CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

RONALD BROWN
PHILIP J. HIGGINS

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Cahiers de topologie et géométrie différentielle catégoriques, tome 22, nº 4 (1981), p. 371-386

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE
Vol. XXII-4 (1981)

3^e COLLOQUE SUR LES CATEGORIES
DEDIE A CHARLES EHRESMANN
Amiens, Juillet 1980

THE EQUIVALENCE OF ∞-GROUPOIDS AND CROSSED COMPLEXES

by Ronald BROWN and Philip J. HIGGINS

INTRODUCTION.

Multiple categories, and in particular n-fold categories and n-categories, have been considered by various authors [8, 11]. The object of this paper is to define ∞-categories (which are n-categories for all n) and ∞-groupoids and to prove the equivalence of categories

$$(\infty\text{-}groupoids) \longleftrightarrow (crossed\ complexes).$$

This result was stated (without definitions) in the previous paper [3] and there placed in a general pattern of equivalences between categories, each of which can be viewed as a higher-dimensional version of the category of groups.

The interest of the above equivalence arises from the common use of n-categories, particularly in situations describing homotopies, homotopies of homotopies, etc..., and also from the fact that ∞-groupoids can be regarded as a kind of half-way house between ω-groupoids and crossed complexes. It is easy to construct from any ω-groupoid a subset which has the structure of an ∞-groupoid and contains the associated crossed complex. In this way we get a diagram of functors

$$\begin{array}{ccc} (\omega\text{-}groupoids) & \longrightarrow & (\infty\text{-}groupoids) \\ & \nwarrow\!\searrow \quad \swarrow\!\nearrow & \\ & (crossed\ complexes) & \end{array}$$

which is commutative up to natural isomorphism. The reverse equivalence

$$(\infty\text{-}groupoids) \longrightarrow (\omega\text{-}groupoids)$$

has, however, proved difficult to describe directly.

Earlier results which point the way to some of the equivalences described here and in [3] are:

(a) the equivalence between 2-categories and double categories with connection described in [10],

(b) the equivalences between double groupoids with connections, $\mathcal{G}$-groupoids (i.e. group objects in the category of groupoids) and crossed modules established in [4, 5], and

(c) the equivalence between simplicial Abelian groups and chain complexes proved in [7, 9].

1. ∞-CATEGORIES AND ∞-GROUPOIDS.

An *n-fold category* is a class A together with n mutually compatible category structures $A^i = (A, d_i^0, d_i^1, \underset{i}{+})$ $(0 \leq i \leq n-1)$ each with A as its class of morphisms (and with d_i^0, d_i^1 giving the initial and final identities for $\underset{i}{+}$). The objects of the category structure A^i are here regarded as members of A, coinciding with the identity morphisms of A^i. The compatibility conditions are:

(1.1) $d_i^\alpha d_j^\beta = d_j^\beta d_i^\alpha$ for $i \neq j$ and $\alpha, \beta \in \{0, 1\}$;

(1.2) $d_i^\alpha (x \underset{j}{+} y) = d_i^\alpha x \underset{j}{+} d_i^\alpha y$ for $i \neq j$ and $\alpha = 0, 1$,

whenever $x, y \in A$ and $x \underset{j}{+} y$ is defined.

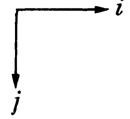
(1.3) (The interchange law) If $i \neq j$, then

$$(x \underset{i}{+} y) \underset{j}{+} (z \underset{i}{+} t) = (x \underset{j}{+} z) \underset{i}{+} (y \underset{j}{+} t)$$

whenever $x, y, z, t \in A$ and both sides are defined.

As in [1], we denote the two sides of (1.3) by

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix}$$



The category structure A^i on A is said to be *stronger* than the structure A^j if every object (identity morphism) of A^i is also an object of A^j. An n-fold category A is then called an *n-category* if the category structures $A^0, A^1, \dots, A^{n-1}$ can be arranged in a sequence of increasing (or decreasing) strength. Different authors choose different orders [8,11]; our exposition will correspond to the order

$$Ob\ A^0 \subset Ob\ A^1 \subset \dots \subset Ob\ A^{n-1}.$$

Adopting this convention, we now define an *∞-category* to be a class A with mutually compatible category structures A^i for all integers $i \geq 0$ satisfying

$$(1.4) \qquad Ob\ A^i \subset Ob\ A^{i+1} \quad \text{for all } i \geq 0.$$

The ∞-categories considered in this paper will also satisfy the extra condition

$$(1.5) \qquad A = \bigcup_{i \geq 0} Ob(A^i).$$

An interesting alternative set of axioms for such ∞-categories, with a more geometric flavour, will be given in Section 2. However, the axioms given above will be used in the later sections for the proof of the main theorem since they make the algebra simpler.

An ∞-groupoid A is an ∞-category satisfying condition (1.5) in which each category structure A^i is a groupoid.

Clearly there is a category Cat^∞ of ∞-categories in which a morphism $f: A \to B$ is a map preserving all the category structures. The full subcategory of Cat^∞ whose objects are ∞-groupoids is denoted by $\mathcal{H}$.

2. THE RELATION OF ∞-GROUPOIDS TO ω-GROUPOIDS.

In this section we explore a direct route from ω-groupoids to ∞-groupoids and use it to reformulate the definitions of ∞-groupoids and ∞-categories. This account is intended to show how ∞-groupoids fit into the pattern of equivalences established in [1,3]; it will not be needed in later sections.

We recall from [1] that an ω-groupoid G is a cubical set with some extra structures. In particular, each G_n carries n groupoid structures $\underset{i}{\oplus}$

with G_{n-1} as set of objects. The face maps $\partial_i^0, \partial_i^1 : G_n \to G_{n-1}$ give the initial and final objects for the groupoid $\underset{i}{\oplus}$, and the degeneracy map ϵ_i, $\epsilon_i : G_{n-1} \to G_n$ embeds G_{n-1} as the set of identity elements of $\underset{i}{\oplus}$. Adopting the conventions of Section 1, we write

$$\eta_i^a = \epsilon_i \partial_i^a : G_n \to G_n$$

and

$$Ob^i(G_n) = \epsilon_i G_{n-1} = \{ x \in G_n \mid \eta_i^a x = x \text{ for } a = 0, 1 \}.$$

The axioms for ω-groupoids now ensure that the groupoid structures

$$(G_n, \eta_i^0, \eta_i^1, \underset{i}{\oplus}), \quad i = 1, 2, \ldots, n$$

are mutually compatible. Thus for $n \geq 0$, G_n carries the structure of n-fold category (with inverses) and $\epsilon_j : G_{n-1} \to G_n$ embeds G_{n-1} as $(n-1)$-fold subcategory of the $(n-1)$-fold category obtained from G_n by omitting the j-th category structure.

Now there is an easy procedure for passing from an n-fold category A to an n-category induced on a certain subset S of A. Let

$$A^i = (A, d_i^0, d_i^1, \underset{i}{+}), \quad i = 0, 1, \ldots, n-1,$$

be the n category structures on A. Write

$$B^i = Ob(A^i) \cap Ob(A^{i+1}) \cap \ldots \cap Ob(A^{n-1}), \quad 0 \leq i \leq n-1,$$

and define

$$S = \{ x \in A \mid d_i^a x \in B^i \text{ for } 0 \leq i \leq n-1, \ a = 0, 1 \}.$$

The compatibility conditions (1.1)-(1.3) imply that each B^i is an n-fold subcategory of A and hence that S is also an n-fold subcategory of A, with category structures $S^i = \{ S, d_i^0, d_i^1, \underset{i}{+} \}$. But, for $x \in S$, $d_i^a x \in B^i \cap S$, so $Ob(S^i) \subset B^i \cap S$; conversely, if $y \in B^i \cap S$ then $y \in B^i \subset Ob(A^i)$, so $d_i^a y = y$. Thus $Ob(S^i) = B^i \cap S$. Since $B^0 \subset B^1 \subset \ldots \subset B^{n-1}$, it follows that S is an n-category.

Before applying this procedure to the n-fold category G_n, we renumber the operations to conform with conventions adopted in Section 1. Write

$$\underset{i}{+} \text{ for } \underset{n-i}{\oplus} \text{ on } G_n \text{ and } d_i^a \text{ for } \eta_{n-i}^a : G_n \to G_n .$$

Then G_n is an n-fold category with respect to the structures

$$A^i = (G_n, d_i^0, d_i^1, \underset{i}{+}), \quad i = 0, 1, \dots, n-1.$$

Also

$$B^i = Ob(A^i) \cap Ob(A^{i+1}) \cap \dots \cap Ob(A^{n-1})$$
$$= \epsilon_{n-i} G_{n-1} \cap \epsilon_{n-i-1} G_{n-1} \cap \dots \cap \epsilon_1 G_{n-1} = \epsilon_1^{n-i} G_i .$$

We therefore define

$$S_n = \{ x \in G_n \mid d_i^a x \in \epsilon_1^{n-i} G_i \text{ for } 0 \le i \le n-1, \ a = 0, 1 \}$$
$$= \{ x \in G_n \mid \partial_j^a x \in \epsilon_1^{j-1} G_{n-j} \text{ for } 1 \le j \le n, \ a = 0, 1 \},$$

and deduce that, for each $n \geq 0$, S_n is an n-fold category with respect to the structures $(S_n, d_i^0, d_i^1, \underset{i}{+})$, $0 \le i \le n-1$. These structures are in fact all groupoids. One verifies easily that the family $(S_n)_{n \geq 0}$ admits all the face operators ∂_i^β of G and also the first degeneracy operator ϵ_1 in each dimension. Since ϵ_1 embeds G_{n-1} in G_n as $(n-1)$-fold subcategory omitting $\underset{1}{\oplus}$, it embeds S_{n-1} in S_n as $(n-1)$-subcategory omitting $\underset{n-1}{+}$. In other words, it preserves the operations $\underset{i}{+}$, $0 \le i \le n-2$ and its image is the set of identities of $\underset{n-1}{+}$. It follows that if we define

$$H = \underset{\to}{\lim} (S_0 \xrightarrow[\epsilon_1]{} S_1 \xrightarrow[\epsilon_1]{} S_2 \hookrightarrow \dots),$$

then the operations $\underset{i}{+}$ (for fixed i) in each dimension combine to give a groupoid structure $H^i = (H, d_i^0, d_i^1, \underset{i}{+})$ on H. Also $Ob(H^i)$ is H_i, the image of S_i in H. Thus we have

(2.1) PROPOSITION. *If G is an ω-groupoid, then G induces on H the structure of ∞-groupoid.* □

Clearly, the structure on H can also be described in terms of the family $S = (S_n)_{n \geq 0}$. The neatest way to do this is to use the operators

$$D_i^a = (\partial_1^a)^{n-i} = \partial_1^a \partial_2^a \dots \partial_{n-i}^a : G_n \to G_i, \quad 0 \le i \le n-1, \ a = 0, 1,$$
$$E_i = \epsilon_1^{n-i} : G_i \to G_n, \quad 0 \le i \le n-1.$$

Since S admits ϵ_1 and all ∂_i^α, there are induced operators

$$D_i^\alpha : S_n \to S_i, \quad E_i : S_i \to S_n, \quad 0 \leq i \leq n-1, \quad \alpha = 0, 1.$$

If $x \in S_n$, we have $\partial_{n-i}^\alpha x = \epsilon_1^{n-i-1} y$ for some $y \in G_i$ and this y is unique, since ϵ_1 is an injection. The effect of D_i^α is to pick out this i-dimensional «essential face» y of x, because

$$D_i^\alpha x = \partial_1^\alpha \partial_2^\alpha \dots \partial_{n-i-1}^\alpha (\partial_{n-i}^\alpha x) = (\partial_1^\alpha)^{n-i-1} (\epsilon_1^{n-i-1} y) = y.$$

If we pass to $H = \underrightarrow{lim}\, S_n$, the operators E_i induce the inclusions $H_i \hookrightarrow H$ and the operators D_i^α induce the $d_i^\alpha : H \to H$, since, for $x \in S_n$, we have $d_i^\alpha x = \epsilon_1^{n-i} y$, where $y = D_i^\alpha X$.

It is easy now to see that the definition of ∞-category given in Section 1 (including condition (1.5)) is equivalent to the following. A (small) *∞-category* consists of

(2.2) A sequence $S = (S_n)_{n \geq 0}$ of sets.

(2.3) Two familes of functions

$$D_i^\alpha : S_n \to S_i, \quad i = 0, 1, \dots, n-1, \quad \alpha = 0, 1,$$
$$E_i : S_i \to S_n, \quad i = 0, 1, \dots, n-1,$$

satisfying the laws

(i) $D_i^\alpha D_j^\beta = D_i^\alpha$ for $i < j$, $\alpha, \beta = 0, 1$,

(ii) $E_j E_i = E_i$ for $i < j$,

(iii) $D_j^\beta E_i = \begin{cases} D_j^\beta & \text{for } j < i \\ 1 & \text{for } j = i \\ E_i & \text{for } j > i. \end{cases}$

(2.4) Category structures $\underset{i}{+}$ on S_n $(0 \leq i \leq n-1)$ for each $n \geq 0$ such that $\underset{i}{+}$ has S_i as its set of objects and D_i^0, D_i^1, E_i as its initial, final and identity maps. These category structures must be compatible, that is:

(i) If $i > j$ and $\alpha = 0, 1$, then

$$D_i^\alpha (x \underset{j}{+} y) = D_i^\alpha x \underset{j}{+} D_i^\alpha y$$

whenever the left hand side is defined.

(ii) $$E_i(x \underset{j}{+} y) = E_i x \underset{j}{+} E_i y$$

in S_n whenever the left hand side is defined.

(iii) (The interchange law) if $i \neq j$ then

$$(x \underset{i}{+} y) \underset{j}{+} (z \underset{i}{+} t) = (x \underset{j}{+} z) \underset{i}{+} (y \underset{j}{+} t)$$

whenever both sides are defined.

The transition from an ∞-category A as defined in Section 1 to one of the above type is made by putting $S_n = Ob(A^n)$ and defining $E_i : S_i \to S_n$ ($i < n$) to be the inclusion map and $D_i^a : S_n \to S_i$ to be the restriction of $d_i^a : A \to A$.

We note finally that, starting from an ω-groupoid G, the ∞-groupoid $S = (S_n)_{n \geq 0}$ described above contains the associated crossed complex $C = \gamma G$ defined in [1] by the rule

$$C_n = \{ x \in G_n \mid \partial_i^a x \in \epsilon_1^{n-1} G_0 \text{ for all } (a, i) \neq (0, 1) \}.$$

The equivalence of categories

$$\gamma : (\omega\text{-}groupoids) \longrightarrow (crossed\ complexes)$$

established in [1] therefore factors through (*∞-groupoids*). We shall show below that the factor

$$a : (\infty\text{-}groupoids) \longrightarrow (crossed\ complexes)$$

is an equivalence, with inverse

$$\beta : (crossed\ complexes) \longrightarrow (\infty\text{-}groupoids).$$

Hence

$$\zeta = \beta\gamma : (\omega\text{-}groupoids) \longrightarrow (\infty\text{-}groupoids)$$

is an equivalence. By results in [2], any ω-groupoid is the homotopy ω-groupoid $\rho(X)$ of a suitable filtered space X. Defining the homotopy ∞-groupoid of X to be $\sigma(X) = \zeta\rho(X)$, we deduce that any (small) ∞-groupoid is of the form $\sigma(X)$ for some X.

3. THE CROSSED COMPLEX ASSOCIATED WITH AN ∞-GROUPOID.

Let H be an ∞-groupoid in the sense of Section 1. Then H has groupoid structures $(H, d_i^0, d_i^1, \underset{i}{+})$ for $i \geq 0$ satisfying the compatibility conditions (1.1), (1.2), (1.3) and the conditions

(3.1) $H_i \subset H_{i+1}$ for $i \geq 0$; $H = \bigcup_{i \geq 0} H_i$

where $H_i = d_i^0 H = d_i^1 H$ is the set of identities (objects) of the i-th groupoid structure. The conditions (3.1) enable us to define the *dimension* of any $x \in H$ to be the least integer n such that $x \in H_n$; we denote this integer by $\dim x$. It is convenient to picture an n-dimensional element x of H as having two vertices $d_0^\alpha x$, two edges $d_1^\alpha x$ joining these vertices, two faces $d_2^\alpha x$ joining the edges, and so on, with x itself joining the two faces $d_{n-1}^\alpha x \in H_{n-1}$. (The actual dimensions of the faces $d_i^\alpha x$ may of course be smaller than i.)

Some immediate consequences of the definitions are

(3.2) (i) For each $x \in H$, $d_j^0 x = d_j^1 x = x$ if $j \geq \dim x$.

(ii) If $i < j$ then $d_i^\alpha d_j^\beta = d_j^\beta d_i^\alpha = d_i^\alpha$ for $\alpha, \beta = 0, 1$.

(iii) If $i < j$ then $d_i^\alpha (x \underset{j}{+} y) = d_i^\alpha x = d_i^\alpha y$ for $\alpha = 0, 1$,

whenever $x \underset{j}{+} y$ is defined.

Here (ii) follows from (i), since $\dim(d_i^\alpha x) \leq i < j$, and (iii) follows from (ii) since, for example,

$$d_i^\alpha (x \underset{j}{+} y) = d_i^\alpha d_j^0 (x \underset{j}{+} y) = d_i^\alpha d_j^0 x = d_i^\alpha x.$$

We shall show that any ∞-groupoid H contains a crossed complex $C = \alpha H$, as described in Section 2. First we recall from [1] the axioms for a crossed complex.

A *crossed complex* C (over a groupoid) consists of a sequence

$$\cdots \to C_n \xrightarrow{\delta} C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\delta} C_1 \underset{\delta^1}{\overset{\delta^0}{\rightrightarrows}} C_0$$

satisfying the following axioms:

(3.3) C_1 is a groupoid over C_0 with δ^0, δ^1 as its initial and final maps. We write $C_1(p,q)$ for the set of arrows from p to q $(p, q \in C_0)$ and $C_1(p)$ for the group $C_1(p,p)$.

(3.4) For $n \geq 2$, C_n is a family of groups $\{C_n(p)\}_{p \in C_0}$ and for $n \geq 3$ the groups $C_n(p)$ are Abelian.

(3.5) The groupoid C_1 operates on the right on each C_n $(n \geq 2)$ by an action denoted $(x, a) \mapsto x^a$. Here, if $x \in C_n(p)$ and $a \in C_1(p,q)$ then $x^a \in C_n(q)$. (Thus $C_n(p) \approx C_n(q)$ if p and q lie in the same component of the groupoid C_1.)

We use additive notation for all groups $C_n(p)$ and for the groupoid C_1, and we use the symbol $0_p \in C_n(p)$, or 0, for all their identity elements.

(3.6) For $n \geq 2$, $\delta : C_n \to C_{n-1}$ is a morphism of groupoids over C_0 and preserves the action of C_1, where C_1 acts on the groups $C_1(p)$ by conjugation: $x^a = -a + x + a$.

(3.7) $\delta\delta = 0 : C_n \to C_{n-2}$ for $n \geq 3$ (and $\delta^0\delta = \delta^1\delta : C_2 \to C_0$, as follows from (3.6).)

(3.8) If $c \in C_2$, then δc operates trivially on C_n for $n \geq 3$ and operates on C_2 as conjugation by c, that is,

$$x^{\delta c} = -c + x + c \quad (x, c \in C_2(p)).$$

The category of crossed complexes is denoted by $\mathcal{C}$.

Given an ∞-groupoid H, we define $C = \alpha H$ by

$$C_0 = H_0, \quad C_1 = H_1 \quad \text{and}$$
$$C_n(p) = \{ x \in H_n \mid d^1_{n-1} x = p \} \quad \text{for } p \in C_0, \ n \geq 2.$$

It follows from (3.2) (ii) that if $x \in C_n(p)$ $(n \geq 2)$ then

$$\text{for } 0 \leq i \leq n-2, \quad d^a_i x = d^a_i d^1_{n-1} x = p.$$

Thus we have the alternative characterisation:

$$C_n(p) = \{ x \in H_n \mid d^a_i x = p \text{ for } 0 \leq i \leq n-1, \ a = 0, 1, \ (i, a) \neq (n-1, 0) \}.$$

For $n \geq 2$, let C_n be the family $\{C_n(p)\}_{p \in C_0}$ and, for $x \in C_n(p)$, define $\delta x = d^0_{n-1} x$. Then $\delta x \in C_{n-1}(p)$ since

$$d^1_{n-2}\, d^0_{n-1}\, x = d^1_{n-2}\, x = p .$$

This defines $\delta : C_n \to C_{n-1}$ for $n \geq 2$, and we define

$$\delta^{\alpha} : C_1 \to C_0 \text{ by } \delta^{\alpha} = d^{\alpha}_0 \; (\alpha = 0, 1).$$

Clearly C_1 is a groupoid over C_0 with respect to the composition $\underset{0}{+}$. Also for each $p \in C_0$ and $n \geq 2$, $C_n(p)$ is a group with respect to each of the compositions $\underset{i}{+}$ for $i = 0, 1, \ldots, n-2$, with zero element p. If $0 \leq i < j \leq n-2$ and $x, y \in C_n(p)$ then the composites

$$\begin{bmatrix} x & p \\ p & y \end{bmatrix} \quad \begin{bmatrix} p & x \\ y & p \end{bmatrix} \qquad \begin{array}{l} \longrightarrow j \\ \downarrow \\ i \end{array}$$

are defined. Evaluating them by rows and by columns we find that

$$x \underset{j}{+} y = x \underset{i}{+} y = y \underset{j}{+} x.$$

Thus, for $n \geq 3$, these group structures in $C_n(p)$ all coincide and are Ablian. We write $x + y$ for $x \underset{0}{+} y$ whenever this is defined in H. Then $C_n(p)$ is a group with respect to $+$. By (1.2), $\delta = d^0_{n-1} : Cn(p) \to C\, n\text{-}1(p)$ is a morphism of groups for $n \geq 2$. Also $\delta\delta = 0$ since for $x \in C_n(p)$ and $n \geq 3$,

$$\delta\delta\, x = d^0_{n-2}\, d^0_{n-1}\, x = d^0_{n-2}\, x = p .$$

Let $x \in C_n(p)$, $n \geq 1$ and let $a \in C_1(p, q)$. We define

$$x^a = -a + x + a.$$

If $n \geq 2$, then

$$\begin{aligned} d^1_{n-1}\, x^a &= -d^1_{n-1}\, a + d^1_{n-1}\, x + d^1_{n-1}\, a \quad \text{by (1.2)} \\ &= -a + p + a = q . \end{aligned}$$

If $n = 1$, then $d^1_0 x = d^1_0 a = q$. Thus, in either case, $x^a \in C_n(q)$ and we obtain an action of C_1 on C_n. This action is preserved by δ since for $n \geq 2$,

$$\delta(x^a) = -d^0_{n-1}\, a + d^0_{n-1}\, x + d^0_{n-1}\, a = -a + \delta x + a.$$

(3.9) LEMMA. *If $n \geq 2$, $x \in C_n(p)$, $u \in H_n$ and $d^0_0 u = p$, then*

$$-u + x + u = x^{d^0_1 u} .$$

PROOF. This follows from evaluating in two ways the composite

$$\begin{bmatrix} -d_1^0 u & x & d_1^0 u \\ -u & p & u \end{bmatrix} \quad \begin{array}{l} \ulcorner\!\longrightarrow 0 \\ \downarrow \\ 1 \end{array} \qquad \square$$

From (3.9) we see that if $x, c \in C_2(p)$, then $-c + x + c = x^{\delta c}$, as required in (3.8). Further, if $x \in C_n(p)$, $n \geq 3$ and $c \in C_2(p)$, then the composite

$$\begin{bmatrix} -c & p & c \\ p & x & p \end{bmatrix} \quad \begin{array}{l} \ulcorner\!\longrightarrow 0 \\ \downarrow \\ 1 \end{array}$$

is also defined, giving $-c + x + c = x$, so in this case (3.9) implies that $x^{\delta c} = x$.

This completes the verification that $C = \{C_n\}_{n \geq 0}$ is a crossed complex, which we denote by aH. We observe that this crossed complex is entirely contained in H, and all its compositions are induced by $\underset{0}{+}$, while its boundary maps are induced by the various d_i^0. The groups $C_n(q)$, $C_n(p)$ are disjoint if $p \neq q$; the groups $C_m(p)$, $C_n(p)$ have only their zero element p in common if $m \neq n$.

We now aim to show that H can be recovered from the crossed complex $C = aH$ contained in it. The key result for this is

(3.10) PROPOSITION. *Let H be an ∞-groupoid with associated crossed complex $C = aH$. Let $n \geq 1$, $x \in H_n$, $d_0^0 x = p$ and $d_0^1 x = q$. Then x can be written uniquely in the form*

$$(*) \qquad x = x_n + x_{n-1} + \dots + x_1, \ \text{where } x_1 \in C_1(p, q),\ x_i \in C_i(p) \text{ for } i \geq 2 \text{ and } + \text{ stands for } \underset{0}{+}.$$

Further, x_i is given by

$$(**) \qquad x_i = d_i^1 x - d_{i-1}^1 x \quad \text{for } 1 \leq i \leq n.$$

PROOF. If $(*)$ holds then, for $1 \leq i \leq n$,

$$\begin{aligned} d_i^1 x &= d_i^1 x_n + d_i^1 x_{n-1} + \dots + d_i^1 x_{i+1} + x_i + x_{i-1} + \dots + x_1 \\ &= x_i + x_{i-1} + \dots + x_1 \end{aligned}$$

since $d_i^1 x_j = p$ for $i < j$. The formula for x_i follows, and this proves uniqueness. For existence, let x_i be defined by $(**)$. Then

$$x_n + x_{n-1} + \ldots + x_i = d_n^1 x - d_0^1 x = x - q = x.$$

Also $x_i \in H_i$, and

$$d_{i-1}^1 x_i = d_{i-1}^1 x - d_{i-1}^1 x = d_0^0 d_{i-1}^1 x = p$$

if $i \geq 2$, that is, $x_i \in C_i(p)$. Similarly, $x_1 \in C_1(p, q)$. □

We now give some basic properties of the decomposition $(*)$ of Proposition (3.10).

(3.11) $d_i^1 x = x_i + x_{i-1} + \ldots + x_1$, $1 \leq i \leq n$.

(3.12) $d_i^0 x = \delta x_{i+1} + x_i + x_{i-1} + \ldots + x_1 = \delta x_{i+1} + d_i^1 x$,

$1 \leq i \leq n-1$.

We have already proved (3.11), and (3.12) is similar. □

(3.13) *If* $z = x \underset{0}{+} y$ *is defined in* H, *then*

$$z_i = \begin{cases} x_1 + y_1 & \text{if } i = 1 \\ x_i + y_i^{-x_1} & \text{if } i \geq 2. \end{cases}$$

PROOF. Clearly

$$z_1 = d_1^1 z = d_1^1 x + d_1^1 y = x_1 + y_1.$$

If $i \geq 2$ then

$$z_i = d_i^1 (x \underset{0}{+} y) - d_{i-1}^1 (x \underset{0}{+} y) = d_i^1 x + d_i^1 y - d_{i-1}^1 y - d_{i-1}^1 x$$
$$= x_i + d_{i-1}^1 x + y_i - d_{i-1}^1 x = x_i + y_i^{-v}$$

by (3.9), where $v = d_1^0 d_{i-1}^1 x$. If $i = 2$, then

$$v = d_1^0 d_1^1 x = d_1^1 x = x_1.$$

If $i \geq 3$, then

$$v = d_1^0 x = \delta x_2 + x_1.$$

But δx_2 acts trivially on C_i for $i \geq 3$, so the result is true in this case also. □

(3.14) *If* $z = x \underset{j}{+} y$ *is defined in* H, *where* $j \geq 1$, *then*

$$z_i = \begin{cases} y_i = x_i & \textit{if } i < j \\ y_i & \textit{if } i = j \\ x_i + y_i & \textit{if } i > j\,. \end{cases}$$

PROOF. First note that $d_j^1 x = d_j^0 y$ and hence $d_0^0 x = d_0^0 y = p$, say. If $i < j$, then

$$\begin{aligned} z_i &= d_i^1 (x \underset{j}{+} y) - d_{i-1}^1 (x \underset{j}{+} y) \\ &= d_i^1 x - d_{i-1}^1 x = d_i^1 y - d_{i-1}^1 y, \quad \text{by (3.2) (iii)} \\ &= x_i = y_i\,. \end{aligned}$$

If $i = j$, then $z_i = d_i^1 y - d_{i-1}^1 y = y_i$.

If $i = j+1$, then $z_i = (d_{j+1}^1 x \underset{j}{+} d_{j+1}^1 y) - d_j^1 y$

$$= \begin{bmatrix} d_{j+1}^1 x & -d_j^1 y \\ d_{j+1}^1 y & -d_j^1 y \end{bmatrix} \qquad \begin{array}{l} \longrightarrow 0 \\ \downarrow \\ j \end{array}$$

But

$$\begin{aligned} d_{j+1}^1 x - d_j^1 y &= x_{j+1} + d_j^1 x - d_j^1 y \\ &= x_{j+1} + d_j^0 y - d_j^1 y = x_{j+1} + d_j^0 y_{j+1} \end{aligned}$$

and $d_{j+1}^1 y - d_j^1 y = y_{j+1}$, so

$$z_i = \begin{bmatrix} x_{j+1} & d_j^0 y_{j+1} \\ p & y_{j+1} \end{bmatrix} \qquad \begin{array}{l} \longrightarrow 0 \\ \downarrow \\ j \end{array}$$

$$= x_{j+1} + y_{j+1}\,.$$

If $i \geq j+2$, then

$$z_i = \begin{bmatrix} d_i^1 x & -d_{i-1}^1 x \\ d_i^1 y & -d_{i-1}^1 y \end{bmatrix} \qquad \begin{array}{l} \longrightarrow 0 \\ \downarrow \\ j \end{array}$$

$$= x_i \underset{j}{+} y_i = x_i + y_i\,. \qquad \square$$

These results show that the ∞-groupoid structure of H can be recovered from the crossed complex structure of $C = \alpha H$, a fact which we make more precise in the next section. We observe that all the equations (3.11)-(3.14) and (**) of (3.10) remain valid for values of i and j greater than the dimensions of x, y, z if we adopt the convention that, for $i > \dim x$, $x_i = d_0^0 x$.

4. THE EQUIVALENCE OF CATEGORIES.

We have constructed, for any ∞-groupoid H, a crossed complex αH, and this construction clearly gives a functor $\alpha : \mathcal{H} \to \mathcal{C}$. We now construct a functor $\beta : \mathcal{C} \to \mathcal{H}$.

Let C be an arbitrary crossed complex. We form an ∞-groupoid $K = \beta C$ by imitating the formulas (3.10)-(3.14). Let K be the set of all sequences

$$x = (\ldots, x_i, x_{i-1}, \ldots, x_1), \quad \text{where } x_1 \in C_1,\ x_i \in C_i(\delta^0 x_1), \quad \text{and } x_i = 0 \text{ for all sufficiently large } i.$$

As for polynomials, we shall write

$$x = (x_n, x_{n-1}, \ldots, x_1) \text{ if } x_i = 0 \text{ for all } i > n.$$

We define maps $d_i^\alpha : K \to K$ by

$$d_0^0 x = (\ldots, 0_p, 0_p, \ldots, 0_p), \quad p = \delta^0 x,$$
$$d_0^1 x = (\ldots, 0_q, 0_q, \ldots, 0_q), \quad q = \delta^1 x,$$
$$d_i^1 x = (x_i, x_{i-1}, \ldots, x_1), \quad i \geq 1,$$
$$d_i^0 x = (\delta x_{i+1} + x_i,\ x_{i-1},\ x_{i-2}, \ldots, x_1), \quad i \geq 1.$$

It is easy to verify the law (1.1). (The crossed module law $\delta\delta = 0$ is needed to prove $d_i^0 d_{i+1}^0 = d_{i+1}^0 d_i^0$.) Also, writing $K_i = d_i^1 K = d_i^0 K$ for $i \geq 0$, we have

$$K_i \subset K_{i+1} \ (i \geq 0) \text{ and } \bigcup_{i \geq 0} K_i = K.$$

Suppose now that we are given $x, y \in K$ such that $d_0^1 x = d_0^0 y$, that is, $\delta^1 x_1 = \delta^0 y_1$. We define

$$x \underset{0}{+} y = (\ldots, x_n + y_n^{-x_1}, \ldots, x_2 + y_2^{-x_1}, x_1 + y_1)$$

which is an element of K. Similarly, if $j \geq 1$ and $d_j^1 x = d_j^0 y$, that is,

$x_i = y_i$ for $i < j$ and $x_j = \delta y_{j+1} + y_j$, we define

$$(x \underset{j}{+} y) = (\dots, x_n + y_n, \dots, x_{j+1} + y_{j+1}, y_j, y_{j-1}, \dots, y_1),$$

again an element of K. In each case it is easy to see that the composition $\underset{j}{+}$ defines a groupoid structure on K with K_j as its set of identities and d_j^0, d_j^1 as its initial and final maps. The law (1.2) follows trivially from these definitions if $a = 1$ or if $i < j$. If $a = 0$ and $i > j$, it reduces immediately to one of the following equations:

$$\delta y_{i+1} + x_i = x_i + \delta y_{i+1}, \quad i > j \geq 1,$$

$$\delta(y_{i+1}^{-x_1}) + x_i = x_i + (\delta y_{i+1})^{-x_1}, \quad i \geq 2,$$

$$\delta(y_2^{-x_1}) + x_1 = x_1 + \delta y_2.$$

These are all easy consequences of the laws for a crossed complex. The interchange law (1.3) is proved in a similar way to complete the verification that $K = \beta C$ is an ∞-groupoid. The construction is clearly functorial.

(4.1) THEOREM. *The functors* $a : \mathcal{H} \to \mathcal{C}$ *and* $\beta : \mathcal{C} \to \mathcal{H}$ *defined above are inverse equivalences.*

PROOF. Given an ∞-groupoid H, the ∞-groupoid $K = \beta a H$ is naturally isomorphic to H by the map

$$(\dots, x_n, x_{n-1}, \dots, x_1) \mapsto \dots + x_n + x_{n-1} + \dots + x_1$$

(the sum on the right being finite since $x_r = 0$ for large r). This is a consequence of Proposition (3.10) and the relations (3.11)-(3.14).

On the other hand, if C is a crossed complex, $H = \beta C$ and $D = a\beta C$, then H_n consists of elements $x = (x_n, x_{n-1}, \dots, x_1)$ and hence

$$D_n = \{ x \in H_n \mid d_{n-1}^1 x \in H_0 \}$$

consists of elements $x = (x_n, 0_p, 0_p, \dots, 0_p)$, where $x_n \in C_n(p)$. It is easy to see that the map $C_n \to D_n$ defined by

$$c \mapsto (c, \underbrace{0_p, 0_p, \dots, 0_p}_{n-1}), \quad c \in C_n(p)$$

gives a natural isomorphism $C \to a\beta C$. □

REFERENCES.

1. R. BROWN & P. J. HIGGINS, On the algebra of cubes, *J. Pure & Appl. Algebra,* 21 (1981), 233-260.

2. R. BROWN & P. J. HIGGINS, Colimit theorems for relative homotopy groups, *J. Pure & Appl. Algebra* 22 (1981), 11-41.

3. R. BROWN & P. J. HIGGINS, The equivalence of ω-groupoids and cubical T-complexes, This issue.

4. R. BROWN & C. B. SPENCER, $\mathcal{G}$-groupoids, crossed modules and the fundamental groupoid of a topological group, *Proc. Kon. Akad. v. Wet.* 79 (1976), 296.

5. R. BROWN & C. B. SPENCER, Double groupoids and crossed modules, *Cahiers Top. et Géom. Diff.* XVII-4 (1976), 343-362.

6. M. K. DAKIN, *Kan complexes and multiple groupoid structures*, Ph. D. Thesis University of Wales, 1977.

7. A. DOLD, Homology of symmetric products and other functors of complexes, *Ann. of Math.* 68 (1958), 54-80.

8. A. & C. EHRESMANN, Multiple functors II, III, *Cahiers Top. et Géom. Diff.* XX (1978), 295-333, 387-443.

9. D. M. KAN, Functors involving c.s.s. complexes, *Trans. A.M.S.* 87 (1958), 330.

10. C. B. SPENCER, An abstract setting for homotopy pushouts and pullbacks, *Cahiers Top. et Géom. Diff.* XVIII-4 (1977), 409-430.

11. O. WYLER, *Multiple functor categories*, Mimeographed Notes, Carnegie-Mellon University, 1972.

R. BROWN:
School of Mathematics and Computer Science
University College of N. Wales
BANGOR LL57 2UW, U.K.

P. J. HIGGINS:
Department of Mathematics
Science Laboratories
South Road
DURHAM DH1 3LE.
U.K.