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PARTIAL COMPLETIONS OF CONCRETE FUNCTORS

by Andrée CHARLES EHRESMANN

INTRODUCTION.

If f and g are differentiable maps between manifolds M and M', the equation $f(m) = g(m)$ may not define a submanifold of M ; two differentiable maps toward M may not have a pullback unless they are transversal. Such difficulties have hindered a categorical study of Differential Geometry; e.g., differentiable categories /50/ are only those internal categories in the category $\mathfrak{D}^n$ of manifolds whose domain and codomain maps are submersions. Is it not possible to embed $\mathfrak{D}^n$ into an «adequate» category? Charles was mainly motivated by this question and inspired by his many works on completions of posets and of local categories /47,55,76,85, 86/ when he wrote his paper / 107 / in the mid-sixties: here he constructs «optimal» extensions of a concrete functor $P: \mathrm{H} \to \mathrm{E}$ into a concrete functor which initially lifts a given class of singleton sources («spreading» functors) or of limit cones («completions») in E. For exemple, the smallest spreading extension of $\mathfrak{D}^n \to$ Ens equips each subset of a manifold M with a structure which has been independently worked out (without categorical aims!) by Ngo Van Qué [6], Aronszajn and Marshall [5].

Later on, several authors tackled analogous problems, often with a view to embedding the category of topological spaces in an initially complete cartesian closed category (Antoine, Chartrelle, Day, Wyler,...); generalizing results of Banaschewski and Burns on completions of posets, Herrlich describes in [2] the smallest - or Mac Neille - and the largest or universal (preserving initial lifts) initial completions of P ; for the bibliography, we refer to [3] where most papers on initial completions (Adamek, Herrlich, Strecker; Börger; Hoffmann ; Tholen ...) are summarized. Recently universal completions of P have been constructed by Adamek-Koubek [1] and, without transfinite induction, by Herrlich [4].

Here all these results are unified: Given a class Γ of cones in E, and a class Δ of initial cones in H, the concrete functor $P: \mathrm{H} \to \mathrm{E}$ is extended into a concrete functor with initial lifts of cones of Γ, and for which the cones of Δ remain initial; two «optimal» solutions, the Mac Neille Γ-completion and the universal (Δ, Γ)-completion of P, are constructed by methods making the most out the ideas of Charles /55, 107/ and Herrlich [2, 4]. If Γ is «not too large» these solutions live in the same universe as P.

HYPOTHESES. There is given a category E (the «base category») and a class Γ of cones in E ; let $Ind\,\Gamma$ be the class formed by the indexing categories of the cones $\gamma \in \Gamma$. Cone always means projective cone.

We denote by $P: \mathrm{H} \to \mathrm{E}$ a concrete (i.e., faithful and amnestic) functor, by $S, S', \ldots$ the objects of H, by $E, E', \ldots$ the objects of E. These notations come from the primitive case where P is the forgetful (Projection) functor from the category of Homomorphisms between Structures of some kind to Ens.

$\mathfrak{U}_0$ and $\mathfrak{U}$ are two universes such that $\mathfrak{U}_0 \in \mathfrak{U}$; the elements of $\mathfrak{U}$ are large sets, or classes, those of $\mathfrak{U}_0$ are small sets. We suppose P lives in $\mathfrak{U}$ (as does for instance the forgetful functor $\mathfrak{D}^n \to$ Ens).

1. PARTIAL COMPLETIONS.

In this section, we recall definitions, and state some results on cones and initial lifts which are used in the sequel.

A. Commutative hull of Γ.

Let γ be a cone in E with vertex E and basis $\phi: \mathrm{I} \to \mathrm{E}$, abbreviated in $\gamma: E \Rightarrow \phi$. If the indexing category I is discrete, γ is called a *source*, and also denoted by $(\gamma(I) \mid I \in \mathrm{I})$; for instance E defines the the singleton source $E^{\wedge} = (Id_E)$.

To the cone γ is associated the source $(\gamma(I) \mid I \in \mathrm{I}_0)$ indexed by the class I_0 of objects of I ; this source is written $\gamma_0 : E \Rightarrow \phi_0$. We denote by Γ_0 the class of all sources γ_0, for $\gamma \in \Gamma$.

If $\gamma_I: \phi(I) \Rightarrow \phi_I$ is a cone indexed by I_I for each $I \in \mathrm{I}_0$, the source

$$(\gamma_I(J).\gamma(I): E \to \phi_I(J) \mid (J,I) \in \sum_{I \in \mathrm{I}_0} \mathrm{I}_{I0})$$

is called the *composite source of* $((\gamma_I)_{\mathrm{I}_0}, \gamma)$, denoted $(\gamma_I)_{\mathrm{I}_0} \circ \gamma$. For instance: $(\phi(I)^{\wedge})_{\mathrm{I}_0} \circ \gamma = \gamma_0$; if $\mathrm{I} = 1$ and $\gamma(1) = f$, then $(\gamma_I)_1 \circ \gamma$ is the source $\gamma_I \circ f = (\gamma_I(J).f \mid J \in \mathrm{I}_{I0})$.

DEFINITION. A class Σ of sources in E is said *commutative* if

1° $E^{\wedge} \in \Sigma$ for each object E of E,

2° $\gamma \in \Sigma$ and $\gamma_I \in \Sigma$ for each $I \in \mathrm{I} = \mathrm{I}_0$ imply $(\gamma_I)_\mathrm{I} \circ \gamma \in \Sigma$.

PROPOSITION 1. *Let Σ be a class of sources in* E*; the smallest commutative class of sources Σ° containing Σ is constructed by transfinite induction. If $Ind\,\Sigma$ and each $\mathrm{I} \in Ind\,\Sigma$ belong to the universe $\mathfrak{U}$, so does $Ind\,\Sigma^\circ$.*

Δ. Σ° is the union of the transfinite increasing sequence $(\Sigma_\lambda)_\lambda$ defined by induction as follows:

$$\Sigma_0 = \{ E^{\wedge} \mid E \in \mathrm{E}_0 \} \cup \Sigma; \quad \Sigma_a = \bigcup_{\lambda < a} \Sigma_\lambda \text{ for each limit ordinal } a,$$

$$\Sigma_{\lambda+1} = \{ (\gamma_I)_\mathrm{I} \circ \gamma \mid \gamma \in \Sigma_\lambda, \gamma_I \in \Sigma_\lambda \;\; \forall I \in \mathrm{I} \}.$$

The construction stops at the limit ordinal larger than the ordinal of I, for each $\mathrm{I} \in Ind\,\Sigma$. ∇

DEFINITION. The smallest commutative class of sources containing Γ_0 is called the *commutative hull of* Γ, denoted by Γ°.

EXAMPLES. The class *Sour* E of all sources in E is commutative and it is the commutative hull of the class *Cone* E of all cones in E. If A is a class of morphisms of E, the class /A/ of singleton sources (a), $a \in A$ has for its commutative hull the class /A'/ corresponding to the sub-category of E generated by $A \cup E_0$.

B. Initial lifts.

Let $P: \mathrm{H} \to \mathrm{E}$ be a concrete functor. A morphism h from S to S' in H is written $g: S \to S'$, where $g = P(h)$. If θ is a cone in H with vertex S and basis Φ and if $\gamma = P\theta$, we also denote θ by $\gamma: S \Rightarrow \Phi$.

A *P-cone* indexed by I is a pair (Φ, γ), where $\Phi : \mathrm{I} \to \mathrm{H}$ is a functor and $\gamma : E \Rightarrow P\Phi$ is a cone. An *initial lift of* (Φ, γ) is an object S of H such that:

1° $\gamma(I) : S \to \Phi(I)$ is in H for each $I \in \mathrm{I}_0$,

2° If $f : E' \to E$ in E and if $\gamma \circ f : S' \Rightarrow \Phi$ is a cone in H, then f lifts into $f : S' \to S$ in H.

As P is concrete, such an S (if it exists) is unique; it is then denoted by $il(\Phi, \gamma)$; so $\gamma : S \Rightarrow \Phi$ is an initial cone for P.

A *P-source* is a P-cone (Φ, γ) indexed by a discrete category I_0; it is often identified to the family of *P-morphisms* (singleton P-sources)

$$(\Phi(I), \gamma(I) \mid I \in \mathrm{I}_0).$$

The dual notion is a *P-sink*.

To the P-cone (Φ, γ) indexed by I is associated the P-source (Φ_0, γ_0), where $\Phi_0 : \mathrm{I}_0 \to \mathrm{H}$ is the restriction of Φ to the objects of I.

PROPOSITION 2. *Let* (Φ, γ) *be a P-cone. It has an initial lift iff the P-source* (Φ_0, γ_0) *has one; in this case,* $il(\Phi, \gamma) = il(\Phi_0, \gamma_0)$. ∇

The following proposition (whose proof is straightforward) is important for the sequel. Let (Φ, γ) be a P-cone indexed by I and (Φ_I, γ_I) be a P-cone indexed by I_I such that $\gamma_I : \Phi(I) \Rightarrow \Phi_I$ is a cone in H, for I in I_0. We denote by $(\Phi_I, \gamma_I)_{\mathrm{I}_0} \circ (\Phi, \gamma)$ the P-source

$$(\Psi, (\gamma_I)_{\mathrm{I}_0} \circ \gamma) \quad \text{where} \quad \Psi : \sum_{I \in \mathrm{I}_0} \mathrm{I}_{I\,0} \to \mathrm{H} : (J, I) \mapsto \Phi_I(J).$$

PROPOSITION 3 *(Commutativity of initial lifts). If* $\Phi(I) = il(\Phi_I, \gamma_I)$ *for for each object* I *of* I, *we have*

$$il(\Phi, \gamma) = il((\Phi_I, \gamma_I)_{\mathrm{I}_0} \circ (\Phi, \gamma))$$

as soon as one of these terms is defined. ∇

DEFINITION. P is called *Γ-complete* if each P-cone (Φ, γ) with $\gamma \in \Gamma$ admits an initial lift.

From Propositions 1, 2, 3, it follows by transfinite induction:

COROLLARY. *If* P *is* Γ_0*-complete, then it is* Γ*-complete and* Γ°*-complete, where* Γ° *is the commutative hull of* Γ.

We consider the category of concrete functors over E, whose objects are the concrete functors $Q: \mathrm{K} \to \mathrm{E}$ and whose morphisms $F: Q \to Q'$, where $Q': \mathrm{K}' \to \mathrm{E}$, are the functors $F: \mathrm{K} \to \mathrm{K}'$ such that $Q'F = Q$. It has a non-full subcategory formed by the Γ*-morphisms*, which are the morphisms $F: Q \to Q'$ such that $F(il(\Phi, \gamma)) = il(F\Phi, \gamma)$ whenever (Φ, γ) is a Q-cone with $\gamma \in \Gamma$ which has an initial lift.

COROLLARY. *If* $F: Q \to Q'$ *is a* Γ_0*-morphism, it is a* Γ*-morphism and if* Q *is* Γ_0*-complete, a* Γ°*-morphism.*

C. Γ-density et Γ-generation.

Here $Q: \mathrm{K} \to \mathrm{E}$ is a concrete functor, H a full subcategory of K and $P: \mathrm{H} \to \mathrm{E}$ is the restriction of Q.

DEFINITION. H is called Γ*-dense for* Q if each object K of K is the initial lift of a P-cone (Ψ, γ) with $\gamma \in \Gamma$ and Ψ valued in H. If H is *Sour* E-dense, it is said *initially dense.*

PROPOSITION 4. *The following conditions are equivalent:*

1° H *is initially dense for* Q.

2° *For each* $K \in \mathrm{K}_0$ *the source* $(k: K \to S \mid S \in \mathrm{H}_0)$ *is initial for* Q.

3° *Let* K, K' *be objects of* K *and* $g: Q(K) \to Q(K')$ *a* E*-morphism; then we have* $g: K \to K'$ *in* K *iff*

$$f: K' \to S \text{ in } \mathrm{K} \text{ and } S \in \mathrm{H}_0 \quad \text{imply} \quad f.g: K \to S \text{ in } \mathrm{K}.$$

If they are satisfied, the insertion $\mathrm{H} \hookrightarrow \mathrm{K}$ *preserves final lifts.* ∇

The dual notion is «finally dense». It will be used in Section 2 through the third characterization above (introduced in /107/ under the name «Q *is* P*-generated*»).

DEFINITION. We call Γ*-hull* (resp. *strict* Γ*-hull*) *of* H *for* Q the smallest full subcategory (resp. smallest subcategory) H' of K containing H and $il(\Psi, \gamma)$ for each Q-cone (Ψ, γ) with $\gamma \in \Gamma$ and Ψ valued in H'. If H' = K, we say that Q *is* Γ*-generated* (resp. *strictly* Γ*-generated*) *by* H.

If Q is Γ-complete, so is its restriction to H'. If H is Γ-dense for Q, then Q is Γ-generated by H (but not conversely).

PROPOSITION 5. *The* Γ*-hull* C *and the strict* Γ*-hull* B *of* H *for* Q *are constructed by transfinite induction; they are in the same universe* $\mathfrak{U}$ *as* H *if so are* $Ind\,\Gamma$ *and each* $\mathrm{I} \in Ind\,\Gamma$.

Δ. C and B are respectively the union of the transfinite increasing sequence $(C_\lambda)_\lambda$ and $(B_\lambda)_\lambda$ defined as follows:

$C_0 = H = B_0$,

$C_a = \bigcup_{\lambda < a} C_\lambda$ and $B_a = \bigcup_{\lambda < a} B_\lambda$ for each limit ordinal a,

$C_{\lambda+1}$ is the full subcategory of K with objects $il(\Psi, \gamma)$ where (Ψ, γ) is a Q-cone, $\gamma \in \Gamma$ and Ψ valued in C_λ,

$B_{\lambda+1}$ is the subcategory of K generated by all the morphisms

$\gamma(I): il(\Psi, \gamma) \to \Psi(I)$ for each $I \in \mathrm{I}_0$,

$g: K_\lambda \to il(\Psi, \gamma)$ whenever $\gamma \circ g: K_\lambda \Rightarrow \Psi_0$ is a cone in K,

where (Ψ, γ) is any Q-cone with $\gamma \in \Gamma$ and $\Psi: \mathrm{I} \to \mathrm{K}$ valued in B_λ. The construction stops at the first limit ordinal greater than the ordinals of I for each $\mathrm{I} \in Ind\,\Gamma$. ∇

COROLLARY. *If* Q *is* Γ*-generated by* H, *then* H *is* Γ°*-dense for* Q.

Proof by induction on C_λ using the commutativity of initial lifts.

D. Γ-completions.

DEFINITION. A Γ*-completion of* $P: \mathrm{H} \to \mathrm{E}$ is defined as a concrete Γ-complete functor $Q: \mathrm{K} \to \mathrm{E}$ of which P is a full restriction. The Γ-completion is Γ*-dense* (resp. *initially dense*) if so is H for Q; it is (strictly) Γ*-generated* if K is the (strict) Γ-hull of H for Q.

An order is defined on the Γ-completions of P as follows:

$Q \leq_\Gamma Q'$ (say Q is Γ-smaller than Q') iff there exists one unique Γ-morphism $F: Q \to Q'$ extending the identity on H.

We are going to construct completions which are optimal for this order.

EXAMPLES. 1. *Sour*E-completions have been considered by several authors, e.g. Herrlich [2, 4] under the name: *initial completions*.

2. If A is a class of morphisms of E, the /A/-completions of P are called A*-completions*; they are the A-spreading functors extending P which are dealt with in /107/.

3. If E is a complete category and *Lim* E is the class of all its small limit-cones, *Lim* E-completions, just called *completions of P*, are constructed in Adamek-Koubek [1] and Herrlich [3]. More generally, if μ is a partial (multiple) choice of limit-cones on E and Γ the class of limit-cones distinguished by μ, we find the *μ-completions* studied in / 107 /.

2. MAC NEILLE COMPLETIONS.

In this section, we construct a Γ-completion of $P: H \to E$ which is both finally dense and Γ-generated; such a Γ-completion is called a *Mac Neille Γ-completion* of P (by analogy with Herrlich's Mac Neille initial completions, named after the Mac Neille completions of posets).

In / 107 /, Charles constructs the Mac Neille A-completion of P for A a subcategory of E (Theorem 2, 3) and, using it, the Mac Neille μ-completion of P (Theorem 5, 6), which he calls «smallest prolongations of P». His method, which generalizes for any class Γ of cones, may be sketched as follows: To H he adds «formal initial lifts» of cones of Γ and as many morphisms as possible for getting a faithful (non-amnestic) functor in which these formal initial lifts become initial lifts; so H is Γ-dense and finally dense for the associated concrete functor $Q: K \to E$ (this condition entirely characterizes Q). In the case $\Gamma = /A/$, this functor is the Mac Neille A-completion of P. For μ-completions or more generally, the construction has to be transfinitely reiterated, because Q is not Γ-complete. (In fact, Charles gets Q as a Mac Neille $\bar{A}$-completion of a certain extension of P.) Now we remark that Q is Γ-complete whenever Γ is equal to its commutative hull Γ° thanks to the commutativity of initial lifts; in this case, an object of K, which is an equivalence class of P-sources, may be identified to the union of these P-sources, hence to a closed source in Herrlich's sense [3]; so, for $\Gamma = Sour\,E$, Q is exactly the Mac Neille initial completion P_4 as constructed by Herrlich in [2].

Whence the idea of the following proof: we first construct the Mac Neille Γ°-completion of P; the Γ-hull of H in it then gives the Mac Neille Γ-completion of P (constructed by induction via Proposition 5).

THEOREM 1. *P admits a Mac Neille Γ-completion* $P_{\Gamma}: H_{\Gamma} \to E$, *which* *lives in the universe* $\mathcal{U}$ *if so do* P, $Ind\,\Gamma$ *and each* $\mathrm{I} \in Ind\,\Gamma$.

Δ. Construction of the Mac Neille Γ°-completion $V: \mathrm{M} \to \mathrm{E}$ of P: If (Φ, γ) is a P-source, we denote by $(\Phi, \gamma)^*$ the *opposite P-sink* [3]:

$$(S, f: P(S) \to E \mid S \in \mathrm{H}_0, \gamma \circ f: S \Rightarrow \Phi),$$

where E is the vertex of γ. Let M_0 be the class of the P-sinks of the form $(\Phi, \gamma)^*$ for some P-source with $\gamma \in \Gamma^{\circ}$; the vertex of γ is denoted by $V(M)$. If M and M' are in M_0, there'll be a morphism $g: M \to M'$ in M mapped by V on g iff

$$(S, f) \text{ in } M \quad \text{implies} \quad (S, g.f) \text{ in } M'.$$

This defines the concrete functor $V: \mathrm{M} \to \mathrm{E}$. We identify H to a full subcategory of M by identifying $S \in \mathrm{H}_0$ to the P-sink

$$(S, id_{P(S)})^* = (S', h \mid h: S' \to S \text{ in } \mathrm{H}).$$

So H becomes finally dense for Q; it is also Γ°-dense, because the object M is the initial lift of each P-source (considered as a V-source!) (Φ, γ) such that $M = (\Phi, \gamma)^*$. Hence V is a Mac Neille Γ°-completion of P if it is Γ°-complete; this is true: let (Ψ, θ) be a V-source with $\theta \in \Gamma^{\circ}$, indexed by I; for each I in I, we have

$$\Psi(I) = (\Phi_I, \gamma_I)^* = il(\Phi_I, \gamma_I) \quad \text{for some } \gamma_I \in \Gamma^{\circ};$$

as Γ° is commutative $(\Phi_I, \gamma_I)_{\mathrm{I}} \circ (\Psi, \theta)$ is a P-source σ whose dual P-sink σ^* is in M_0, so that $\sigma^* = il(\sigma) = il(\Psi, \theta)$ (by Proposition 3).

– Let H_{Γ} be the Γ-hull of H for V, and $P_{\Gamma}: \mathrm{H}_{\Gamma} \to \mathrm{E}$ the restriction of V. It is Γ-complete (Corollary, 1), and H is still finally dense, H_{Γ} being a full subcategory of M. Hence P_{Γ} is a Mac Neille Γ-completion of P. If $Ind\,\Gamma$ and all $\mathrm{I} \in Ind\,\Gamma$ are in the universe $\mathcal{U}$, then so does $Ind\,\Gamma^{\circ}$ (Proposition 1), which implies M and H_{Γ} are also in $\mathcal{U}$. ∇

REMARK. M is a full subcategory of the Mac Neille initial completion P_4 so that H_{Γ} may also be defined as the Γ-hull of H for P_4. However M is in $\mathcal{U}$ while P_4 may not; conditions for it to be in $\mathcal{U}$ are given in [3].

The «optimality» of P_{Γ} will be deduced from the following proposition, which generalizes Theorems 3 and 6 of /107/ and has a similar proof.

THEOREM 2. *Let* $Q: \mathrm{K} \to \mathrm{E}$ *be a* Γ*-generated* Γ*-completion of* $P: \mathrm{H} \to \mathrm{E}$ *and* $Q': \mathrm{K}' \to \mathrm{E}$ *be a finally dense* Γ*-completion of* $P': \mathrm{H}' \to \mathrm{E}$. *Let* $F: P \to P'$ *be a morphism satisfying the «lifting-cones» condition:*

If $\Phi: \mathrm{I} \to \mathrm{H}$ *is a functor with* $\mathrm{I} \in Ind\,\Gamma$, *each cone in* H' *with basis* $F\Phi$ *is the image by* F *of a cone in* H *with basis* Φ.

Then F *extends in a unique* Γ*-morphism* $F': Q \to Q'$.

Δ. $\mathrm{K} = \bigcup_\lambda \mathrm{C}_\lambda$ (Proposition 5) and F' is defined by induction on C_λ: If F' is defined on C_λ and if $K = il(\Phi, \gamma)$ with $\gamma \in \Gamma$ and Φ valued in C_λ, we take $F'(K) = il(F'\Phi, \gamma)$; since Q' is finally dense, $F'(K)$ does not depend on the choice of (Φ, γ), and $g: K \to K'$ in $\mathrm{C}_{\lambda+1}$ implies that $g: F'(K) \to F'(K')$ in K'. Whence F' is defined on $\mathrm{C}_{\lambda+1}$. ∇

COROLLARY 1. *The Mac Neille* Γ*-completion* P_Γ *of* P *is the* Γ*-smallest finally dense* Γ*-completion of* P *and the* Γ*-largest* Γ*-generated one; in particular, two Mac Neille completions are isomorphic.*

COROLLARY 2. *The Mac Neille* Γ*-completion* P_Γ *of* P *is fully embedded in any finally dense* Γ*-completion* Q' *of* P.

Indeed, if $F = Id_{\mathrm{H}}$ and Q finally dense, F' above is a full embedding.

REMARK. Corollary 2 says that P_Γ is also the smallest finally dense Γ-completion of P for the preorder on completions:

$Q < Q'$ iff there exists a full embedding $Q \to Q'$ (not a Γ-morphism!) extending the identity on H.

For this preorder, Herrlich proves that P_4 is in fact the smallest initial completion of P ; this stronger result comes from the duality Theorem for initially complete functors, which has no analogon for a general Γ.

As in /107/, Theorem 2 is easily adapted to characterize the strict Γ-hull $\mathrm{H}_\Gamma^{\vdash}$ of H for V (or for P_Γ). We say that a Γ-completion $Q: \mathrm{K} \to \mathrm{E}$ of P *is weakly dense* if $K = K'$ whenever:

for each $S \in \mathrm{H}_0$, we have: $g: S \to K$ iff $g: S \to K'$ in K.

THEOREM 3. *The restriction* $P_\Gamma^{\vdash}: \mathrm{H}_\Gamma^{\vdash} \to \mathrm{E}$ *of* P_Γ *is the unique (up to isomorphism)* Γ*-completion of* P *which is both weakly dense and strictly* Γ*-generated; it is the* Γ*-smallest weakly dense* Γ*-completion and its* Γ*-*

largest strictly Γ-generated one. ∇

3. UNIVERSAL PARTIAL COMPLETIONS.

In this section Δ denotes a given class of initial cones in H for $P: \mathrm{H} \to \mathrm{E}$ such that $P\delta \in \Gamma$ for each $\delta \in \Delta$. Let $\Delta_0 = \{ \delta_0 \mid \delta \in \Delta \}$; by Proposition 1, Δ_0 is a class of initial sources for P.

DEFINITION. A Γ-completion $Q: \mathrm{K} \to \mathrm{E}$ of P is called a (Δ, Γ)*-completion of* P if the insertion $\mathrm{H} \hookrightarrow \mathrm{K}$ sends (all δ in) Δ on initial cones. It is called a *universal* (Δ, Γ)*-completion* if it also satisfies:

Let $Q': \mathrm{K}' \to \mathrm{E}$ be a Γ-complete concrete functor and $F: P \to Q'$ be a morphism sending Δ on initial cones; then there exists one unique Γ-morphism $F': Q \to Q'$ extending F.

The universal (Δ, Γ)-completion is unique (up to isomorphism) if it exists, and it is strictly Γ-generated.

EXAMPLES. If $\Delta = \emptyset$, a universal (Δ, Γ)-completion is called a *free* Γ-completion. If Δ is the class of all initial cones δ in H with $P\delta \in \Gamma$, a universal (Δ, Γ)-completion is just called a *universal* Γ*-completion.* The free and universal Γ-completions are proved to exist in /107/ for Γ associated to a subcategory A of E or to a partial choice μ of limits (Theorem 10), but no explicit construction is given in this last case. Herrlich describes the free initial completion P_2 and the universal initial completion P_3 in [1] and, in [4] the universal completion P^* (for $\Gamma = Lim\ \mathrm{E}$), whose objects are the «complete sources». Adapting his method as in Section 2, we'll obtain the universal (Δ_0, Γ^0)-completion U of P, with objects the Δ-complete P-sources; the Γ-hull of H for U is the universal (Δ, Γ)-completion of P.

DEFINITION. A P-source $\sigma = (S_I, f_I \mid I \in \mathrm{I})$ is said Δ*-complete* if it contains the P-morphisms

(a) $(S, h.f_I)$ for each $h: S_I \to S$ in H,

(b) (S', g) if there exists $(d_J : S' \to S'_J \mid J \in \mathrm{J})$ in Δ_0 with

$$(S'_J, d_J . g) \text{ in } \sigma \text{ for each } J \in \mathrm{J}.$$

(Intuitively, σ is closed under left composition by H and factors through the initial sources of Δ_o .)

PROPOSITION 6. *Each P-source* $\sigma = (\Phi, \gamma)$ *is included in a smallest* Δ-*complete P-source, denoted by* $\Delta\sigma = (\Delta\Phi, \Delta\gamma)$, *which is constructed by transfinite induction. We have* $il\ \sigma = il\ \Delta\sigma$ *as soon as one of them is defined.*

Δ. The P-source $\Delta\sigma$ is the union of the transfinite sequence $(\sigma_\lambda)_\lambda$ where $\sigma_0 = \sigma$, $\sigma_a = \bigcup_{\lambda < a} \sigma_\lambda$ for a limit ordinal a , and $\sigma_{\lambda+1}$ is deduced from σ_λ by adding elements of the form (a) and (b) above. For the last assertion, we prove by induction on σ_λ that, if the P-source $\sigma \circ g = (\Phi, \gamma \circ g)$ lifts into a source $\gamma \circ g : S \Rightarrow \Phi$ in H , then the P-source $\Delta\sigma \circ g$ lifts into a source with basis $\Delta\Phi$ in H . ∇

THEOREM 4. *P has a universal* (Δ, Γ)-*completion* $U_\Gamma : L_\Gamma \to E$ *which is* Γ-*generated, hence* Γ^o-*dense. It lives in the universe* $\mathfrak{U}$ *if so do* $Ind\, \Gamma$ *and each of its elements.*

Δ. 1. Construction of a (Δ_o, Γ^o)-completion $U : L \to E$ of P . Let L_o be the class formed by the Δ-complete P-sources L of the form (Proposition 6) $\Delta(\Phi, \gamma)$ for some $\gamma \in \Gamma^o$; let $U(L)$ be the vertex of γ . If L' is also in L_o, then $g : L' \to L$ is a morphism in L mapped by U on g iff $(\Phi, \gamma \circ g)$ is included in L' (which implies $L \circ g \subset L'$). We identify H to a full subcategory of L by identifying the object S to

$$\Delta(S, id_{P(S)}) = (S', f \mid f : S \to S' \text{ in H}).$$

As we have $h : L \to S$ in L iff (S, h) is in L , it follows that L is the initial lift of (Φ, γ) considered as a U-source, and that $\delta_o \in \Delta_o$ remains an initial source for U . The fact that U is Γ^o-complete is proved as in Theorem 1, thanks to the commutativity of initial lifts and of Γ^o .

2. Universality of U. Let $Q : K \to E$ be a Γ^o-complete functor, and $F : P \to Q$ a morphism sending Δ_o on initial sources. If there is a Γ^o-morphism $F' : U \to Q$ extending F , it maps L on $il(F\Phi, \gamma)$ and $g : L' \to L$ on $g : F'(L') \to F'(L)$. So we have just to prove that this F' is well-defined i.e., that

$$il(F\Phi', \gamma') = il(F\Phi, \gamma) \quad \text{if} \quad \Delta(\Phi', \gamma') = L = \Delta(\Phi, \gamma).$$

Indeed $(F\Phi, \gamma)$ generates a $F\Delta$-complete Q-source σ and $il\,\sigma = il(F\Phi, \gamma)$ (Proposition 6). From the construction of $\Delta(\Phi, \gamma) = (\Delta\Phi, \Delta\gamma)$, we deduce by induction that $(F\Delta\Phi, \Delta\gamma) \supset (F\Phi', \gamma')$ is included in $\Delta(F\Phi, \gamma) = \sigma$. Therefore

$$\Delta(F\Phi', \gamma') = \sigma \quad \text{and} \quad il(F\Phi', \gamma') = il\,\sigma = il(F\Phi, \gamma).$$

3. Universal (Δ, Γ)-completion of P. Let $U_\Gamma : \mathrm{L}_\Gamma \to \mathrm{E}$ be the restriction of U to the Γ-hull of H for U; it is a (Δ, Γ)-completion of P. To prove the universality, let $Q': \mathrm{K}' \to \mathrm{E}$ be a Γ-complete functor, and $G: P \to Q'$ a morphism sending Δ on initial cones. We consider the universal Γ°-completion $U': \mathrm{L}' \to \mathrm{E}$ of Q'; as $Q' \hookrightarrow U'$ is a Γ°-morphism, hence a Γ-morphism, $G: P \to U'$ still sends Δ on initial cones and, by Part 2, it extends into a unique Γ°-morphism $G': U \to U'$. If G' maps L_Γ into the full subcategory K' of L' its restriction $G'': U_\Gamma \to Q'$ will be the unique Γ-morphism extending G. This is proved by induction on C_λ, where $\mathrm{L}_\Gamma = \bigcup_\lambda \mathrm{C}_\lambda$ (Proposition 5): suppose G' maps C_λ into K'; if L is an object of $\mathrm{C}_{\lambda+1}$, we have $L = il(\Psi, \gamma')$, where $\gamma' \in \Gamma$ and Ψ valued in C_λ; as $G'\Psi$ is valued in K' and Q' is Γ-complete, the Q'-cone $(G'\Psi, \gamma')$ has an initial lift for Q', which remains an initial lift for the universal Γ°-completion U', hence is equal to $G'(L) = il(G'\Psi, \gamma')$. It follows that G' maps $\mathrm{C}_{\lambda+1}$ into the full subcategory K'. ∇

REMARKS. 1. Suppose $\Gamma = Lim\ \mathrm{E}$. Then the universal completion U_Γ of P is (Γ-)dense (not only Γ-generated); this has been proved by Adamek-Koubek [1] via a construction which is transfinite only for morphisms, and by Herrlich [4] thanks to his one-step construction. His proof rests on the two facts:

A functor is complete as soon as it lifts products and equalizers;

Let $\sigma = \Delta(\Phi, \gamma)$ with $\gamma \in Lim\ \mathrm{E}$; any P-source included in σ is also included in a P-source $(\bar\Phi, \bar\gamma) \subset \sigma$ with $\bar\gamma$ a limit-cone [4];

it is easily adapted, whatever be Δ, to prove that L_Γ reduces to the full subcategory C_1 of L (Part 3 above), whence:

COROLLARY. *The universal $(\Delta, Lim\ \mathrm{E})$-completion of P is $(Lim\ \mathrm{E}$-$)$ dense.*

2. The free initial completion of P is the largest initial completion, while its universal one is the largest preserving initial lifts initial completion [2]. This maximality property is no more valid for a general class Γ ; we only prove as in Theorem 4 the

PROPOSITION 7. *Let* $Q: \mathrm{K} \to \mathrm{E}$ *and* $Q': \mathrm{K}' \to \mathrm{E}$ *be* (Δ_0, Γ°)*-completions of* P. *If* Q *satisfies*

(1) For each K *in* K_0 *there exists a P-source* (Φ, γ) *uith* $\gamma \in \Gamma^\circ$ *such that* $\Delta(\Phi, \gamma) = (S', h \mid h: K \to S'$ *in* $\mathrm{K})$ *and* $K = il(\Phi, \gamma)$,

then there exists a morphism $Q \to Q'$ *extending the identity on* H.

COROLLARY. *The universal* (Δ_0, Γ°)*-completion* U *of* P *is the largest* (Δ_0, Γ°)*-completion which satisfies (1).*

Another maximality property of the universal (Δ, Γ)-completion is given in Comment 91-1 [0], Part IV-1.

3. The problem of «lifting singleton sources» may be translated in the world of internal functors in a category, leading to universal internal $(\Delta, /A/)$-completions ; cf. /95, 96/ and Synopsis n° 5 [0], Part III-2.

4. Many authors consider only concrete functors which are transportable. The preceding results are easily adapted to this case.

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