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ALGEBRAIC ASPECTS OF TOPOS THEORY

by J. LAMBEK and P. J. SCOTT

When working with fields, one sometimes has to go into the algebraic category of (commutative) rings. For example, to adjoin an indeterminate x to a field F one forms the ring $F[x]$, from which a field $F(x)$ may be obtained by constructing its ring of quotients. For similar reasons the notion of «dogma» was introduced in [13] to capture the algebraic aspect of toposes:

$$\frac{\text{dogma}}{\text{topos}} = \frac{\text{ring}}{\text{field}} .$$

Dogmas are essentially the same as Volger's «semantical categories» [17] and Bénabou's «formal toposes». They are also related to models of «type theory» in the sense of Chruch [3] and Henkin [9] and models of «illative combinatory logic» in the sense of Curry and Feys [5].

Every one knows that groups and rings are sets with operations satisfying certain identities. In the same way categories and dogmas may be viewed as graphs with operations and identities, as are cartesian closed categories [11] and monoidal closed categories [10]. Thus, a *category* is a graph with operations (one nullary and one binary)

$$1_A : A \to A \qquad \frac{f: A \to B \quad g: B \to C}{gf: A \to C}$$

satisfying the identities

$$1_B f \cdot=\cdot f, \quad f 1_A \cdot=\cdot f, \quad (hg)f \cdot=\cdot h(gf)$$

for all $f: A \to B$, $g: B \to C$ and $h: C \to D$. A *cartesian* category is essentially a category with canonical finite products, more precisely, a category with the following additional structure: three nullary operations and one binary operation

$$0_A : A \to 1 \qquad \pi_{A,B} : A \times B \to A \qquad \pi'_{A,B} : A \times B \to B$$

$$\frac{f : C \to A \quad g : C \to B}{\langle f, g \rangle : C \to A \times B}$$

satisfying the identities

$$k \cdot = \cdot\ 0_A \qquad \pi_{A,B} \langle f, g \rangle \cdot = \cdot\ f \qquad \pi'_{A,B} \langle f, g \rangle \cdot = \cdot\ g$$

$$\langle \pi_{A,B} h, \pi'_{A,B} h \rangle \cdot = \cdot\ h$$

for all $k : A \to 1$, $f : C \to A$, $g : C \to B$ and $h : C \to A \times B$.

A topos is, among other things, a cartesian closed category, that is, a cartesian category with exponents B^A and a natural isomorphism $Hom(C, B^A) \approx Hom(C \times A, B)$. For technical reasons we require exponents only for a specified object Ω and shall write Ω^A as PA. A *predogma* (for want of a better name) is a cartesian category with the following additional structure: a nullary and a unary operation

$$\epsilon_A : PA \times A \to \Omega \qquad \frac{f : C \times A \to \Omega}{f^* : C \to PA}$$

satisfying the identities

$$\epsilon_A \langle f^* \pi_{C,A}, \pi'_{C,A} \rangle \cdot = \cdot\ f \qquad (\epsilon_A \langle g \pi_{C,A}, \pi'_{C,A} \rangle)^* \cdot = \cdot\ g$$

for all $f : C \times A \to \Omega$ and $g : C \to PA$.

A dogma is a predogma with a nullary operation $\delta_A : A \times A \to \Omega$ satisfying a number of identities, for example

$$\delta_A \langle f, f \rangle \cdot = \cdot\ \delta_1 \langle 1_1, 1_1 \rangle$$

for all $f : C \to A$. In what follows, we shall write

$$f \underset{A}{=} g \text{ for } \delta_A \langle f, g \rangle \qquad \text{and} \qquad \mathrm{T} \text{ for } \delta_1 \langle 1_1, 1_1 \rangle$$

so that the above sample identity may be written

$$f \underset{A}{=} f \cdot = \cdot\ \mathrm{T}\ .$$

It is important to distinguish the internal equality symbol $\underset{A}{=}$ from the extemal one $\cdot = \cdot$.

A complete list of identities satisfied by δ_A will be found in [13]. Even there the identities are made intelligible by first factoring

$$Hom(-,\Omega): \mathfrak{A}^{op} \to Ens$$

through the category of preordered sets. They could have been simplified further by factoring through the category of Heyting algebras. Having tried unsuccessfully *) to recapture these identities from Guitart's «contravariant standard constructions» [8], we shall here propose another approach: to factor the functor $Hom(-,\Omega)$ through the category of «deductive sets».

A *deductive set* $(X, \vdash)$ consists of a set X together with a binary relation $\vdash$ between finite subsets $\Gamma = \{f_1, \dots, f_n\}$ of X and elements g of X satisfying the following conditions:

$$(1)\quad \{f\} \vdash f \qquad\qquad (2)\quad \frac{\Gamma \vdash f \quad \Gamma\cup\{f\} \vdash g}{\Gamma \vdash g}$$

$$(3)\quad \frac{\Gamma \vdash g}{\Gamma\cup\{f\} \vdash g} \qquad\qquad (4)\quad \frac{f \vdash g \quad g \vdash f}{f \cdot = \cdot\ g}\ .$$

A *morphism* $\phi : (X, \vdash) \to (Y, \vdash)$ of deductive sets is a mapping $X \to Y$ such that

$$\frac{\{f_1, \dots, f_n\} \vdash g}{\{\phi(f_1), \dots, \phi(f_n)\} \vdash \phi(g)}\ .$$

Examples of deductive sets are meet-semilattices and implication algebras with largest element T, where e.g. $\{f_1, f_2\} \vdash g$ stands for

$$f_1 \wedge f_2 \leq g \quad \text{or} \quad f_1 \Rightarrow (f_2 \Rightarrow g) \cdot = \cdot\ \mathrm{T} \quad \text{respectively.}$$

To say that $Hom(-,\Omega)$ factors through the category of deductive sets then amounts to saying that for each object A, $Hom(A,\Omega)$ is equipped with a relation $\vdash_A$ satisfying condition (1) to (4) (with $\vdash$ replaced by $\vdash_A$) and also

*) However, in the meantime, Guitart has succeeded in augmenting his system to the so-called «algebraic universes», which are adequate for mathematics and which are more closely related to dogmas.

$$(5)\qquad \frac{\{f_1,\dots,f_n\} \vdash_A g}{\{f_1 h,\dots,f_n h\} \vdash_B gh}$$

for all $f_1,\dots,f_n,\ g: A\to\Omega$ and $h: B\to A$. Rules (1) to (5) will be called *structural rules*. Predogmas for which $Hom(-,\Omega)$ factors through the category of deductive sets appear to be a special case of the «categories with deduction» discussed in the seminar of Bénabou.

We are finally in a position to define dogmas. A *dogma* is a predogma for which $Hom(-,\Omega)$ factors through the category of deductive sets and which contains a nullary operation $\delta_A: A\times A\to\Omega$ satisfying the following additional conditions, where we have written $f \underset{A}{=} g$ for $\delta_A\langle f,g\rangle$:

$$(6)\quad \Gamma \vdash_A f \underset{B}{=} f \qquad\qquad (7)\quad f \underset{B}{=} g \vdash_A hf \underset{C}{=} hg$$

$$(8)\quad f \underset{B}{=} g \vdash_A \langle f, 1_A\rangle \underset{B\times A}{=} \langle g, 1_A\rangle$$

for all $f,\ g: A\to B$ and $h: B\to C$;

$$(9)\quad \frac{\Gamma, f \vdash_A g \quad \Gamma, g\vdash_A f}{\Gamma \vdash_A f \underset{\Omega}{=} g} \qquad\qquad (10)\quad f \underset{\Omega}{=} g,\ f \vdash_A g$$

for all $f,\ g: A\to\Omega$;

$$(11)\qquad \frac{\{f_1\pi_{A,B},\dots,f_n\pi_{A,B}\} \vdash_{A\times B} f \underset{\Omega}{=} g}{\{f_1,\dots,f_n\} \vdash_A f^* \underset{PB}{=} g^*}$$

for all $f_1,\dots,f_n: A\to\Omega$ and $f,\ g: A\times B\to\Omega$. Rules (6) to (11) will be called *rules of equality*.

It may be shown that the definition of «dogma» given here agrees with that of [13] by comparing the present symbol $\vdash_A$ with the symbol $\leq_A$ there. On the one hand, $\{f_1,\dots,f_n\} \vdash_A g$ may be interpreted as

$$f_1 \wedge \dots \wedge f_n \leq_A g,$$

and, on the other hand, $f \leq_A g$ may be regarded as $\{f\} \vdash_A g$.

It is perhaps more instructive to see how formulas of type theory are interpreted in a dogma $\mathfrak{A}$. For this purpose let us assume that type theory is based on the notion of equality as presented in [15]. To spell things out, we are given types 1 and Ω and assume that PA and $A\times B$ are types if A and B are. For each type there is given a countable set of variables of that type; moreover, one has the following terms, listed under their respective types:

1	Ω	PA	$A\times B$
$*$	$a \in \alpha$ $a = a'$	$\{x \in A \mid \phi(x)\}$	$\langle a, b\rangle$

where it is assumed that a, a' and the variable x are of type A, α is of type PA, b is of type B and $\phi(x)$ is of type Ω.

It was shown in [13] how to adjoin an indeterminate arrow $x: 1 \to A$ to a dogma $\mathfrak{A}$ to obtain the dogma $\mathfrak{A}[x]$. Let $\theta(x_1, \ldots, x_n)$ be a term of type A depending on the variables x_i of type A_i, we shall interpret it as an arrow in the dogma

$$\mathfrak{A}[x_1, \ldots, x_n] = \mathfrak{A}[x_1] \ldots [x_n]$$

as follows: x_i is interpreted as the indeterminate $x_i: 1 \to A_i$, $*$ as the unique arrow $1 \to 1$, $a \in \alpha$ as $\epsilon_A\langle a, \alpha\rangle$, $a = a'$ as $\delta_A\langle a, a'\rangle$ and $\langle a, b\rangle$ as the arrow $1 \to A\times B$ with the same name, where a, α, a' and b are assumed to have been interpreted already. It remains to interpret $\{x \in A \mid \phi(x)\}$ as an arrow $1 \to PA$ in $\mathfrak{A}$, assuming that $\phi(x)$ has already been interpreted as an arrow $1 \to \Omega$ in $\mathfrak{A}[x]$.

Now dogmas, like predogmas and even cartesian or cartesian closed categories, have the following property of «functional completeness» [12]: given any arrow $\phi(x): 1 \to B$ in $\mathfrak{A}[x]$, there is a unique arrow $f: A \to B$ in $\mathfrak{A}$ such that $\phi(x) \cdot \underset{x}{=} \cdot fx$, where $\cdot \underset{x}{=} \cdot$ denotes (external) equality in $\mathfrak{A}[x]$. When $\mathfrak{A}$ is a dogma or predogma, we may take $B = \Omega$, then f corresponds to an arrow $1 \to PA$, which we denote by $\{x \in A \mid \phi(x)\}$.

Other logical symbols may be defined in terms of equality as follows

T	for $* = *$,
$p \wedge q$	for $\langle p, q\rangle = \langle \mathrm{T}, \mathrm{T}\rangle$,
$p \Rightarrow q$	for $(p \wedge q) = p$,

$\forall_{x \in A}\, \phi(x)$	for	$\{x \in A \mid \phi(x)\} = \{x \in A \mid \mathrm{T}\}$,
$\perp$	for	$\forall_{t \in \Omega}\, t$,
$p \vee q$	for	$\forall_{t \in \Omega}\, (((p \Rightarrow t) \wedge (q \Rightarrow t)) \Rightarrow t)$,
$\exists_{x \in A}\, \phi(x)$	for	$\forall_{t \in \Omega} (\forall_{x \in A} (\phi(x) \Rightarrow t) \Rightarrow t)$,
$\{a\}$	for	$\{x \in A \mid x = a\}$,
$\exists !_{x \in A} \phi(x)$	for	$\exists_{x \in A} (\{x \in A \mid \phi(x)\} = \{x\})$.

Thus any fomula of ordinary type theory may be interpreted in any dogma. Of course, if we want to allow for Peano arithmetic, we must adjoin a natural number object to the dogma in question [13].

Type theory is not just a language, but a deductive system. Given a set $X = \{x_1, \dots, x_m\}$ of variables of types $A_1, \dots, A_m$ respectively, one writes

$$\phi_1(X), \dots, \phi_m(X) \vdash_X \psi(X)$$

to indicate that $\psi(X)$ may be deduced from the assumptions $\phi_1(X), \dots$ $\dots, \phi_m(X)$ according to the rules of intuitionistic type theory. We shall show how such an entailment is interpreted in a dogma. Using cartesian products, one may replace X by a single variable x of type

$$A = A_1 \times A_2 \times \dots \times A_m .$$

Using functional completeness, one may replace $\phi_i(x)$ by $f_i\, x$, $\psi(x)$ by gx, where f_i and g are arrows $A \to \Omega$. We shall say that the given entailment *holds* in the dogma $\mathfrak{A}$ if $\{f_1, \dots, f_m\} \vdash_A g$.

It may be checked that if the entailment $\phi_1(x), \dots, \phi_n(X) \vdash_X \psi(X)$ is provable in intuitionistic type theory, then it holds in any dogma. *)

As is well-known, a topos is a predogma in which the functor $Hom(-, \Omega)$ is naturally equivalent to the subobject functor. Why is a topos a dogma? In a topos one defines $\delta_A : A \times A \to \Omega$ as the characteristic morphism of the monomorphism $\langle 1_A, 1_A \rangle : A \to A \times A$. Given f_i, $g : A \to \Omega$, we shall say that $\{f_1, \dots, f_n\} \vdash_A g$ means the following: for all $h : B \to A$,

*) The converse of this statement is also true, in view of the existence of the free dogma generated by the empty graph, which is discussed later.

$$\text{if } f_i h \cdot = \cdot \top O_B \text{ for } i = 1, \dots, n, \text{ then } gh \cdot = \cdot \top O_B.$$

It is then easily checked that conditions (1) to (11) hold in a topos. This definition is closely related to the so-called Kripke-Joyal semantics [16]. It immediately allows us to interpret intuitionist type theory in any topos.

Actually we obtain a functor $U: Top \to Dog$ from the category of toposes to that of dogmas. The morphisms in Top are the so-called «logical morphisms». The morphisms in Dog are functors that preserve the dogma structure on the nose. They were called «orthodox» functors in [13]; but we may as well call them *logical functors*, as they specialize to the logical morphisms in Top. The functor U thus is an inclusion and we may regard Top as a full subcategory of Dog.

To obtain a left adjoint F to the functor U one wants to construct a topos $F(\mathfrak{A})$ for each dogma $\mathfrak{A}$ together with a logical functor $H_{\mathfrak{A}}$: $\mathfrak{A} \to UF(\mathfrak{A})$ so that, for each logical functor $G: \mathfrak{A} \to U(\mathfrak{B})$, where $\mathfrak{B}$ is a topos, there is a unique morphism

$$G': F(\mathfrak{A}) \to \mathfrak{B} \text{ in } Top \text{ with } U(G')H_{\mathfrak{A}} = G.$$

Volger [17] succeeded in doing this with some waving of hands: G' was not exactly a functor, only a lax functor with $G'(gf) \approx G'(g)G'(f)$, and it was only unique up to isomorphism.

How does one get around this difficulty? Bill Lawvere has suggested that one should meet the two-category structure of Top head on. According to Max Kelly, the two-morphisms in Top are all isomorphisms, so one should study lax functors with coherent isomorphisms and redefine the word «adjoint» for this context. Rather than getting bogged down in two-category theory, another solution was proposed in [13].

Let Top_0 be the subcategory of Top whose objects are toposes which have *canonical subobjects*. By this we mean that in each equivalence class of monomorphisms $A \rightarrowtail B$ there is a unique «canonical» one and that canonical monomorphisms satisfy the following obvious conditions: identity arrows and compositions of canonical monos are canonical, and, if

$f: C \to A$ and $g: D \to B$ are canonical monos, so are

$$f \times g: C \times D \to A \times B \quad \text{and} \quad Pf: PC \to PA,$$

where P is the covariant powerset functor. Furthermore, let the morphisms of Top_0 be logical morphisms which preserve canonical subobjects. Then the functor $U_0: Top_0 \to Dog$ is no longer full, but it has a left adjoint F_0, without waving of hands. The details are found in [13].

Some people feel unhappy when being asked to confine attention to toposes with canonical subobjects. So let us point out that all toposes occuring in nature (sets, presheaves, sheaves,...) have canonical subobjects. For example, we can surely distinguish a genuine subset of the set N of natural numbers from a monomorphism into N. Of course, toposes ingeniously constructed by mathematicians need not have canonical subobjects. Nevertheless, every topos is equivalent to one with canonical subobjects. To see this, one only has to take the arrows with target Ω in the old topos as objects of the new topos. More precisely, regarding the topos $\mathfrak{A}$ as a dogma $U(\mathfrak{A})$, the topos $F_0 U(\mathfrak{A})$ has canonical subobjects and is equivalent to $\mathfrak{A}$. At any rate, one has the following:

THEOREM. *The category Top contains a subcategory Top_0 such that every object of Top is equivalent by a logical functor to an object of Top_0 and such that the restriction U_0 to Top_0 of the forgetful functor U: $Top \to Grph$ has a left adjoint F_0.*

From now on we shall write Top for Top_0 and replace U_0 and F_0 by U and F respectively.

The adjunction functor $H_{\mathfrak{A}}: \mathfrak{A} \to UF(\mathfrak{A})$ has the following properties proved in [13]:

(1) $H_{\mathfrak{A}}$ is faithful if and only if the «singleton» morphism $\delta_A^*: A \to PA$ is a monomorphism for each object A. This is also equivalent to saying that whenever $\vdash \forall_{x \in A} fx = gx$ holds in $\mathfrak{A}$, then $f \cdot = \cdot\, g$ in $\mathfrak{A}$. We may express this by the slogan: internal equality implies external equality.

It should be remarked that the formula $\forall_{x \in A} fx = gx$ appearing above is not in the language of pure type theory, but in the applied language which admits all objects of $\mathfrak{A}$ as types and also terms produced by arrows of $\mathfrak{A}$, according to the rule that if a is a term of type A and $f: A \to B$ an arrow in $\mathfrak{A}$ then fa is a term of type B.

(2) $H_{\mathfrak{A}}$ is full and faithful if and only if, for each object A, the morphism $\delta_A^*: A \to PA$ is an equalizer of two arrows into some object of the form PB. This is equivalent to saying that $\mathfrak{A}$ *has description*: if

$$\vdash \forall_{x \in A} \exists !_{y \in B} \phi(x, y)$$

holds in $\mathfrak{A}$, then there is a unique arrow $f: A \to B$ such that

$$\vdash \forall_{x \in A} \phi(x, fx)$$

holds in $\mathfrak{A}$. (It follows from this that internal equality implies external equality, as is seen by taking

$$\phi(x, y) \equiv fx = y \wedge gx = y.\ )$$

(3) $H_{\mathfrak{A}}$ is an equivalence if and only if $\mathfrak{A}$ is a topos.

We shall mention two uses to which the functor $F: Dog \to Top$ can be put.

I. Construction of the free topos generated by a graph $\mathfrak{X}$.

As has already been mentioned by Volger [17], since dogmas are equational over graphs, one may construct the free dogma $F_d(\mathfrak{X})$ generated by a graph $\mathfrak{X}$ by the method of [10]. The free topos generated by $\mathfrak{X}$ is then given by $FF_d(\mathfrak{X})$. Free toposes generated by graphs can also be constructed directly, using the language of type theory, as in Boileau [1], Coste [4], Fourman [6] and Lambek-Scott [14].

II. Adjunction of an indeterminate arrow $x: 1 \to A$ *to a topos* $\mathfrak{A}$.

First form $\mathfrak{A} \to \mathfrak{A}[x]$, this is only a dogma, then form

$$\mathfrak{A}[x] \to F(\mathfrak{A}[x]) = \mathfrak{A}(x)$$

to get a topos. There is some analogy to the situation for fields:

$$F \to F[x] \to Q(F[x]) = F(x),$$

hence the notation. However, the analogy is not complete, since $F(x)$ does not have the expected universal property, while $\mathfrak{A}(x)$ does: let H

be the morphism $\mathcal{A} \to \mathcal{A}(x)$ in *Top*, then, for any $F: \mathcal{A} \to \mathcal{A}'$ in *Top* and any $a: 1 \to F(A)$ in $\mathcal{A}'$, there exists a unique $F': \mathcal{A}(x) \to \mathcal{A}'$ in *Top* such that $F'H = F$ and $F'(x) \cdot = \cdot\ a$.

It will be of interest to compare the polynomial topos $\mathcal{A}(x)$ with the slice topos $\mathcal{A}/A$, whose objects are arrows $f: B \to A$, B being any object of $\mathcal{A}$, and whose arrows are commutative triangles. It was noticed for Grothendieck toposes in [7] and for elementary toposes by Joyal (unpublished) that $\mathcal{A}/A$ behaves much like $\mathcal{A}(x)$. First there is a logical functor $H: \mathcal{A} \to \mathcal{A}/A$ which sends the object B of $\mathcal{A}$ onto the object $\pi_{A,B}: A \times B \to A$ of $\mathcal{A}/A$. Moreover, $\mathcal{A}/A$ contains the arrow

$$\xi: H(1) \to H(A) \quad \text{given by} \quad \langle \pi_{A,1}, \pi_{A,1} \rangle : A \times 1 \to A \times A,$$

which behaves like an indeterminate in the following way. Given any logical functor $F: \mathcal{A} \to \mathcal{A}'$ and any arrow $a: 1 \to F(A)$ in $\mathcal{A}'$, there exists a logical functor $F': \mathcal{A}/A \to \mathcal{A}'$, unique up to natural isomorphism, such that

$$F'H \approx F \quad \text{and} \quad F'(\xi) \cdot = \cdot\ a.$$

F' is constructed at the object $f: B \to A$ by forming the pullback:

$$\begin{array}{ccc} F'(f) & \longrightarrow & F(B) \\ \downarrow & & \downarrow {\scriptstyle F(f)} \\ 1 & \xrightarrow{\ a\ } & F(A) \end{array}$$

One could make F' unique by stipulating that $F'(f)$ is a canonical subobject of $F(B)$, but even then $F'H$ cannot be equal to F. For, if we apply the above construction to the object

$$H(C) = \pi_{A,C}: A \times C \to A,$$

$F'H(C)$ will be a canonical subobject of $F(A \times C) = F(A) \times F(C)$, but the monomorphism $F(C) \to F(A) \times F(C)$ is not canonical. We don't know whether the SGA 4-Joyal construction can be fixed up to exhibit the expected universal property on the nose. Failing this, our construction of $\mathcal{A}(x)$ via dogmas will have to serve.

Finally, let us refer to the very interesting concept of «graphical algebra» discussed by Albert Burroni [2]. This notion includes not only categories, cartesian categories, cartesian closed categories, monoidal closed categories, predogmas and dogmas, all of which we have regarded as algebraic over graphs, but supposedly also categories with canonical limits or colimits and toposes with canonical subobjects. This suggests that *Top* is monadic over the category of graphs *).

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*) NOTE ADDED IN MARCH 1981. In his revised manuscript [2] now called «Algèbres graphiques», Burroni actually asserts this. One of the present authors has proved that *Top* is monadic over *Cat*.

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