*

PROOF. We consider a $\mathcal{B}$, not necessarily small, which admits filtered colimits, a finite indexing type $H: \mathcal{K} \to \mathcal{V}$, and a functor $Q: \mathcal{K} \otimes \mathcal{B} \to \mathcal{A}$ such that each $Q(k,-): \mathcal{B} \to \mathcal{A}$ is finitary; and what we assert is that

$$\{H?, Q(?,-)\}: \mathcal{B} \to \mathcal{A}$$

is finitary. To give Q, however, is equivalently to give a finitary $R: \mathcal{B} \to [\mathcal{K}, \mathcal{A}]$; and $\{H?, Q(?,-)\}$, being the composite of R with $\{H,-\}: [\mathcal{K}, \mathcal{A}] \to \mathcal{A}$, is finitary by the first assertion of (4.10).

(4.14) *For any cocomplete* $\mathcal{A}$, *the full subcategory* $\mathcal{A}_f$ *is closed under finite colimits.*

PROOF. Since $\mathcal{A}(-, A)$ sends colimits in $\mathcal{A}$ to limits in $\mathcal{V}$, this follows from the case $\mathcal{A} = \mathcal{V}$ of (4.13).

(4.15) REMARK. Note that we used Hypothesis (4.4) in the proof of (4.9).

5. LOCALLY FINITELY PRESENTABLE SYMMETRIC MONOIDAL CLOSED CATEGORIES

In accordance with Hypothesis (4.4), we suppose for this section that $\mathcal{V}_0$ is l.f.p., with strong generator $\mathcal{H} \subset \mathcal{V}_{0f}$.

(5.1) *For an object G of the cocomplete $\mathcal{V}$-category $\mathcal{A}$, the following are equivalent:*

(i) G is f.p. in $\mathcal{A}$;

(ii) $x \otimes G$ is f.p. in $\mathcal{A}_0$ for all $x \in \mathcal{V}_{0f}$;

(iii) $x \otimes G$ is f.p. in $\mathcal{A}_0$ for all $x \in \mathcal{H}$.

PROOF. To say that G is f.p. in $\mathcal{A}$ is to say that $\mathcal{A}(G, -) \colon \mathcal{A} \to \mathcal{V}$, or equivalently $\mathcal{A}(G, -)_0 \colon \mathcal{A}_0 \to \mathcal{V}_0$, is finitary. Since

$$\mathcal{V}_0(x, \mathcal{A}(G, -)_0) \cong \mathcal{V}_0(x \otimes G, -),$$

the result follows from (1.3) and (3.6).

(5.2) *The following are equivalent:*

(i) $\mathcal{V}_{0f} \subset \mathcal{V}_{f0}$;

(ii) $x \otimes y \in \mathcal{V}_{0f}$ whenever $x, y \in \mathcal{V}_{0f}$;

(iii) $x \otimes y \in \mathcal{V}_{0f}$ whenever $x, y \in \mathcal{H}$.

PROOF. (i) implies (ii) by (5.1) and (ii) implies (iii) trivially. Given (iii), we first use (iii) $\Rightarrow$ (i) of (5.1) to deduce that $\mathcal{H} \subset \mathcal{V}_{f0}$; then use (i) $\Rightarrow$ (ii) of (5.1) to deduce that $x \otimes y \in \mathcal{V}_{0f}$ if $x \in \mathcal{V}_{0f}$ and $y \in \mathcal{H}$; and finally use (iii) $\Rightarrow$ (i) of (5.1), with the symmetry of $\otimes$, to deduce that $\mathcal{V}_{0f} \subset \mathcal{V}_{f0}$.

(5.3) *The following are equivalent:*

(i) $\mathcal{V}_{f0} \subset \mathcal{V}_{0f}$;

(ii) $I \in \mathcal{V}_{0f}$;

(iii) $V = \mathcal{V}_0(I, -) \colon \mathcal{V}_0 \to \mathcal{S}et$ is finitary;

(iv) $\mathcal{A}_{f0} \subset \mathcal{A}_{0f}$ for every cocomplete $\mathcal{V}$-category $\mathcal{A}$.

PROOF. (ii) and (iii) are equivalent by definition; (iv) implies (i) trivially; (i) implies (ii) since $I \in \mathcal{V}_{f0}$ by (2.2); and (ii) implies (iv) by the part (i) $\Rightarrow$ (ii) of (5.1).

(5.4) The example (2.11) shows that neither $\mathcal{V}_{0f} \subset \mathcal{V}_{f0}$ nor $\mathcal{V}_{f0} \subset \mathcal{V}_{0f}$ is automatic when $\mathcal{V}_0$ is l.f.p.; for the $\mathcal{V}_0 = [M, \mathcal{S}et]$ of that example is l.f.p. by (3.5). The same example shows that $\mathcal{V}_{f0} \subset \mathcal{V}_{0f}$ does not imply $\mathcal{V}_{0f} \subset \mathcal{V}_{f0}$; for the former holds precisely when M is finitely generated as a group, and the latter precisely when M is finite. The author sees no reason to suppose that $\mathcal{V}_{of} \subset \mathcal{V}_{fo}$ implies $\mathcal{V}_{fo} \subset \mathcal{V}_{of}$, but has no counter-example to this. The author is endebted to R. Börger for discussions concerning this example.

(5.5) We shall say that $\mathcal{V}$ is *locally finitely presentable as a (symmetric monoidal) closed category* if $\mathcal{V}_o$ is l.f.p. and if $\mathcal{V}_{of} = \mathcal{V}_{fo}$. By (5.2) and (5.3), a closed $\mathcal{V}$ with l.f.p. $\mathcal{V}_o$ is l.f.p. as a closed category precisely when $\mathcal{V}_{of}$ is *closed under the monoidal structure*, in the sense that $I \in \mathcal{V}_{of}$ and $x \otimes y \in \mathcal{V}_{of}$ when $x, y \in \mathcal{V}_{of}$; while it in fact suffices for the latter that $x \otimes y \in \mathcal{V}_{of}$ whenever $x, y \in \mathcal{H}$, where $\mathcal{H} \subset \mathcal{V}_{of}$ is a strong generator for $\mathcal{V}_o$.

(5.6) EXAMPLES. If $\mathcal{V}_o$ is the category of algebras for a one-sorted finitary algebraic theory, and if the theory is *commutative* in the sense of Linton [12], then $\mathcal{V}_o$ admits a symmetric monoidal closed structure $\mathcal{V}$ in which the tensor product represents the bi-homomorphisms and the unit object I represents the forgetful functor to $\mathcal{S}et$. In this case $F1 = I$ is an f.p. strong generator in $\mathcal{V}_o$ by (3.5), so that $\mathcal{V}$ is l.f.p. as a closed category by the last remarks of (5.5). The closed categories $\mathcal{V} = \mathcal{S}et$, $\mathcal{S}et_*$, $\mathcal{A}b$, R-$\mathcal{M}od$ of sets, pointed sets, abelian groups, and R-modules for a commutative ring R, are particular examples.

(5.7) EXAMPLES. The category $\mathcal{V}_o = [M, \mathcal{S}et]$ is an example of (5.6) when the group M is abelian. The tensor product here, however, is not the cartesian one; and (5.4) shows that $[M, \mathcal{S}et]$ with its cartesian closed structure is l.f.p. as a closed category only when M is finite.

(5.8) EXAMPLES. Consider the closed categories $\mathcal{G}$-R-$\mathcal{M}od$, $\mathcal{DG}$-R-$\mathcal{M}od$, $\mathcal{G}ph$ of graded R-modules, differential graded R-modules, and graphs - the first two with their classical closed structures, and the last with its cartesian one. In each case $\mathcal{V}_o$ is l.f.p. by (3.5), the f.p. objects $F_i 1$ forming a strong generator. In the first two cases

$$F_0 1 = I \quad \text{and} \quad F_i 1 \otimes F_j 1 = F_{i+j} 1,$$

so that $\mathcal{V}$ is l.f.p. as a closed category by (5.5). In the last case

$$F_0 1 = (0 \rightrightarrows 1) \quad \text{and} \quad F_1 1 = (1 \rightrightarrows 2);$$

the various products $F_i 1 \times F_j 1$ are all finite and hence f.p. by (2.8), as is the unit object $1 = (1 \rightrightarrows 1)$; so that again $\mathcal{V}$ is l.f.p. as a closed category.

(5.9) EXAMPLES. Consider the cartesian closed categories $\mathcal{C}at$, $\mathcal{P}reord$, $\mathcal{O}rd$, $\mathcal{G}pd$, $\mathcal{E}quiv$ of (small) categories, preorders, orders, groupoids, and sets-with-an-equivalence-relation. In each case $\mathcal{V}_0$ is l.f.p. by (3.5), an f.p. strong generator being given by 2 in the first three cases and by I in the other two. Since 2×2 and $\mathrm{I} \times \mathrm{I}$ are finite, and hence f.p. by (2.9), as is the unit object 1, it follows by (5.5) that $\mathcal{V}$ is in each case l.f.p. as a closed category.

(5.10) EXAMPLES. A finite complete ordered set, such as 2, with its cartesian closed structure, is clearly l.f.p. as a closed category.

(5.11) For the closed categories $\bar{\mathrm{R}}_+$, $\mathcal{CG}\mathcal{T}op$, $\mathcal{B}an$ of the extended non-negative reals, of compactly-generated topological spaces, and of Banach spaces (see [*] Section 1.1), it is not even true that $\mathcal{V}_0$ is l.f.p.

(5.12) HYPOTHESIS. For the rest of this article we strengthen the hypothesis (4.4) by supposing throughout that *$\mathcal{V}$ is l.f.p. as a closed category.* Accordingly, both $x \in \mathcal{V}_{0f}$ and $x \in \mathcal{V}_{f0}$ mean the same as $x \in \mathcal{V}_f$.

6. LEFT EXACT AND FLAT FUNCTORS INTO $\mathcal{V}$

(6.1) If $\mathcal{T}$ is a small $\mathcal{V}$-category, we call the functor $F: \mathcal{T} \to \mathcal{V}$ *flat* if $F*-: [\mathcal{T}^{op}, \mathcal{V}] \to \mathcal{V}$ is left exact, where $*$ denotes the indexed colimit, as in [*] Section 3.1.

(6.2) *A flat functor $F: \mathcal{T} \to \mathcal{V}$ preserves any finite limit that happens to exist in $\mathcal{T}$; in particular, a flat functor is left exact if $\mathcal{T}$ is finitely complete.*

PROOF. Let $H: \mathcal{K} \to \mathcal{V}$ be a finite indexing type, and let $T: \mathcal{K} \to \mathcal{T}$ be such that the limit $\{H, T\}$ exists in $\mathcal{T}$. Let $Y: \mathcal{T} \to [\mathcal{T}^{op}, \mathcal{V}]$ denote the Yoneda embedding. Since Y preserves limits by [*] Section 3.3, we have a canonical isomorphism $Y\{H, T\} \cong \{H, YT\}$; and since F is flat, we have a canonical isomorphism $F*\{H, YT\} \cong \{H, F*YT\}$. Combining these gives

$F*Y\{H,T\} \cong \{H, F*YT\}$; and since $F*Yt \cong Ft$ by (3.9) and (3.10) of [*], we have the desired result $F\{H,T\} \cong \{H, FT\}$.

(6.3) *Any representable* $\mathcal{T}(t,-)$ *is flat,* since by (3.10) of [*] the functor $\mathcal{T}(t,-)*-: [\mathcal{T}^{op}, \mathcal{C}] \to \mathcal{C}$ is isomorphic, for any $\mathcal{C}$, to the evaluation E_t, which by [*] Section 3.3 preserves all limits that exist. Since $-*T$ is cocontinuous by [*] Section 3.3, it follows from (4.12) that *any filtered colimit in* $[\mathcal{T}, \mathcal{V}]$ *of flat functors is flat,* so that in particular *every filtered colimit* F *of representables is flat.* In fact (4.12) gives more: *for such an* F *and any l.f.p. and f.c.* $\mathcal{C}$, *the functor* $F*-: [\mathcal{T}^{op}, \mathcal{C}] \to \mathcal{C}$ *is left exact.*

(6.4) When $\mathcal{V} = \mathcal{S}et$ it is a classical result (see [*] Thm 5.38, or (6.7) below) that every flat $F: \mathcal{T} \to \mathcal{V}$ is a filtered colimit of representables. The author has no proof of this for a general $\mathcal{V}$ satisfying the Hypothesis (5.12). It is however true when $\mathcal{T}$ is finitely complete ((6.11) below) and for certain special $\mathcal{V}$, if $\mathcal{T}$ at least admits certain cotensor products ((6.10) below).

(6.5) Consider an arbitrary $F: \mathcal{T} \to \mathcal{V}$ with $\mathcal{T}$ small. The composite of $V = \mathcal{V}_0(I,-): \mathcal{V}_0 \to \mathcal{S}et$ with the underlying functor $F_0: \mathcal{T}_0 \to \mathcal{V}_0$ of F is an ordinary functor $VF_0: \mathcal{T}_0 \to \mathcal{S}et$. Recall from [*] Section 1.10 that the comma-category $1/VF_0$, whose objects are pairs (t,a) with $t \in \mathcal{T}$ and $a \in VF_0t$, is called the category $el(VF_0)$ of *elements* of VF_0. Since a map $\mathcal{T}(t,-) \to F$ from a representable into F corresponds by Yoneda to an element $a \in VF_0t$, we have a canonical inductive cone $\mu_{(t,a)}: \mathcal{T}(t,-) \to F$ in $[\mathcal{T}, \mathcal{V}]_0$, with vertex F, whose base is the ordinary functor

$$el(VF_0)^{op} \xrightarrow[d^{op}]{} \mathcal{T}_0{}^{op} \xrightarrow[Y_0]{} [\mathcal{T}, \mathcal{V}]_0,$$

where d is the projection sending (t,a) to t ; and this cone induces a canonical map $\rho: colim(Y_0 d^{op}) \to F$. When $\mathcal{V} = \mathcal{S}et$ it is a classical result - see [*] Section 3.3 - that μ is a colimit cone; so that ρ is an isomorphism. For a general $\mathcal{V}$ - even for one satisfying (5.12) - this is false. In fact, when $\mathcal{T}$ is the *unit* $\mathcal{V}$-category $\mathcal{I}$ with one object 0, and with $\mathcal{I}(0,0) = I$, so that $[\mathcal{I}, \mathcal{V}] \cong \mathcal{V}$ and F is just an object of $\mathcal{V}$, the cone μ consists of all maps $I \to F$ in $\mathcal{V}_0$; and this is a colimit cone for all F

precisely when $\{I\}$ is *dense* in $\mathcal{V}_0$ - which is false even for $\mathcal{V} = \mathcal{A}b$. In the example $\mathcal{V} = \mathcal{C}at$ it is easily seen, when $\mathcal{T} = \mathcal{G} = 1$, that the map ρ from the colimit into F is the inclusion into F of the discrete category with the same objects; and similarly for $\mathcal{V} = \mathcal{G}pd$.

(6.6) PROPOSITION. *Suppose that* $\mathcal{V}$ *is cartesian closed, and that* $V: \mathcal{V}_0 \to \mathcal{S}et$ *preserves coproducts and regular epimorphisms; as is true in the examples* $\mathcal{V} = \mathcal{S}et$, $\mathcal{G}ph$, $\mathcal{C}at$, $\mathcal{P}ord$, $\mathcal{O}rd$, $\mathcal{G}pd$, $\mathcal{E}quiv$. *Then, if* $F: \mathcal{T} \to \mathcal{V}$ *is flat, the category* $el(VF_0)^{op}$ *is filtered.*

PROOF. Let $T: \mathcal{P} \to el(VF_0)$ where $\mathcal{P}$ is a finite ordinary category. To give T is to give a functor $S = dT: \mathcal{P} \to \mathcal{T}_0$ and a cone $(\gamma_p: 1 \to VF_0Sp)$, which is equally an element $\gamma \in lim\, VF_0S$. The flatness of F gives

$$F * lim_p\, \mathcal{T}(-, Sp) \cong lim_p (F * \mathcal{T}(-, Sp)),$$

and this latter by Yoneda is $lim_p\, FSp = lim\, F_0S$. By the formula of [*] Section 3.10 expressing $F * G$ as a coend, we have a regular epimorphism $f: \Sigma_{t \in \mathcal{T}} F_0t \times lim_p\, \mathcal{T}(t, Sp) \to lim\, F_0S$. Applying V to f, and using the hypotheses of the proposition and the preservation of limits by V, we get a surjection $Vf: \Sigma_t VF_0t \times lim_p\, \mathcal{T}_0(t, Sp) \to lim\, VF_0S$. Let some inverse image of γ under Vf be $(\gamma \in VF_0t,\ \beta \in lim_p\, \mathcal{T}_0(t, Sp))$. We can regard β as a projective cone in $\mathcal{T}_0$ over S with vertex t. Since $VF_0\beta_p: VF_0t \to VF_0Sp$ maps a to γ_p by construction, β is equally a projective cone in $el(VF_0)$ over T with vertex (t, a). Thus $el(VF_0)^{op}$ is filtered.

(6.7) Combining (6.6), (6.5), and (6.3), we regain the classical result in the case $\mathcal{V} = \mathcal{S}et$ that $F: \mathcal{T} \to \mathcal{S}et$ *is flat if and only if* $(el\, F)^{op}$ *is filtered, and if and only if* F *is a filtered colimit of representables.*

(6.8) We recall also the other classical result that, *when* $\mathcal{V} = \mathcal{S}et$ *and* $\mathcal{T}$ *is finitely complete,* $F: \mathcal{T} \to \mathcal{S}et$ *is flat precisely when it is left exact.* For flatness implies left exactness by (6.2); while if $\mathcal{T}$ is finitely complete and F is left exact, it is immediate that $el\, F$ is finitely complete, so that $(el\, F)^{op}$ is finitely cocomplete and *a fortiori* filtered.

(6.9) PROPOSITION. *Let* $\mathcal{T}$ *admit the cotensor products* $x \pitchfork -$ *for* $x \in \mathcal{V}_f$, *and let* $F: \mathcal{T} \to \mathcal{V}$ *preserve them. Then if* $VF_0: \mathcal{T}_0 \to \mathcal{S}et$ *is flat, the can-*

onical cone μ of (6.5) is a colimit cone in $[\mathcal{T},\mathcal{V}]$, expressing F as a filtered colimit of representables in $[\mathcal{T},\mathcal{V}]$; whence F is flat by (6.3).

PROOF. Since colimits in $[\mathcal{T},\mathcal{V}]$ are formed pointwise by [*] Section 3.3, $\mathcal{V}$ being cocomplete by [*] Section 3.10, we have only to prove that $\mu_{(t,a,s)}: \mathcal{T}(t,s) \to Fs$ is, for each $s \in \mathcal{T}$, a colimit cone in $\mathcal{V}$; which by [*] Section 3.8 is to say that $\mu_{(t,a,s)}$ is a colimit cone in $\mathcal{V}_0$. Since $\mathcal{V}_0$ is l.f.p. by (4.4), it suffices by (3.6) to show that

$$\mathcal{V}_0(x, \mathcal{T}(t,s)) \to \mathcal{V}_0(x, Fs)$$

is a colimit cone in $\mathcal{S}et$ for each $x \in \mathcal{V}_{0f}$; which by (5.12) means for each $x \in \mathcal{V}_f$. For $x \in \mathcal{V}_f$, however, $\mathcal{V}_0(x, \mathcal{T}(t,s)) \cong \mathcal{T}_0(t, x \pitchfork s)$ by the definition of the cotensor product $x \pitchfork s$; while

$$\mathcal{V}_0(x, Fs) \cong V[x, Fs] = V(x \pitchfork Fs),$$

which since F preserves $x \pitchfork -$ is isomorphic to $VF(x \pitchfork s)$. Writing r for $x \pitchfork s$, we are reduced to proving that $\mathcal{T}_0(t,r) \to VF_0 r$ is a colimit cone in $\mathcal{S}et$; which is true by (6.5).

(6.10) COROLLARY. *When $\mathcal{V}$ satisfies the conditions of (6.6) and $\mathcal{T}$ admits the cotensor products $x \pitchfork t$ for $x \in \mathcal{V}_f$, $F: \mathcal{T} \to \mathcal{V}$ is flat precisely when F is a filtered colimit of representables.*

PROOF. A filtered colimit of representables is flat by (6.3). If F is flat, it preserves the $x \pitchfork t$ for $x \in \mathcal{V}_f$ by (6.2), since these are finite limits by the proof of (4.3). Moreover $el(VF_0)^{op}$ is filtered by (6.6), so that VF_0 is flat by (6.7). The result now follows by (6.9).

(6.11) THEOREM. *Let $F: \mathcal{T} \to \mathcal{V}$ where the small $\mathcal{T}$ is finitely complete. Then the following are equivalent:*

(i) F is left exact;

(ii) the canonical cone $\mu_{(t,a)}: \mathcal{T}(t,-) \to F$ of (6.5) expresses F as a filtered colimit in $[\mathcal{T},\mathcal{V}]$ of representables;

(iii) F is some filtered colimit in $[\mathcal{T},\mathcal{V}]$ of representables;

(iv) F lies in the closure of $\mathcal{T}^{op} \subset [\mathcal{T},\mathcal{V}]$ under filtered colimits (which by [*] Thm 5.35 is the free filtered-colimit completion of $\mathcal{T}^{op}$);

(v) $F * -: [\mathcal{T}^{op}, \mathcal{A}] \to \mathcal{A}$ is left exact for any l.f.p. and f.c. $\mathcal{A}$;

(vi) F is flat.

PROOF. If F is left exact, so is F_0 by (4.7), whence VF_0 is left exact since $V = \mathcal{V}_0(I, -)$ preserves all limits; so that VF_0 is flat by (4.7) and (6.8). Hence (i) $\Rightarrow$ (ii) by (6.9). It is trivial that (ii) $\Rightarrow$(iii) and (iii)$\Rightarrow$(iv); while (iv) implies (v) by the arguments of (6.3) along with (4.12). The implication (v) $\Rightarrow$ (vi) is trivial on taking $\mathcal{A} = \mathcal{V}$; and we have (vi)$\Rightarrow$(i) by (6.2).

(6.12) THEOREM. *Let $M : \mathcal{T} \to \mathcal{S}$ where $\mathcal{T}$ and $\mathcal{S}$ are finitely complete and $\mathcal{T}$ is small. Let $Lan_M F : \mathcal{S} \to \mathcal{V}$ be the left Kan extension along M of $F : \mathcal{T} \to \mathcal{V}$. Then if F is left exact, so is $Lan_M F$.*

PROOF. By [*] Section 4.1, $Lan_M F$ is the composite of $\tilde{M} : \mathcal{S} \to [\mathcal{T}^{op}, \mathcal{V}]$, where $\tilde{M} S = \mathcal{S}(M-, S)$, and $- * F : [\mathcal{T}^{op}, \mathcal{V}] \to \mathcal{V}$. But $\tilde{M}$ is trivially left exact, while $- * F$ is left exact since F is flat by (6.11) and $*$ is symmetric by [*] Section 3.1.

(6.13) REMARK. Up to this point we have used, besides Hypothesis 4.4, only the part $\mathcal{V}_{0f} \subset \mathcal{V}_{f0}$ of Hypothesis (5.12) - namely, in the proof of (6.9).

7. $\mathcal{A}_f$ AS A SMALL DENSE SUBCATEGORY OF THE LOCALLY FINITELY PRESENTABLE $\mathcal{A}$

(7.1) *The set of finite indexing types is small; so that by [*] Section 3.5 the closure under finite colimits of a small full subcategory $\mathcal{G}$ in a finitely-cocomplete $\mathcal{A}$ is again small, and in particular the free finite-colimit completion of a small $\mathcal{G}$ (its finite colimit-closure in $[\mathcal{G}^{op}, \mathcal{V}]$) is small.*

PROOF. By (4.1), it is a matter of observing that $\mathcal{V}_f$ is small. For $\mathcal{V} = \mathcal{S}et$ we have this by (2.6). This justifies the argument in (7.2) below when $\mathcal{V} = \mathcal{S}et$, which then gives for any $\mathcal{V}$ satisfying (5.12) the smallness of $\mathcal{V}_{0f}$, and hence of $\mathcal{V}_f$. This now gives the general case of (7.1), which is needed for the general case of (7.2).

(7.2) THEOREM. *Let $\mathcal{A}$ be l.f.p., let $\mathcal{G} \subset \mathcal{A}_f$ be a strong generator of $\mathcal{A}$, and let $Z : \mathcal{A}_f \to \mathcal{A}$ be the inclusion. Then*

(i) $\mathcal{A}_f$ is the closure of $\mathcal{G}$ in $\mathcal{A}$ under finite colimits, and $\mathcal{A}_f$ is small and finitely cocomplete;

(ii) for each $A \in \mathfrak{A}$ *the totality of maps* $g: G \to A$ *with* $G \in \mathfrak{A}_f$ *expresses* A *as the filtered colimit in* $\mathfrak{A}$ *of the functor* $Z_0 d: Z_0/A \to \mathfrak{A}_0$, *where* $d: Z_0/A \to \mathfrak{A}_{f0}$ *is the projection from the comma-category;*

(iii) the colimits in (ii) are Z-absolute, and thus present $\mathfrak{A}_f$ *as a small dense subcategory of* $\mathfrak{A}$;

(iv) the full embedding $\tilde{Z}: \mathfrak{A} \to [\mathfrak{A}_f^{op}, \mathfrak{V}]$ *is finitary, and has the left adjoint* $-*Z: [\mathfrak{A}_f^{op}, \mathfrak{V}] \to \mathfrak{A}$;

(v) $\mathfrak{A}$ *is complete;*

(vi) the replete image of $\tilde{Z}$ *is the category* $Lex[\mathfrak{A}_f^{op}, \mathfrak{V}]$ *of left-exact functors* $\mathfrak{A}_f^{op} \to \mathfrak{V}$, *so that* $\tilde{Z}$ *induces an equivalence* $\mathfrak{A} \simeq Lex[\mathfrak{A}_f^{op}, \mathfrak{V}]$.

PROOF. Write $\bar{\mathcal{G}}$ for the closure of $\mathcal{G}$ in $\mathfrak{A}$ under finite colimits; it is small by (7.1), and of course finitely cocomplete; and by (4.14) it is contained in $\mathfrak{A}_f$. Write $K: \bar{\mathcal{G}} \to \mathfrak{A}$ for the inclusion; since $\mathcal{G}$ is a strong generator so *a fortiori* is $\bar{\mathcal{G}} \supset \mathcal{G}$, so that by (3.1) the functor $\tilde{K}: \mathfrak{A} \to [\bar{\mathcal{G}}^{op}, \mathfrak{V}]$ is finitary, conservative, and right-adjoint; its left adjoint is in fact $-*K$ by the definition of the latter. For any $A \in \mathfrak{A}$, let β denote the canonical cone consisting of all maps $g: G \to A$ with $G \in \bar{\mathcal{G}}$, so that $\beta_{G,g} = g$; the indexing category K_0/A of this cone is filtered, since by (4.7) $\bar{\mathcal{G}}_0$ is closed under finite conical colimits in $\mathfrak{A}_0$. The image $\tilde{K}\beta$ of the cone β under $\tilde{K}$ has, since $\tilde{K}K \cong Y$, the components $\tilde{K}g: \bar{\mathcal{G}}(-, G) \to \tilde{K}A$, and is in fact precisely the canonical cone μ of (6.5) for the functor $\tilde{K}A: \bar{\mathcal{G}}^{op} \to \mathfrak{V}$; for the maps $\bar{\mathcal{G}}(-, G) \to \tilde{K}A$ correspond by Yoneda to the elements of

$$V(\tilde{K}A)G = V\mathfrak{A}(KG, A) = \mathfrak{A}_0(G, A),$$

or the maps $g: G \to A$. But $\tilde{K}A = \mathfrak{A}(K-, A)$ is a left exact functor $\bar{\mathcal{G}}^{op} \to \mathfrak{V}$, since K preserves finite colimits and $\mathfrak{A}(-, A)$ converts them into limits. Hence $\tilde{K}\beta = \mu$ is a colimit cone by (6.11). But the conservative and finitary $\tilde{K}$ reflects filtered colimits; so that β is already a colimit cone in $\mathfrak{A}$. If $A \in \mathfrak{A}_f$, it follows from (2.12) that some $g: G \to A$ with $G \in \bar{\mathcal{G}}$ is a retraction, with say $gi = 1$. Then $g: G \to A$ is the coequalizer of 1, $ig: G \to G$, and hence by (4.3) A belongs to $\bar{\mathcal{G}}$, since $\bar{\mathcal{G}}$ is closed under finite colimits. As we already have that $\bar{\mathcal{G}} \subset \mathfrak{A}_f$, we conclude that $\bar{\mathcal{G}} = \mathfrak{A}_f$, and that K coincides with Z. This completes the proof of (i) and of (ii). We also have (iii), since we have seen that the colimit β is preserved by $\tilde{Z}$; so that $\mathcal{A}_f$ is dense in $\mathcal{A}$ by [*] Thm 5.19, and $\tilde{Z}$ is fully faithful. We now have (iv), and also (v) because $\mathcal{A}$ is reflective in the complete $[\mathcal{A}_f{}^{op}, \mathcal{V}]$. As for (vi), we have already observed that each $\tilde{Z}A$ is left exact. For the converse, let $F: \mathcal{A}_f{}^{op} \to \mathcal{V}$ be left exact. Then, by (6.11), F is a filtered colimit of representables in $[\mathcal{A}_f{}^{op}, \mathcal{V}]$. But the representables lie in the replete image of $\tilde{Z}$, since $\tilde{Z}Z \cong Y$; and this image is closed under filtered colimits in $[\mathcal{A}_f{}^{op}, \mathcal{V}]$, since $\tilde{Z}$ is finitary.

(7.3) COROLLARY. *For a cocomplete* $\mathcal{A}$, *the following are equivalent:*

(i) $\mathcal{A}$ *is locally finitely presentable;*

(ii) $\mathcal{A}_f$ *is small and strongly generating;*

(iii) $\mathcal{A}_f$ *is small and dense;*

(iv) $\mathcal{A}$ *is a full reflective subcategory of some* $[\mathcal{T}, \mathcal{V}]$ *with* $\mathcal{T}$ *small and with the inclusion* $\mathcal{A} \to [\mathcal{T}, \mathcal{V}]$ *finitary. (Here the cocompleteness of* $\mathcal{A}$ *is automatic.)*

(7.4) There are simple generalizations in which «finite» is replaced throughout - except in (5.12) - by «of cardinal $< \alpha$», where α is a small regular cardinal. An ordinary category is an *α-category* if its set of morphisms has cardinal $< \alpha$. A cone or a conical limit is *α-small*, or is an *α-cone* or an *α-limit*, if its indexing category is an α-category. An ordinary category $\mathcal{L}$ is *α-filtered* if every functor from an α-category into $\mathcal{L}$ is the base of some inductive cone; and a (small) conical limit is *α-filtered* if its indexing category is α-filtered. Generalizing (1.2), *α-filtered colimits commute in* $\mathcal{S}et$ *with α-limits.* A $\mathcal{V}$-functor is *α-ary* if it preserves α-filtered colimits, and *has a rank* if it is α-ary for some (small) α; whereupon its rank is the least such α. The object G of $\mathcal{A}$ is *α-presentable* if $\mathcal{A}(G,-): \mathcal{A} \to \mathcal{V}$ *is* α-ary, and is *presentable* if it is α-presentable for some α. We write $\mathcal{A}_\alpha$ for the full subcategory of α-presentable objects; and as in (2.6), the α-presentable sets are the α-small ones. The $\mathcal{V}$-category $\mathcal{A}$ is *locally α-presentable* if it is cocomplete and has a strong generator $\mathcal{G} \subset \mathcal{A}_\alpha$; and it is *locally presentable* if it is locally α-presentable for some (small) α. An indexing type $H: \mathcal{K} \to \mathcal{V}$ is *α-small* if $ob\mathcal{K}$ has fewer than α isomorphism classes, each $\mathcal{K}(k, k')$ lies in $\mathcal{V}_\alpha$, and H factorizes through $\mathcal{V}_\alpha$; a limit indexed by such an H is called an *α-limit*; when $\mathcal{V} = \mathcal{S}et$ these reduce to the conical α-limits above; and for a general $\mathcal{V}$ with $\mathcal{V}_0$ l.f.p. they reduce as in (3.4) to conical α-limits and the $x \pitchfork$- with $x \in \mathcal{V}_\alpha$. A $\mathcal{V}$-category is *α-complete* if it admits α-limits, and a $\mathcal{V}$-functor is *α-left-exact* if it preserves them; the α-analogues of the results of Section 4 are all valid. When $\mathcal{V} = \mathcal{S}et$ and $\mathcal{A}$ is locally α-presentable with strong generator $\mathcal{G} \subset \mathcal{A}_\alpha$, the proof in (7.2) carries over, and in particular exhibits $\mathcal{A}_\alpha$ as the closure of $\mathcal{G}$ in $\mathcal{A}$ under α-colimits. Applying this to the l.f.p. and hence locally α-presentable $\mathcal{V}_0$, we see that $\mathcal{V}_{0\alpha}$ is the closure of $\mathcal{V}_{0f}$ under α-colimits. Always supposing that $\mathcal{V}$ satisfies the hypothesis (5.12), we conclude that $x \otimes y \in \mathcal{V}_{0\alpha}$ if $x, y \in \mathcal{V}_{0\alpha}$; and since $I \in \mathcal{V}_{0f} \subset \mathcal{V}_{0\alpha}$, we get from the α-versions of (5.2) and (5.3) the equivalence of $x \in \mathcal{V}_{0\alpha}$ with $x \in \mathcal{V}_\alpha$. Now everything above carries over to the case of general α. There is one comment to be made: since every A in the locally α-presentable $\mathcal{A}$ is a colimit of objects in $\mathcal{A}_\alpha$, and since this colimit is small and hence β-small for some $\beta \geq \alpha$, the object A is, in fact, β-presentable - so that every object in $\mathcal{A}$ is presentable. We continue to write only of the finitary case, leaving the reader to make the easy generalizations - except when the result requires the general case.

(7.5) PROPOSITION. *When $\mathcal{A}$ is l.f.p. so is $\mathcal{A}_0$, and $\mathcal{A}_{0f} = \mathcal{A}_{f0}$. Conversely, a cocomplete $\mathcal{A}$ is l.f.p. if $\mathcal{A}_0$ is l.f.p. and if $\mathcal{A}_{0f} \subset \mathcal{A}_{f0}$.*

PROOF. When $\mathcal{A}$ is l.f.p., $\mathcal{A}_0$ is cocomplete since $\mathcal{A}$ is, by [*] Section 3.8. Moreover each $A \in \mathcal{A}_0$ is by (7.2) a filtered colimit in $\mathcal{A}_0$ of objects $G \in \mathcal{A}_{f0}$; and these colimits are preserved by the $\mathcal{A}_0(G, -)$ with $G \in \mathcal{A}_{f0}$, since $\mathcal{A}_{f0} \subset \mathcal{A}_{0f}$ by (5.3). By [*] Thm 5.19, therefore, the small subcategory $\mathcal{A}_{f0} \subset \mathcal{A}_{0f}$ is dense, and *a fortiori* strongly generating, in $\mathcal{A}_0$; so that $\mathcal{A}_0$ is l.f.p. By (7.2) again, $\mathcal{A}_{0f}$ is the closure of $\mathcal{A}_{f0}$ in $\mathcal{A}_0$ under finite colimits; but this is $\mathcal{A}_{f0}$ itself, since $\mathcal{A}_{f0}$ is closed under finite colimits by (7.2) and (4.7).

Conversely, if $\mathcal{A}_0$ is l.f.p., and $\mathcal{A}$ is cocomplete with $\mathcal{A}_{0f} \subset \mathcal{A}_{f0}$, every $A \in \mathcal{A}$ is the filtered colimit in $\mathcal{A}_0$, and hence in $\mathcal{A}$, of objects

$G \in \mathcal{A}_{0f}$. Because $\mathcal{A}_{0f} \subset \mathcal{A}_{f0}$, this colimit is preserved by the $\mathcal{A}(G,-)$ with $G \in \mathcal{A}_{0f}$. By [*] Thm 5.19, therefore, the full subcategory of $\mathcal{A}$ determined by the objects in $\mathcal{A}_{0f}$ is dense in $\mathcal{A}$; and it is contained in $\mathcal{A}_f$, whence $\mathcal{A}$ is l.f.p.

(7.6) PROPOSITION. *When $\mathcal{A}$ is l.f.p., a functor $T: \mathcal{A} \to \mathcal{B}$ is finitary exactly when $1: TZ \to TZ$ exhibits T as the left Kan extension of TZ along the inclusion $Z: \mathcal{A}_f \to \mathcal{A}$. When, moreover, $\mathcal{B}$ admits filtered colimits, $Lan_Z H$ exists for all $H: \mathcal{A}_f \to \mathcal{B}$; and Lan_Z gives an equivalence between $[\mathcal{A}_f, \mathcal{B}]$ and the $\mathcal{V}$-category $Fin[\mathcal{A},\mathcal{B}]$ of all finitary functors $\mathcal{A} \to \mathcal{B}$, with the restriction $[Z, 1]$ along Z as its equivalence-inverse.*

PROOF. Given (7.2), this follows from Thm 4.98, Thm 4.99, and Lemma 5.18, of [*].

(7.7) REMARK. When $\mathcal{A}$ is l.f.p. so is $\mathcal{A}_0$ by (7.5); and then by (1.3), if $\mathcal{B}$ admits filtered colimits, $T: \mathcal{A} \to \mathcal{B}$ is finitary exactly when T_0: $\mathcal{A}_0 \to \mathcal{B}_0$ is so. It follows from (7.6) that the identity map expresses T as $Lan_Z TZ$ if and only if the identity map expresses T_0 as $Lan_{Z_0} T_0 Z_0$. Thus for any $H: \mathcal{A}_f \to \mathcal{B}$ where $\mathcal{B}$ admits filtered colimits, we have $(Lan_Z H)_0 \cong Lan_{Z_0} H_0$. When $\mathcal{B}$ is cocomplete, it therefore follows from [*] 4.2 an isomorphism

$$\int^{G \in \mathcal{A}_f} \mathcal{A}(ZG, A) \otimes HG \cong \int^{G \in \mathcal{A}_{f0}} \mathcal{A}_0(ZG, A) . HG$$

between functors $\mathcal{A}_0 \to \mathcal{B}_0$. This strikes the author as surprising. When $\mathcal{V} = \mathcal{C}at$ and $\mathcal{A} = \mathcal{V}$, for instance, both $\otimes$ and . are the cartesian product, if we regard $\mathcal{A}_0(ZG, A)$ as the discrete category formed by the objects of $\mathcal{A}(ZG, A) = [ZG, A]$; so that here the difference between $\mathcal{A}(ZG, A) \times HG$ and $\mathcal{A}_0(ZG, A) \times HG$ is exactly balanced out by the extra relations (involving the 2-cells) which occur in the passage in the quotient on the left side.

(7.8) *When $\mathcal{A}$ is l.f.p., a functor $S: \mathcal{A} \to \mathcal{B}$ is a left adjoint precisely when it is cocontinuous.*

PROOF. Since $\mathcal{A}$ has by (7.2) the small dense subcategory $\mathcal{A}_f$, the result follows from [*] Thm 5.33.

(7.9) *When both $\mathcal{A}$ and $\mathcal{B}$ are locally presentable, a functor $T: \mathcal{A} \to \mathcal{B}$ has* *a left adjoint S precisely when it is cocontinuous and has a rank.*

PROOF. For one direction, it suffices to illustrate by the case where $\mathcal{A}$ and $\mathcal{B}$ are l.f.p. and the continuous T is finitary. Then T has a left adjoint by [∗] Thm 5.32, since filtered colimits present $\mathcal{A}_f$ as dense in $\mathcal{A}$, since $\mathcal{B}$ is the closure under small colimits of $\mathcal{B}_f$, since T preserves filtered colimits, and since each $\mathcal{B}(H,-)$ with $H \in \mathcal{B}_f$ also preserves filtered colimits.

For the other direction, it suffices to illustrate by the case where $\mathcal{A}$ and $\mathcal{B}$ are l.f.p. The small subcategory $S(\mathcal{B}_f)$ of $\mathcal{A}$ is then by (7.4) contained in $\mathcal{A}_\alpha$ for some regular cardinal α. Then $S(\mathcal{B}_\alpha) \subset \mathcal{A}_\alpha$, since by the α-analogue of (7.2) the closure of $\mathcal{B}_f$ in $\mathcal{B}$ under α-colimits is $\mathcal{B}_\alpha$, and $\mathcal{A}_\alpha$ is closed under α-colimits. To show that T preserves α-filtered colimits, it suffices by (3.6) to prove that all $\mathcal{B}(H, T-)$ with $H \in \mathcal{B}_\alpha$ does so; but $\mathcal{B}(H, T-) \cong \mathcal{A}(SH, -)$ does so because $SH \in \mathcal{A}_\alpha$.

(7.10) REMARK. We have now used the full strength of Hypothesis (5.12): as remarked in (4.15), the proof of (4.9) used Hypothesis (4.4), that $\mathcal{V}_0$ is l.f.p.; as remarked in (6.13), the proof of (6.9) used the part $\mathcal{V}_{0f} \subset \mathcal{V}_{f0}$ of (5.12); and now the remaining part $\mathcal{V}_{f0} \subset \mathcal{V}_{0f}$ of (5.12) has been used in the proof of (7.1).

8. SOME PROPERTIES OF LOCALLY FINITELY PRESENTABLE $\mathcal{A}$ WHEN $\mathcal{V} = \mathcal{S}et$

(8.1) Because of (7.5), many of the properties of an l.f.p. $\mathcal{A}$ - including various characterizations of $\mathcal{A}_f$ - can be carried over directly from the classical case $\mathcal{V} = \mathcal{S}et$, as studied by Gabriel-Ulmer in [7]. Because the present article is in other respects so near to being self-contained, we take the liberty of recalling some of these here, with sketches of their proofs.

(8.2) Because we are concerned in this section only with $\mathcal{A}$ that are cocomplete and finitely complete, a *regular epimorphism* (in the sense of, say, [8]) is the same thing as a coequalizer of some pair of maps, being the coequalizer of its kernel-pair. The *regular factorization* $f = nq$ of a map $f: A \to B$ is its factorization through the coequalizer q of the kernel-pair of f. When $\mathfrak{A}$ has a generator $\mathcal{G}$, this q is equally the coequalizer of the evident maps $\phi, \psi : \Sigma_{G \in \mathcal{G}} M(G,f).G \to A$, where $M(G,f)$ is the set of those pairs $u, v: G \to A$ with $fu = fv$. Because $\mathfrak{A}$ is finitely complete, the *strong epimorphisms* of [8] coincide with the *extremal epimorphisms* - those maps that factorize through no proper subobject of their codomain. The regular epimorphisms are extremal; and the converse is true if and only if the regular epimorphisms are closed under composition, which is further equivalent to the assertion that the n of any regular factorization $f = nq$ is monomorphic (cf. [8]).

(8.3) (Cf. [7] Section 6.6) *If* $\mathfrak{A}$ *has a generator* $\mathcal{G} \subset \mathfrak{A}_f$ (not necessarily a strong one - $\mathfrak{A}$ could be topological spaces, which is not l.f.p.), *write* $\mathcal{G}^{\sigma}$ *for the closure of* $\mathcal{G}$ *in* $\mathfrak{A}$ *under finite coproducts. Let* $f: A \to B$ *be a regular epimorphism with* $A \in \mathfrak{A}_f$. *Then* $B \in \mathfrak{A}_f$ *precisely when* f *is the coequalizer of maps* $u, v: H \to A$ *with* $H \in \mathcal{G}^{\sigma}$.

PROOF. One direction is trivial by (2.5). For the other, f is by (8.2) the coequalizer of $\phi, \psi : K \to A$ where K is a coproduct of objects of $\mathcal{G}$. The coprojections $i_H : H \to K$ of the finite sub-coproducts of its summands express K as a filtered colimit of objects $H \in \mathcal{G}^{\sigma}$. If B_H is the coequalizer of ϕi_H and ψi_H, we have induced maps $p_H : B_H \to B$; and these express B as a filtered colimit, since colimits commute with colimits, and since the filtered colimit of the functor constant at A is A. The connecting maps $B_H \to B_{H'}$ for $H \subset H'$ being clearly epimorphisms, and each B_H being f.p. by (2.5), some p_H is an isomorphism by (2.12).

(8.4) (Cf. [7] Section 6.6) For any $f: A \to B$ in $\mathfrak{A}$, write $f = n_1 q_1$ for its regular factorization, where $n_1 : A_1 \to B$. Now let $n_1 = n_2 q_2$ be the regular factorization of n_1, with $n_2: A_2 \to B$, and write p_1 for $q_1: A \to A_1$ and p_2 for $q_2 q_1 : A \to A_2$. If we continue thus transfinitely, defining p_α: $A \to A_\alpha$ as the colimit of the p_β for $\beta < \alpha$ when α is a limit ordinal, we get a sequence of factorizations $n_\alpha p_\alpha$ of f, in each of which p_α is an extremal epimorphism. The sequence becomes stationary at some α if and only if n_α is monomorphic, and then $n_\alpha p_\alpha$ is the (unique) factorization of f into an extremal epimorphism and a monomorphism. *If* $\mathfrak{A}$ *has a generator*

$\mathcal{G} \subset \mathcal{A}_f$, *the sequence becomes stationary at the first infinite ordinal* ω. For, in proving n_ω a monomorphism, it suffices to consider pairs $u, v: G \to A_\omega$ with $n_\omega u = n_\omega v$ and $G \in \mathcal{G}$. Because G is f.p. and A_ω is the *filtered* colimit of the A_i with $i < \omega$, such u, v factorize through $A_i \to A_\omega$ for some $i < \omega$, say via $x, y: G \to A_i$. Now, since $n_i x = n_i y$, we have $q_{i+1} x = q_{i+1} y$, giving $u = v$ as desired.

It is clear from (8.2) that an object A of such a category $\mathcal{A}$ has but a small set of regular-epimorphic quotients; and now it follows from the above that *an object* A *of such an* $\mathcal{A}$ *has but a small set of extremal epimorphic quotients. This is true in particular of any l.f.p.* $\mathcal{A}$, *and of any locally presentable* $\mathcal{A}$ by a trivial extension.

(8.5) A generator $\mathcal{G}$ of $\mathcal{A}$ is said to be *projective* when each $\mathcal{G}(G,-)$: $\mathcal{A} \to \mathcal{S}et$ preserves regular epimorphisms. When this is so it follows easily that the n in any regular factorization nq is a monomorphism, so that regular and extremal epimorphisms coincide by (8.2). When the projective generator $\mathcal{G}$ is a *strong* generator, say with inclusion $K: \mathcal{G} \to \mathcal{A}$, the existence of the conservative, right-adjoint $\tilde{K}: \mathcal{A} \to [\mathcal{G}^{op}, \mathcal{S}et]$ shows that $\mathcal{A}$ has the further property of being a *regular category*, in the sense that regular epimorphisms are stable under pullback; for this is trivially true in $\mathcal{S}et$ and hence in $[\mathcal{G}^{op}, \mathcal{S}et]$, and a conservative right adjoint clearly reflects extremal epimorphisms. The classical construction of regular quotients, via congruences, in the category of algebras of a (many-sorted) finitary algebraic theory, shows that the $\{F_x 1\}$ of (3.5) constitute a projective strong generator, so that such a category is regular. We can infer that $\mathcal{C}at_0$ is not such a category; for there the extremal epimorphism from 2 to the one-object category given by the monoid $\{1, e\}$ with $e^2 = e$ is not regular.

(8.6) We have from [7] Satz 7.14 a result stronger than that at the end of (8.4): *if* $\mathcal{A}$ *is locally presentable, any* $A \in \mathcal{A}$ *has but a small set of epimorphic quotients - that is,* $\mathcal{A}$ *is cowellpowered.* ($\mathcal{A}$ is trivially *wellpowered*, since it has a strong generator.) This may be seen by first passing to the comma-category $A/\mathcal{A}$ of objects under A. This is cocomplete; and the projection $d: A/\mathcal{A} \to \mathcal{A}$ is conservative, has the left adjoint $B \mapsto A+B$, and preserves all connected - and hence all filtered - colimits. Accordingly $A/\mathcal{A}$ is locally α-presentable if $\mathcal{A}$ is, by (3.3). Now $f: A \to B$ is epimorphic in $\mathcal{A}$ exactly when $f: 1_A \to f$ is epimorphic in $A/\mathcal{A}$; but 1_A is the initial object in $A/\mathcal{A}$.

It suffices, then, to prove that the initial object 0 of a locally-presentable $\mathcal{A}$ has but a small set of epimorphic quotients. Call an epimorphic image K of 0 an *atom*, and write $\mathcal{K}$ for the full subcategory of $\mathcal{A}$ given by the atoms. Clearly $\mathcal{K}$ is a preordered set, since any two maps $K \to B$ coincide if $K \in \mathcal{K}$. To prove $\mathcal{K}$ small it will more than suffice to prove $\mathcal{K}$ locally presentable; for then it is complete by (7.2) and well-powered by (8.6), and every object is a subobject of the terminal object. In fact, if we choose some regular $\alpha \geq \aleph_1$ such that $\mathcal{A}$ is locally α-presentable, then $\mathcal{K}$ too is locally α-presentable.

First, $\mathcal{K}$ is closed under colimits in $\mathcal{A}$, since if each $0 \to K_i$ is epimorphic, so is $0 = colim\, 0 \to colim\, K_i$; hence $\mathcal{K}$ is cocomplete. We show that $\mathcal{K} \cap \mathcal{A}_\alpha$, which is clearly contained in $\mathcal{K}_\alpha$, is a strong generator for $\mathcal{K}$ - in fact, dense in $\mathcal{K}$. {Then, since $\mathcal{K} \cap \mathcal{A}_\alpha$ is closed in $\mathcal{K}$ under α-colimits, it is in fact by (7.2) (i) the whole of $\mathcal{K}_\alpha$.} Write $Z: \mathcal{A}_\alpha \to \mathcal{A}$ and $Z': \mathcal{K} \cap \mathcal{A}_\alpha \to \mathcal{A}$ for the inclusions; we shall show that for $K \in \mathcal{K}$ the comma-category Z/K has Z'/K as a final (full) subcategory - which will be α-filtered since Z/K is. Then, since K is by (7.2) the canonical Z/K-indexed colimit in $\mathcal{A}$ of the objects in $\mathcal{A}_\alpha$, it is also the canonical Z'/K-indexed colimit, in $\mathcal{A}$ and hence in $\mathcal{K}$, of the objects in $\mathcal{K} \cap \mathcal{A}_\alpha$. This latter α-filtered colimit is preserved by $\mathcal{K}(H,-)$ for all $H \in \mathcal{K} \cap \mathcal{A}_\alpha$; so that by [*] Thm 5.19 these colimits present $\mathcal{K} \cap \mathcal{A}_\alpha$ as dense in $\mathcal{K}$.

Since Z/K is filtered, to prove the finality of Z'/K it suffices by [*] Prop. 4.71 to show that any $f: G \to K$ with $G \in \mathcal{A}_\alpha$ factorizes through some $H \in \mathcal{K} \cap \mathcal{A}_\alpha$. To say that K is an atom is equally to say that the two coprojections $i_K, j_K: K \to K+K$ coincide. It follows that the composites of $i_G, j_G: G \to G+G$ with $f+f: G+G \to K+K$ coincide. Since K is the α-filtered colimit of all the $h: G' \to K$ with $G' \in \mathcal{A}_\alpha$, so $K+K$ is the α-

filtered colimit of the $h+h: G'+G' \to K+K$. Since $\mathcal{A}(G,-)$ preserves this colimit, f is the composite of some $g_1 : G \to G_1$ and some $f_1 : G_1 \to K$, with $G_1 \in \mathcal{A}_a$, such that $(g_1+g_1)i_G = (g_1+g_1)j_G$. Now repeat this process, replacing $f: G \to K$ by $f_1: G_1 \to K$, to get $g_2 : G_1 \to G_2$ with

$$(g_2+g_2)i_{G_1} = (g_2+g_2)\, j_{G_1}$$

and $f_2: G_2 \to K$ with $f_1 = f_2 g_2$; and so on. Finally let the colimit of

$$G \xrightarrow[g_1]{} G_1 \xrightarrow[g_2]{} G_2 \xrightarrow[g_3]{} G_3 \to \cdots$$

be $q_n : G_n \to H$, inducing $\bar{f}: H \to K$. Then $H \in \mathcal{A}_a$, since $\mathcal{A}_a$ is closed under countable colimits. Moreover, since

$$i_H q_n = (q_n + q_n) i_{G_n} = (q_{n+1} + q_{n+1})(g_{n+1} + g_{n+1}) i_{G_n},$$

and since $(g_{n+1}+g_{n+1})i_{G_n} = (g_{n+1}+g_{n+1})j_{G_n}$, we have $i_H q_n = j_H q_n$ for all n; so that $i_H = j_H$, and $H \in \mathcal{K}$. Thus $H \in \mathcal{K} \cap \mathcal{A}_a$, as required.

(8.7) When we regard a small set $\mathcal{G}$ of objects of $\mathcal{A}$ as a *discrete* category $|\mathcal{G}|$, with inclusion $Z : |\mathcal{G}| \to \mathcal{A}$, the counit ϵ of the adjunction

$$- * Z \dashv \tilde{Z} : \mathcal{A} \to [\,|\mathcal{G}|^{op}, \mathcal{S}et]$$

has components $\epsilon_A : \Sigma_{G \in \mathcal{G}} \mathcal{A}(G, A).G \to A$, whose components in turn are given by $\epsilon_{A,G,f} = f$ for $f: G \to A$. By the definitions of [$*$] Section 3.6, $\mathcal{G}$ is a generator [resp. a strong generator] when $\tilde{Z}$ is faithful [resp. conservative], which by [$*$] Section 3.4 is the case precisely when each ϵ_A is an epimorphism [resp. an extremal epimorphism]. We call $\mathcal{G}$ a *regular generator* when each ϵ_A is a regular epimorphism. Then, in the notation of (8.2), we have for each $A \in \mathcal{A}$ a coequalizer diagram

$$(8.8) \qquad \Sigma_{G \in \mathcal{G}} M(G, \epsilon_A).G \underset{\psi}{\overset{\phi}{\rightrightarrows}} \Sigma_{G \in \mathcal{G}} \mathcal{A}(G,A).G \xrightarrow[\epsilon_A]{} A,$$

so that our definition of regular generator agrees with that in [$*$] Section 4.10. It was observed there that each $\mathcal{A}(G,-)$ with $G \in \mathcal{G}$ preserves the colimit (8.8), sending it in fact to a *split* coequalizer diagram in $\mathcal{S}et$. Finally, a small dense full subcategory $\mathcal{G}$ is *a fortiori* a regular generator (cf. [$*$] Section 5.3), since we have the regular epimorphism

$$\Sigma_{G \in \mathcal{G}} \mathcal{A}(G,A).G \to \int^{G \in \mathcal{G}} \mathcal{A}(G,A).G \approx A.$$

(8.9) We have as in [*] Prop. 5.24 that, *if* $\mathcal{G}$ *is a regular generator in* $\mathcal{A}$, *the closure* $\mathcal{G}^{\Sigma}$ *of* $\mathcal{G}$ *under small coproducts is dense in* $\mathcal{A}$; this follows from [*] Thm 5.19, for the coequalizer diagrams (8.8) are preserved by the $\mathcal{A}(H,-)$ for $H \in \mathcal{G}^{\Sigma}$, a product of split coequalizer diagrams being a split coequalizer diagram. More importantly, we have as in [*] Prop. 5.24 that, *if* $\mathcal{G}$ *is a regular generator contained in* $\mathcal{A}_f$ *- so that* $\mathcal{A}$ *is l.f.p. - the closure* $\mathcal{G}^{\sigma}$ *of* $\mathcal{G}$ *under finite coproducts is already dense in* $\mathcal{A}$; this follows from [*] Thm 5.19 because the infinite coproducts in (8.8) are the filtered colimits of their finite sub-coproducts, and these filtered colimits are preserved by the $\mathcal{A}(H,-)$ with $H \in \mathcal{G}^{\sigma}$, since $\mathcal{G}^{\sigma} \subset \mathcal{A}_f$ by (2.5). {In fact, $\mathcal{G}$ itself is dense in $\mathcal{A}$ if coproducts in $\mathcal{A}$ are universal; for this, see [7] Satz 3.7.}

(8.10) It is immediate that, if $\mathcal{G}$ is a generator or a strong generator, so is any small $\mathcal{G}' \supset \mathcal{G}$. For regular generators this is still true but no longer immediate. It suffices to show that, when the ϵ_A of (8.7) are regular epimorphisms, so is $(\epsilon_A, f): (\Sigma\,\mathcal{A}(G,A).G) + C \to A$ for any $f: C \to A$. However the composite of (ϵ_A, f) with

$$1 + \epsilon_C : (\Sigma\,\mathcal{A}(G,A).G) + (\Sigma\,\mathcal{A}(G,C).G) \to (\Sigma\,\mathcal{A}(G,A).G) + C$$

is, by the naturality of ϵ, the composite of ϵ_A with

$$(1, \Sigma\,\mathcal{A}(G,f).G) : (\Sigma\,\mathcal{A}(G,A).G) + (\Sigma\,\mathcal{A}(G,C).G) \to \Sigma\,\mathcal{A}(G,A).G.$$

Since the latter map is a retraction, its composite with ϵ_A is a regular epimorphism by [8] Prop. 2.1; thus $(\epsilon_A, f)(1+\epsilon_C)$ is a regular epimorphism; and, since $1+\epsilon_C$ is an epimorphism, (ϵ_A, f) is a regular epimorphism by [8] Prop. 2.2.

(8.11) (Cf. [7] Satz 7.6) *Let* $\mathcal{G} \subset \mathcal{A}_f$ *be a regular generator of the l.f.p.* $\mathcal{A}$, *and write* $\mathcal{G}^{e}$ *for the set of objects which are coequalizers of pairs* $u, v : H \to K$ *with* $H, K \in \mathcal{G}^{\sigma}$. *Then* $\mathcal{A}_f$ *consists of the retracts of the objects of* $\mathcal{G}^{e}$; *and* $\mathcal{A}_f$ *coincides with* $\mathcal{G}^{e}$ *if all extremal epimorphisms in* $\mathcal{A}$ *are regular.*

PROOF. By (2.5), $\mathcal{A}_f$ certainly contains all the retracts of the objects of

$\mathcal{G}^e$. For the converse, let $A \in \mathcal{A}_f$, and express A as in (8.8) as the coequalizer of maps $\phi, \psi : P \to Q$ with $P, Q \in \mathcal{G}^\Sigma$. In the functor category $[\rightrightarrows, \mathcal{A}]$ consider the canonical cone of vertex (ϕ, ψ) given by all maps $(i, j) : (u, v) \to (\phi, \psi)$ of the form

$$\begin{tikzcd} H \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d, "i"'] & K \arrow[d, "j"] \\ P \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & Q , \end{tikzcd}$$

where $H, K \in \mathcal{G}^\sigma$. It is easy to see that the comma category indexing this cone is filtered; for, since the coproduct P is the filtered colimit of its finite sub-coproducts, every $f : B \to P$ with $B \in \mathcal{A}_f$ factorizes through one of these finite sub-coproducts, which lies in $\mathcal{G}^\sigma$; and similarly for maps $g : B \to Q$. The same reasoning shows that the $i : H \to P$ which occur in this cone include all the coprojections from finite sub-coproducts of P; while the possibility of taking $H = 0$ shows that the $j : K \to Q$ which occur include all the coprojections from finite sub-coproducts of Q; and from these observations it follows that this filtered cone is a colimit-cone. If $A_{u,v} \in \mathcal{G}^e$ denotes the coequalizer of u, v in the diagram above, we conclude that the coequalizer A of ϕ, ψ is the filtered colimit of the $A_{u,v}$; so that, by (2.12), the f.p. A is a retract of some $A_{u,v}$. If the composite extremal epimorphism $K \to A_{u,v} \to A$ is in fact a regular epimorphism, we have by (8.3) that $K \in \mathcal{G}^e$.

(8.12) When $\mathcal{A}$ is the category of algebras for an X-sorted finitary algebraic theory, as in (2.8), it follows from (8.5) that extremal epimorphisms are regular and that the $\{F_x 1\}_{x \in X}$ constitute a regular generator. By (8.11), therefore, $\mathcal{A}_f$ consists of the coequalizers of the finite coproducts of the $F_x 1$, as stated in (2.7) and (2.8). When $\mathcal{A} = \mathcal{C}at_0$, the ordinary category of small categories, it is easy to verify (cf. [$*$] Section 5.3) that the ordered set $3 = \{0, 1, 2\}$ is dense in $\mathcal{A}$. *A fortiori* the set $\mathcal{G}$ of all free categories Fg on some finite graph g is dense, since 3 is such a category. Hence $\mathcal{G}$ is certainly a regular generator, and $\mathcal{G} \subset \mathcal{A}_f$ by (2.9). Thus, by

(8.11), any $A \in \mathcal{A}_f$ is a retract of an object of $\mathcal{G}^e$; so that there is some extremal epimorphism $p: Fg \to A$. If h is the image in A of the finite graph g under p, then h too is a finite graph, with the same objects as A, which generates A, the induced $q: Fh \to A$ being an extremal epimorphism. But such a q, which is the identity on objects, is easily seen to be a regular epimorphism. Hence by (8.3) we conclude that $A \in \mathcal{G}^e$ as asserted in (2.9). Similar arguments apply to the category $\mathcal{G}pd_0$ of small groupoids, and to the subcategories of $\mathcal{C}at_0$ and $\mathcal{G}pd_0$ mentioned in (3.5).

9. FINITARY ESSENTIALLY-ALGEBRAIC $\mathcal{V}$-THEORIES

(9.1) Whenever $\mathcal{V}_0$ is l.f.p., the closed category $\mathcal{V}$ is *locally bounded* in the sense of [*] Section 6.1, as is pointed out there; so that all the results of [*] Ch. 6 apply. Our aim now is to give more delicate results under our present stronger hypothesis of (5.12), that $\mathcal{V}$ is l.f.p. as a closed category. In addition, some of the results of [*] admit simpler proofs under this hypothesis.

(9.2) By a *finitary essentially-algebraic $\mathcal{V}$-theory* we mean a small finitely-complete $\mathcal{V}$-category $\mathcal{T}$. If $\mathcal{F}$ denotes the set of finite indexing types, this is the same thing as an $\mathcal{F}$-theory in the sense of [*] Section 6.3. We shall henceforth omit, as understood, the adjective «essentially-algebraic», and call $\mathcal{T}$ a *finitary $\mathcal{V}$-theory,* or just a *finitary theory*. A *model* of $\mathcal{T}$ in $\mathcal{B}$ is a left-exact functor $\mathcal{T} \to \mathcal{B}$; and these form a $\mathcal{V}$-category

$$Mod[\mathcal{T}, \mathcal{B}] = Lex[\mathcal{T}, \mathcal{B}] \subset [\mathcal{T}, \mathcal{B}].$$

A model of $\mathcal{T}$ in $\mathcal{V}$ is called a *$\mathcal{T}$-algebra* ; and these form a $\mathcal{V}$-category

$$\mathcal{T}\text{-}Alg = Mod[\mathcal{T}, \mathcal{V}] = Lex[\mathcal{T}, \mathcal{V}] \subset [\mathcal{T}, \mathcal{V}].$$

Similarly, for a regular cardinal α, we define an *α-ary $\mathcal{V}$-theory* to be a small α-complete $\mathcal{V}$-category $\mathcal{T}$; and now a model or an algebra is to be α-left-exact. The results below for finitary theories extend at once to α-ary ones.

(9.3) For any small $\mathcal{V}$-category $\mathcal{K}$, let $K: \mathcal{K} \to \bar{\mathcal{K}}$ be the full embedding into its free completion under finite limits ; see [*] Section 5.7. Then, by

(7.1) above, $\bar{\mathcal{K}}$ is a finitary theory. By [*] Thm 5.35, *restriction along K gives for every finitely-complete $\mathcal{B}$ an equivalence $Mod[\bar{\mathcal{K}}, \mathcal{B}] \simeq [\mathcal{K}, \mathcal{B}]$, whose equivalence-inverse is Ran_K (right Kan extension along K).* When $\mathcal{K}$ is the unit $\mathcal{V}$-category $\mathcal{I}$, it follows from [*] Thm 5.35 together with (3.4) and (7.2) above that $\bar{\mathcal{K}} = \mathcal{V}_f{}^{op}$; and now, since $Mod[\mathcal{V}_f{}^{op}, \mathcal{B}] \simeq \mathcal{B}$ for any finitely-complete $\mathcal{B}$, we may call $\mathcal{V}_f{}^{op}$ *the finitary theory of an object.*

(9.4) For any $\mathcal{V}$-theory $\mathcal{T}$, the representable $\mathcal{T}(t,-): \mathcal{T} \to \mathcal{V}$ preserves all limits and is therefore a $\mathcal{T}$-algebra. We conclude from [*] Prop. 5.16 that *the Yoneda embedding $Y: \mathcal{T}^{op} \to [\mathcal{T}, \mathcal{V}]$ factorizes through $\mathcal{T}$-Alg, say via the fully-faithful $Z: \mathcal{T}^{op} \to \mathcal{T}$-$Alg$; the full inclusion $\mathcal{T}$-$Alg \to [\mathcal{T}, \mathcal{V}]$ is isomorphic to $\tilde{Z}$ where $\tilde{Z}F = \mathcal{T}$-$Alg(Z-, F)$; and Z is a dense embedding of $\mathcal{T}^{op}$ in $\mathcal{T}$-Alg.* It also follows from (4.12) that *$\mathcal{T}$-Alg is closed in $[\mathcal{T}, \mathcal{V}]$ under filtered colimits, so that $\mathcal{T}$-Alg admits filtered colimits and $\tilde{Z}$ is finitary.* (We know from [*] Thm 6.11 that $\mathcal{T}$-Alg is in fact reflective in $[\mathcal{T}, \mathcal{V}]$, and hence complete and cocomplete; but in the present hypotheses we get a simpler proof of this in (9.7) below.)

(9.5) By a *map $M: \mathcal{T} \to \mathcal{T}'$ of finitary $\mathcal{V}$-theories* we mean a left-exact functor. Since $FM: \mathcal{T} \to \mathcal{V}$ is left exact when $F: \mathcal{T}' \to \mathcal{V}$ is, the functor $[M, 1]: [\mathcal{T}', \mathcal{V}] \to [\mathcal{T}, \mathcal{V}]$ restricts to a functor $M^*: \mathcal{T}'$-$Alg \to \mathcal{T}$-Alg; such a functor between categories of algebras induced by a map of theories is called an *algebraic functor*. Note that *M^* is finitary*; for $[M, 1]$ preserves all colimits, while $\mathcal{T}'$-Alg and $\mathcal{T}$-Alg are closed under filtered colimits. Now by (6.12), *the left adjoint $Lan_M: [\mathcal{T}, \mathcal{V}] \to [\mathcal{T}', \mathcal{V}]$ of $[M, 1]$ restricts to a functor $M_*: \mathcal{T}$-$Alg \to \mathcal{T}'$-Alg left adjoint to M^*.* Since

$$Lan_M \mathcal{T}'(s,-) = \tilde{M} - * \mathcal{T}'(s,-) \cong (\tilde{M}-)s = \mathcal{T}(Ms,-),$$

we have commutativity to within isomorphism in each square of the diagram

$$(9.6) \qquad \begin{tikzcd} \mathcal{T}^{op} \arrow[r, "Z"] \arrow[d, "M^{op}"'] & \mathcal{T}\text{-}Alg \arrow[r, "\tilde{Z}"] \arrow[d, "M_*"] & {[\mathcal{T}, \mathcal{V}]} \arrow[d, "Lan_M"] \\ \mathcal{T}'^{op} \arrow[r, "Z'"'] & \mathcal{T}'\text{-}Alg \arrow[r, "\tilde{Z}'"'] & {[\mathcal{T}', \mathcal{V}]} \end{tikzcd} .$$

The commutativity of the left square here is, by [*] Prop. 6.16, a general fact about maps of $\mathcal{F}$-theories for any $\mathcal{F}$ and any $\mathcal{V}$; that of the right square, however, depending on (6.12), is special to l.f.p. $\mathcal{V}$ and finitary (or more generally α-ary) theories.

(9.7) For the finitary theory $\mathcal{T}$, let $K: \mathcal{T} \to \bar{\mathcal{T}}$ be, as in (9.3), the embedding into the free finite-limit completion of $\mathcal{T}$. Since $\mathcal{T}$ is finitely complete, there is a left-exact $L: \bar{\mathcal{T}} \to \mathcal{T}$, unique to within isomorphism, for which $LK \cong 1: \mathcal{T} \to \mathcal{T}$. The composite of $L^*: \mathcal{T}\text{-}Alg \to \bar{\mathcal{T}}\text{-}Alg$ (which is just composition with L) and the equivalence $\bar{\mathcal{T}}\text{-}Alg \simeq [\mathcal{T}, \mathcal{V}]$ of (9.3) (which is given by composition with K) is just the inclusion $\tilde{Z}$: $\mathcal{T}\text{-}Alg \to [\mathcal{T}, \mathcal{V}]$. Since L^* has a left adjoint by (9.5), so has $\tilde{Z}$; so that *$\mathcal{T}$-Alg is a reflective full subcategory of $[\mathcal{T}, \mathcal{V}]$.*

(9.8) THEOREM. *For a finitary theory $\mathcal{T}$, the category $\mathcal{T}$-Alg is l.f.p.; and $(\mathcal{T}\text{-}Alg)_f$ is the replete image of the embedding $Z: \mathcal{T}^{op} \to \mathcal{T}\text{-}Alg$, and is hence equivalent to $\mathcal{T}^{op}$. Moreover a category $\mathcal{A}$ is equivalent to $\mathcal{T}$-Alg for some finitary theory $\mathcal{T}$ precisely when $\mathcal{A}$ is l.f.p., whereupon $\mathcal{T} \simeq \mathcal{A}_f^{op}$.*

PROOF. $\mathcal{T}$-Alg is l.f.p. by (3.1) and (9.7), since $\tilde{Z}$ is finitary by (9.4). The representables $\mathcal{T}(t,-)$ being f.p. in $[\mathcal{T}, \mathcal{V}]$ by (2.2), their images under the left adjoint $-*Z$ of $\tilde{Z}$ are f.p. in $\mathcal{T}$-Alg by (2.4). But they are their own images, since the representables already lie in $\mathcal{T}$-Alg; whence $\mathcal{T}^{op} \subset (\mathcal{T}\text{-}Alg)_f$. Since $\mathcal{T}^{op}$ is dense in $\mathcal{T}$-Alg by (9.4), it follows from (7.2) that $(\mathcal{T}\text{-}Alg)_f$ is the closure of $\mathcal{T}^{op}$ in $\mathcal{T}$-Alg under finite colimits. This, however, is just the repletion of $\mathcal{T}^{op}$; for $Z: \mathcal{T}^{op} \to \mathcal{T}\text{-}Alg$ preserves finite colimits. To see this, it suffices to observe that $\mathcal{T}\text{-}Alg(Z-, F)$: $\mathcal{T} \to \mathcal{V}$ preserves finite limits for each $F \in \mathcal{T}\text{-}Alg$; but $\mathcal{T}\text{-}Alg(Z-, F)$ is just $\tilde{Z}F$, or F seen as an object of $[\mathcal{T}, \mathcal{V}]$ - and the algebra F is by definition left exact. The final assertion now follows from (7.2).

(9.9) THEOREM. *Let $\mathcal{T}$ be a finitary theory and let $\mathcal{B}$ be cocomplete. A functor $S: \mathcal{T}\text{-}Alg \to \mathcal{B}$ has a left adjoint T precisely when it is cocontinuous; and then its restriction $G = SZ: \mathcal{T}^{op} \to \mathcal{B}$ is a comodel for $\mathcal{T}$, in the sense that $G^{op}: \mathcal{T} \to \mathcal{B}^{op}$ is a model. Moreover $1: SZ \to SZ$ then expresses S as $Lan_Z G$. On the other hand, a given functor $G: \mathcal{T}^{op} \to \mathcal{B}$ is a comodel precisely when $\tilde{G}: \mathcal{B} \to [\mathcal{T}, \mathcal{V}]$ factorizes through the full em-*

bedding $\bar{Z}: \mathcal{T}\text{-}Alg \to [\mathcal{T}, \mathcal{V}]$, *say as* $\bar{G} \cong \bar{Z}\, T$; *then* T *has the left adjoint* $S = Lan_Z G$, *and* $SZ \cong G$. *In this way we obtain an equivalence of* $\mathcal{V}$-*categories*

$$Cocts[\mathcal{T}\text{-}Alg, \mathcal{B}] = Ladj[\mathcal{T}\text{-}Alg, \mathcal{B}] \simeq Com[\mathcal{T}^{op}, \mathcal{B}] = Mod[\mathcal{T}, \mathcal{B}^{op}]^{op},$$

the equivalence from left to right being composition with Z *and that from right to left being* Lan_Z.

PROOF. When account is taken of [*] Thm 6.11, this is a special case of [*] Thm 5.56. The latter result, however, is very general; accordingly, we give a simple direct proof in the present case. Z being fully faithful, so is Lan_Z by [*] Thm 4.99, and $(Lan_Z G)Z \cong G$ for any G by [*] Prop. 4.23. We have therefore only to identify the image under Lan_Z of the comodels G. Since SZ is trivially a comodel if S is cocontinuous, this image contains the cocontinuous S by [*] Thm 5.29; and a cocontinuous S is left adjoint by (7.8) and (9.8). For the other direction, we observe that G is a comodel when G preserves finite colimits, which is to say that $\mathcal{B}(G-, B): \mathcal{T} \to \mathcal{V}$ preserves finite limits for each $B \in \mathcal{B}$, or that $\bar{G}: \mathcal{B} \to [\mathcal{T}, \mathcal{V}]$ factorizes through $\mathcal{T}\text{-}Alg$ as $\bar{G} \cong \bar{Z}\, T$ for some T. But then T has as left adjoint S the restriction $(- * G)\bar{Z} = \bar{Z} - * G$ of the left adjoint $- * G$ of $\bar{G}$; so that $S = Lan_Z G$ by [*] Section 4.1. Thus for each model G, $Lan_Z G$ is cocontinuous.

(9.10) THEOREM. *Let* $\mathcal{T}$ *be a finitary theory, let* $\mathcal{B}$ *be cocomplete, and let the adjunction* $S \dashv T: \mathcal{B} \to \mathcal{T}\text{-}Alg$ *be related to the comodel* $G: \mathcal{T}^{op} \to \mathcal{B}$ *as in (9.9), by* $S \cong Lan_Z G$ *and* $G \cong SZ$. *Then* T *is finitary if and only if* $G(\mathcal{T}^{op}) \subset \mathcal{B}_f$.

PROOF. If T is finitary, (2.4) gives $S((\mathcal{T}\text{-}Alg)_f) \subset \mathcal{B}_f$, and hence $G(\mathcal{T}^{op}) \subset \mathcal{B}_f$, by (9.9) and (9.8). For the converse, since $\bar{Z}$ is finitary and conservative, T is finitary by (1.3) if $\bar{G}$ is so. Since colimits in $[\mathcal{T}, \mathcal{V}]$ are formed pointwise, $\bar{G}$ is finitary when each $E_t\bar{G}: \mathcal{B} \to \mathcal{V}$ is finitary. However $E_t\bar{G} = \mathcal{B}(Gt, -)$, which is finitary since $Gt \in \mathcal{B}_f$.

(9.11) PROPOSITION (cf. [7] Satz 7.8). *Suppose that the* $G: \mathcal{T}^{op} \to \mathcal{B}$ *of (9.10) not only takes its values in* $\mathcal{B}_f$, *but is fully faithful. Then* S *is* *fully faithful, and its image is the closure of* $G(\mathcal{T}^{op})$ *in* $\mathcal{B}$ *under small colimits.*

PROOF. S is fully faithful precisely when the unit $\eta: 1 \to TS$ is an isomorphism. Since both T and S are finitary, and since $\mathcal{T}$-Alg is by (6.11) the closure of $\mathcal{T}^{op}$ under filtered colimits, it suffices to prove that $\eta Z: Z \to TSZ$ is an isomorphism; or equivalently that $\tilde{Z}\eta Z: \tilde{Z}Z \to \tilde{Z}TSZ$ is an isomorphism in $[\mathcal{T}, \mathcal{V}]$. But $\tilde{Z}Z \cong Y$, and $\tilde{Z}TSZ \cong \tilde{G}G$ by (9.9); and $\tilde{Z}\eta Z$ is the canonical map $Y \to \tilde{G}G$, which is an isomorphism (see [*] Section 4.2) if G is fully faithful. Since $\mathcal{T}$-Alg is the closure of $\mathcal{T}^{op}$ under filtered colimits, the image of S is contained in the closure in $\mathcal{B}$ of $G(\mathcal{T}^{op})$ under small colimits, and therefore coincides with this closure.

(9.12) EXAMPLE (cf. [7], *loc. cit.*). With $\mathcal{V} = \mathcal{S}et$, let $\mathcal{B}$ be the *dual* of the category of compact Hausdorff spaces. By [7] Section 6.5, $\mathcal{B}_f$ consists of the finite spaces. Taking $\mathcal{T} = \mathcal{S}et_f \simeq (\mathcal{B}_f)^{op}$ and G to be the inclusion, we get by (9.11) an identification of $(\mathcal{S}et_f\text{-}Alg)^{op}$ with the small-limit closure in $\mathcal{B}^{op}$ of the finite spaces: that is, the category of totally-disconnected compact spaces. On the other hand, $\mathcal{S}et_f\text{-}Alg$ is easily seen to be the category of boolean algebras.

(9.13) PROPOSITION. *If* $\mathcal{T}$ *and* $\mathcal{T}'$ *are finitary theories, a functor* $T: \mathcal{T}'\text{-}Alg \to \mathcal{T}\text{-}Alg$ *is algebraic - that is, of the form* M^* *for some map* $M: \mathcal{T}' \to \mathcal{T}$ *of theories - precisely when it is finitary and has a left adjoint.*

PROOF. One direction comes from (9.5), and the other from (9.9) and (9.10) on taking $\mathcal{B} = \mathcal{T}'\text{-}Alg$.

(9.14) We can improve (9.13) by observing that *every* $\mathcal{V}$*-natural transformation* $M_1^* \to M_2^*$ *between algebraic functors is induced by a unique* $\mathcal{V}$*-natural transformation* $M_1 \to M_2$. In fact the situation is still richer than this. Denote by $\mathcal{V}$-$\mathcal{T}h$ the ($\mathcal{V}$-$\mathcal{C}at$)-category whose objects are the finitary $\mathcal{V}$-theories and whose hom-object $\mathcal{V}\text{-}\mathcal{T}h(\mathcal{T}, \mathcal{T}')$ is the $\mathcal{V}$-category $Lex[\mathcal{T}, \mathcal{T}']$, a full subcategory of $[\mathcal{T}, \mathcal{T}']$. Denote by $\mathcal{V}$-$\mathcal{L}fp$ the ($\mathcal{V}$-$\mathcal{C}at$)-category whose objects are the l.f.p. $\mathcal{V}$-categories and whose hom-object $\mathcal{V}\text{-}\mathcal{L}fp(\mathcal{A}, \mathcal{A}')$ is the full sub-$\mathcal{V}$-category of $[\mathcal{A}, \mathcal{A}']$ given by the finitary $\mathcal{V}$-functors with left adjoints. Then (9.9) and (9.10) give an equivalence

$$(9.15) \qquad \mathcal{V}\text{-}\mathcal{T}\!h(\mathcal{T},\mathcal{T}') \simeq \mathcal{V}\text{-}\mathcal{L}\!f\!p(\mathcal{T}'\text{-}Alg, \mathcal{T}\text{-}Alg)\,;$$

so that, by (9.8), we have in fact a *biequivalence*

$$(9.16) \qquad (\mathcal{V}\text{-}\mathcal{T}\!h)^{op} \approx \mathcal{V}\text{-}\mathcal{L}\!f\!p\,.$$

The functor sending the $\mathcal{V}$-theory $\mathcal{T}$ to $\mathcal{T}$-*Alg* may be called the *semantics* functor *Sem*, and that sending the l.f.p. $\mathcal{A}$ to $\mathcal{A}_f^{op}$ may be called the *structure* functor *Str*. We can in fact define *Str* on a category $\mathcal{V}$-$\mathcal{F}\!in$ bigger than $\mathcal{V}$-$\mathcal{L}\!f\!p$; the objects are now all cocomplete $\mathcal{A}$ with $\mathcal{A}_f$ small, and the hom-object is still the full subcategory of $[\mathcal{A},\mathcal{A}']$ given by the finitary functors with left adjoints; this hom-object is in general not a $\mathcal{V}$-category, but a $\mathcal{V}'$-category for some larger $\mathcal{V}'$ in a higher universe. If $Str\,\mathcal{A}$ is still defined as $\mathcal{A}_f^{op}$, we have a *biadjunction*

$$(9.17) \qquad Str \dashv Sem:\ (\mathcal{V}\text{-}\mathcal{T}\!h)^{op} \to \mathcal{V}\text{-}\mathcal{F}\!in\,,$$

since (9.9) and (9.10) give $\mathcal{V}\text{-}\mathcal{T}\!h(\mathcal{T},\mathcal{A}_f^{op}) \simeq \mathcal{V}\text{-}\mathcal{F}\!in(\mathcal{A},\mathcal{T}\text{-}Alg)$.

(9.18) It follows from (7.5) that *if* $\mathcal{T}$ *is a finitary* $\mathcal{V}$*-theory, then* $\mathcal{T}_0$ *is a finitary* $\mathcal{S}et$*-theory and* $(\mathcal{T}\text{-}Alg)_0 \simeq \mathcal{T}_0\text{-}Alg$. Using (6.11), (7.2), (7.5) and (9.8), we can describe the equivalence in detail: the $\mathcal{T}$-algebra $F\colon \mathcal{T} \to \mathcal{V}$ is sent to the $\mathcal{T}_0$-algebra $VF_0\colon \mathcal{T}_0 \to \mathcal{S}et$; while the equivalence-inverse sends the $\mathcal{T}_0$-algebra $H\colon \mathcal{T}_0 \to \mathcal{S}et$ to the colimit F in $[\mathcal{T},\mathcal{V}]$ of the canoncial functor $Y_0 d^{op}\colon (el H)^{op} \to [\mathcal{T},\mathcal{V}]_0$ of (6.5). Not every finitary $\mathcal{S}et$-theory need admit an enrichment to a finitary $\mathcal{V}$-theory, since an l.f.p. ordinary category need admit no enrichment to a $\mathcal{V}$-category. Thus, when $\mathcal{V} = \mathcal{A}b$, the l.f.p. category of all groups admits no additive structure. When $\mathcal{V} = \mathcal{C}at$ or $\mathcal{G}pd$, however, every finitary $\mathcal{S}et$-theory $\mathcal{S}$ admits at least one enrichment to a finitary $\mathcal{V}$-theory $\mathcal{T}$; it suffices to take $\mathcal{T}$ to be $\mathcal{S}$, made into a 2-category with only identity 2-cells; then $\mathcal{T}$ trivially admits the cotensor product $2 \pitchfork t$, and is hence finitely complete as a 2-category and *a fortiori* as a $\mathcal{G}pd$-category. Such an $\mathcal{S}$ may admit two *different* enrichments to a finitary $\mathcal{V}$-theory: taking $\mathcal{V} = \mathcal{C}at$ again, we have the $\mathcal{V}$-theory $\mathcal{C}at_f^{op}$ of an object; and if $\mathcal{S}$ is the underlying category $\mathcal{C}at_{0f}^{op}$ of this, the $\mathcal{T}$ constructed above is not $\mathcal{C}at_f^{op}$. In fact,

$(\mathcal{C}at_f{}^{op})\text{-}Alg = \mathcal{C}at$, while $\mathcal{T}\text{-}Alg$ is the 2-category of all small categories, all functors, but only identity 2-cells.

10. FINITARY $\mathcal{V}$-SKETCHES

(10.1) By a *finitary* $\mathcal{V}$-*sketch* $(\mathcal{S}, \Phi)$, or just Φ for short, we mean a small $\mathcal{V}$-category $\mathcal{S}$ together with a small set $\Phi = \{\Phi_\gamma : H_\gamma \to \mathcal{S}(A_\gamma, P_\gamma -)\}_{\gamma \in \Gamma}$, where the $H_\gamma : \mathcal{K}_\gamma \to \mathcal{V}$ are finite indexing types, where $P_\gamma : \mathcal{K}_\gamma \to \mathcal{S}$, and where $A_\gamma \in \mathcal{S}$. A *model* of Φ in $\mathcal{B}$ is a functor $F : \mathcal{S} \to \mathcal{B}$ such that each cylinder

$$H_\gamma \xrightarrow[\Phi_\gamma]{} \mathcal{S}(A_\gamma, P_\gamma -) \xrightarrow[F]{} \mathcal{B}(FA_\gamma, FP_\gamma -)$$

is a limit-cylinder in $\mathcal{B}$; and a model of Φ in $\mathcal{V}$ is called a Φ-algebra; see [∗] Section 6.3. The results of [∗] Ch. 6 all apply; and in fact the proof of [∗] Thm 6.5 can be simplified in our present case, replacing the appeal to Theorem 10.2 of [9] by one to the simpler Theorem 6.2 of [9]. In particular, from [∗] Section 6.4, we get

(10.2) THEOREM. *Consider the composite of the Yoneda embedding* $\mathcal{S}^{op} \to [\mathcal{S}, \mathcal{V}]$ *and the reflexion of* $[\mathcal{S}, \mathcal{V}]$ *onto* $\Phi\text{-}Alg$, *and let* $\mathcal{T}^{op}$ *be the closure of its image in* $\Phi\text{-}Alg$ *under finite colimits, so that we have a functor* $K : \mathcal{S} \to \mathcal{T}$. *Then* K *is a model of* Φ, *and is fully faithful precisely when the cylinders* Φ_γ *are already limi-cylinders in* $\mathcal{S}$. *The sketch* Φ *and the finitary theory* $\mathcal{T}$ *have the same models in any finitely-complete* $\mathcal{B}$, *restriction along* K *giving an equivalence* $Mod[\mathcal{T}, \mathcal{B}] \simeq \Phi\text{-}Mod[\mathcal{S}, \mathcal{B}]$, *whose equivalence-inverse is given by right Kan extension along* K.

(10.3) If $\mathcal{T}$ and $\mathcal{T}'$ are finitary $\mathcal{V}$-theories, we have as in [∗] Section 6.5 the sketch $(\mathcal{T} \otimes \mathcal{T}', \Phi)$, where Φ is so chosen that a Φ-model in $\mathcal{B}$ is a functor $F : \mathcal{T} \otimes \mathcal{T}' \to \mathcal{B}$ with each $F(t, -)$ and each $F(-, s)$ left exact. The corresponding finitary theory in the sense of (10.2), which was denoted in [∗] by $\mathcal{T} \otimes_{\mathcal{F}} \mathcal{T}'$ when the appropriate set of indexing types was $\mathcal{F}$, may be denoted here by $\mathcal{T} \otimes_{lex} \mathcal{T}'$.

(10.4) PROPOSITION. *If* $\mathcal{A}$ *is l.f.p. and* $\mathcal{T}$ *is a finitary theory, then* $Mod[\mathcal{T}, \mathcal{A}]$ *is l.f.p.*

PROOF. $\mathcal{A}$ is equivalent by (9.8) to $\mathcal{T}'$-*Alg* for some finitary theory $\mathcal{T}'$. Now $Mod[\mathcal{T},\mathcal{A}] \simeq Mod[\mathcal{T}, Mod[\mathcal{T}',\mathcal{V}]]$ is equivalent by [∗] Section 6.5 to the l.f.p. category $(\mathcal{T}\otimes_{\mathrm{lex}}\mathcal{T}')$-*Alg*.

(10.5) Let $\mathcal{T}$ and $\mathcal{T}'$ be finitary $\mathcal{V}$-theories as in (10.3), and let $K: \mathcal{T}\otimes\mathcal{T}' \to \mathcal{T}\otimes_{\mathrm{lex}}\mathcal{T}'$ be the «generic model» of Φ, as in (10.2). At the level of ordinary categories, let $(\mathcal{T}_0 \times \mathcal{T}'_0, \Psi)$ be the sketch whose algebras in $\mathcal{C}$ are the functors $\mathcal{T}_0 \times \mathcal{T}'_0 \to \mathcal{C}$ left exact in each variable separately, and let $L: \mathcal{T}_0 \times \mathcal{T}'_0 \to \mathcal{T}_0 \otimes_{\mathrm{lex}} \mathcal{T}'_0$ be the generic model. Let $N: \mathcal{T}_0 \times \mathcal{T}'_0 \to (\mathcal{T}\otimes\mathcal{T}')_0$ be the canonical functor of [∗] Section 1.4, and recall that, if $F: \mathcal{T}\otimes\mathcal{T}' \to \mathcal{A}$ is any $\mathcal{V}$-functor, the partial functors of $F_0 N$ are $(F(t,-))_0$ and $(F(-,s))_0$. We see at once that the composite $K_0 N: \mathcal{T}_0 \times \mathcal{T}'_0 \to (\mathcal{T}\otimes_{\mathrm{lex}}\mathcal{T}')_0$ is left exact in each variable separately, and therefore factorizes (to within isomorphism) through L to give a map $M: \mathcal{T}_0 \otimes_{\mathrm{lex}} \mathcal{T}'_0 \to (\mathcal{T}\otimes_{\mathrm{lex}}\mathcal{T}')_0$ of theories. This map is not an equivalence of theories: the theory $\mathcal{V}_f^{op}$ of an object is the identity for the $\otimes_{\mathrm{lex}}$ of $\mathcal{V}$-theories, so that

$$(\mathcal{V}_f^{op} \otimes_{\mathrm{lex}} \mathcal{V}_f^{op})_0 \simeq \mathcal{V}_{f0}^{op} = \mathcal{V}_{0f}^{op},$$

which is different from $\mathcal{V}_{f0}^{op} \otimes_{\mathrm{lex}} \mathcal{V}_{f0}^{op}$; when $\mathcal{V} = \mathcal{C}at$, the first is the theory of a category, while the second is the theory of a double category. More generally, we have for any finitary $\mathcal{V}$-theory $\mathcal{T}$ a map of theories

$$M: \mathcal{V}_{0f}^{op} \otimes_{\mathrm{lex}} \mathcal{T}_0 \to (\mathcal{V}_f^{op} \otimes \mathcal{T})_0 \simeq \mathcal{T}_0 ;$$

the corresponding functor $\mathcal{V}_{0f}^{op} \times \mathcal{T}_0 \to \mathcal{T}_0$ left exact in each variable is that sending (x,t) to $x \pitchfork t$. The relation between theory-maps

$$\mathcal{V}_{0f}^{op} \otimes_{\mathrm{lex}} \mathcal{S} \to \mathcal{S}$$

and $\mathcal{V}$-enrichments of the finitary $\mathcal{S}et$-theory $\mathcal{S}$ will be discussed in the second part of this article.

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Department of Mathematics
UNIVERSITY OF SYDNEY
N.S.W. 2006
AUSTRALIE