CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

HEINRICH KLEISLI

Coshape-invariant functors and Mackey's induced representation theorem

Cahiers de topologie et géométrie différentielle catégoriques, tome 22, nº 1 (1981), p. 105-110

<http://www.numdam.org/item?id=CTGDC_1981__22_1_105_0>

CAHIERS DE TOPOLOGIE
ET GEOMETRIE DIFFERENTIELLE

3^e COLLOQUE SUR LES CATEGORIES
DEDIE A CHARLES EHRESMANN
Vol. XXII-1 (1981)

COSHAPE-INVARIANT FUNCTORS AND MACKEY'S INDUCED REPRESENTATION THEOREM

by Heinrich KLEISLI *)

In this paper it is shown how to obtain Mackey's induced representation theorem by means of a simple categorical argument. This is done in the situation where we are given a finite group G, a subgroup H, and where we consider G/H as a discrete measure space (see, for instance, [C], page 88). The method employed indicates generalisations to the situation where G and H are non-discrete topological groups. Therefore, the categorical argument will again be described in detail, although it is already fully explained in [F-K].

1. A THEOREM ON COSHAPE-INVARIANT FUNCTORS.

Let $\mathcal{V}$ be a (fixed) closed category. All categories, functors, etc., are to be regarded as $\mathcal{V}$-categories, $\mathcal{V}$-functors, etc. All Kan extensions are supposed to be pointwise, that is, given by their Kan formula (see [D], Theorem I.4.3, formula (1)).

DEFINITION 1.1. Let $K: \mathcal{P} \to \mathcal{T}$ be a functor, $\mathcal{P}$ a small category, and $\mathcal{V}$ a complete closed category. Then the *coshape of* K is the category ${}_K\mathcal{S}$ which has the same objects as $\mathcal{T}$ and where

$$ {}_K\mathcal{S}(X, Y) = Nat(\mathcal{T}(K-, X), \mathcal{T}(K-, Y)), $$

the $\mathcal{V}$-object of natural transformations between the functors $\mathcal{T}(K-, X)$ and $\mathcal{T}(K-, Y)$.

The identity map between the objects of $\mathcal{T}$ and of ${}_K\mathcal{S}$ can easily be made into a functor $D: \mathcal{T} \to {}_K\mathcal{S}$.

*) This paper was written while the author was on leave of absence and visiting Queen's University, Canada.

DEFINITION 1.2. We say that a functor $F: \mathcal{T} \to \mathcal{V}$ is *coshape-invariant* (with respect to K) if it factors through D.

THEOREM 1.3. *Assume that the base category $\mathcal{V}$ is complete. Let $K: \mathcal{P} \to \mathcal{T}$ be a functor between small categories, and let $F: \mathcal{T} \to \mathcal{V}$ be a coshape-invariant functor $F = \bar{F} D$. If the functor $\bar{F}$ can be extended along the embedding $E: {}_K\mathcal{S} \to \mathcal{V}^{\mathcal{P}^{op}}$ to a cocontinuous functor $\hat{F}$, then F is a left Kan extension along K.*

Let $Y: \mathcal{P} \to \mathcal{V}^{\mathcal{P}^{op}}$ denote the Yoneda functor. By the definition of the generalised tensor product (see [A], page 2) and by the Yoneda lemma we have

$$\mathcal{V}^{\mathcal{P}^{op}}(G \otimes_{\mathcal{P}} Y, H) = Nat(G\cdot, \mathcal{V}^{\mathcal{P}^{op}}(Y\cdot, H)) \approx \mathcal{V}^{\mathcal{P}^{op}}(G, H),$$

whence $G \otimes_{\mathcal{P}} Y \approx G$. Now, $\hat{F}$ is cocontinuous, so that

$$\hat{F}\cdot = \hat{F}(\cdot \otimes_{\mathcal{P}} Y) \approx \cdot \otimes_{\mathcal{P}} \hat{F} Y,$$

and therefore,

$$F = \hat{F} E D\cdot \approx \mathcal{T}(\cdot, K) \otimes_{\mathcal{P}} \hat{F} Y = Lan_K \hat{F} Y.$$

Since $E: {}_K\mathcal{S} \to \mathcal{V}^{\mathcal{P}^{op}}$ is an embedding, the left Kan extension $Lan_E \bar{F}$ is an ordinary extension of $\bar{F}$ along E, which can be computed by means of the Kan formula

$$(Lan_E \bar{F})\cdot = \mathcal{V}^{\mathcal{P}^{op}}(E, \cdot) \otimes_{{}_K\mathcal{S}} \bar{F}.$$

This yields the following corollary of Theorem 1.3.

COROLLARY 1.4. *In the situation of Theorem 1.3, if $Lan_E \bar{F}$ is cocontinuous, then F is a left Kan extension along K.*

2. APPLICATION TO COSHAPE-INVARIANT REPRESENTATIONS.

We shall apply Corollary 1.4 to the following special situation. As base category $\mathcal{V}$ we choose the category *k-Mod* of k-vector spaces. Let $T = kG$ and $P = kH$ be the group algebras over k of a discrete group G and of a subgroup H of G, and let $K: P \to T$ denote the canonical embedding. We consider P and T as single object categories (enriched over *k-Mod*) and $K: P \to T$ as a (enriched) functor.

Then the coshape of K is again a (enriched) single object category, given by the endomorphism algebra $S = End_{P^{op}} T$ of T considered as right P-module. The algebra homomorphism $D: T \to S$ which associates with each element t of T the left multiplication $x \mapsto tx$ in T yields the corresponding functor D.

A linear representation $R: G \to GL(V)$ of the group G by linear transformations on the vector space V can be considered as a kG-module that is, a (enriched) functor $R: kG \to k\text{-}Mod$. Hence the following definition.

DEFINITION 2.1. We say that a linear representation $R: kG \to k\text{-}Mod$ is *coshape-invariant* (with respect to K) if it factors through $D: kG \to S$; in other words, if the representation module admits an S-module structure extending its kG-module structure.

A linear representation $R: kG \to k\text{-}Mod$ is a left Kan extension along K of a linear representation $Q: kH \to k\text{-}Mod$ if it is of the form

$$R = Hom_{kG}(K, kG) \otimes_{kH} Q = kG \otimes_{kH} Q;$$

in other words, if it is induced by Q.

Corollary 1.4 of Theorem 1.3 yields now the following result.

THEOREM 2.2. *Let G be a group and H a subgroup of finite index and denote by $K: kH \to kG$ the canonical embedding of the corresponding group algebras. A necessary and sufficient condition for a linear representation $R: G \to GL(V)$ to be induced by a linear representation $Q: H \to GL(V)$ is that the functor $R: kG \to k\text{-}Mod$ is coshape-invariant with respect to K.*

The necessity of the condition is obvious. In order to show that it is also sufficient, by Corollary 1.4 it suffices to check that $Lan_E \bar{R}$ is cocontinuous, where $\bar{R}$ is an extension of R along D.

The Kan formula for $Lan_E \bar{R}$ becomes

$$(Lan_E \bar{R})- = Hom_{kH^{op}}(E, -) \otimes_S \bar{R} \approx Hom_{kH^{op}}(kG, -) \otimes_S \bar{R},$$

where $Hom_{kH^{op}}(kG, M)$ has the right S-module structure induced by the natural S-module structure on kG. By hypothesis, the subgroup H of G has finite index so that kG considered as right kH-module is finitely generated and free. Hence, the functor $Hom_{\overline{k}H^{op}}(kG, -)$ is cocontinuous, and therefore also the functor $Lan_E \overline{R}$.

It remains to be shown that Theorem 2.2 is a version of Mackey's induced representation Theorem, namely in the situation where G/H is a discrete measure space.

A *spectral measure* P *on the discrete measure space* G/H is a function P on G/H with values in $End_k V$ where V is a finite dimensional k-vector space (that is, an element of $\mathcal{V} = k\text{-}Mod$), satisfying the following properties:

(2.1) $$P(x)^2 = P(x) \text{ for all } x \text{ in } G/H,$$

(2.2) $$P(x)P(y) = 0 \quad \text{for } x \neq y,$$

(2.3) $$\sum_{x \in G/H} P(x) = Id.$$

If we choose $k = \mathbb{C}$, we can find a scalar-product on V such that all endomorphisms $P(x)$ are projectors, that is, Hermitian idempotent operators.

Now let V be the representation module of a linear representation $R: G \to GL(V)$. Mackey speaks of an *imprimitivity system* P *of* R *based on* G/H if a spectral measure P satisfies the following additional property:

(2.4) $$R(g)P(x) = P(gx)R(g) \text{ for all } g \text{ in } G \text{ and } x \text{ in } G/H.$$

Mackey's induced representation Theorem (see [C], Theorem 10) says that a linear representation $R: G \to GL(V)$ is induced by a linear representation of a subgroup H of G if and only if it possesses an imprimitivity system P based on G/H. It is obtained from Theorem 2.2 by means of the following easily proved lemma.

LEMMA 2.3. *Let* G *be a group and* H *a subgroup. Denote by* kG *and* kH *the corresponding group algebras. Every* kH^{op}*-endomorphism* s *of* kG *can be written in a unique way as sum*

$$s = \sum_{x \in G/H} D(a_x) \circ \pi_x,$$

where the π_x *are projectors given by*

$$\pi_x(g) = \delta_{x, gH}\, g \quad \textit{for all } g \textit{ in } G.$$

Suppose the linear representation R to be coshape-invariant, that is, $R = \bar{R} D$ for some (enriched) functor $\bar{R}: S \to k\text{-}Mod$. Defining P by setting

$$P(x) = \bar{R}(\pi_x) \quad \text{for all} \quad x \in G/H,$$

we obtain an imprimitivity system of R based on G/H.

Conversely, assume that R possesses such an imprimitivity system. Then we define $\bar{R}: S \to k\text{-}Mod$ as follows. Let $s = \sum_{x \in G/H} D(a_x) \circ \pi_x$ be an element of $S = End_{kH^{op}} kG$. Put

$$\bar{R}(s) = \sum_{x \in G/H} R(a_x) \circ P(x),$$

and verify the functor properties (of an enriched functor). Since

$$D(a) = \sum_{x \in G/H} D(a) \circ P(x),$$

we also have $\bar{R} D = R$.

REMARK 2.4. There are other versions of Theorem 2.2, where G and H are non-discrete topological groups, and where, of course, the hypothesis that H is a subgroup of finite index is replaced by another condition assuring the sufficiency of coshape-invariance for a linear representation to be induced. The new hypothesis depends on the choice of the closed category $\mathcal{V}$ (replacing $k\text{-}Mod$) and the form of the single object categories P and T (replacing the group algebras kH and kG).

REFERENCES.

A. C. AUDERSET, Adjonctions et monades au niveau des 2-catégories, *Cahiers Topo. et Géom. Diff.* XV-1 (1974), 3-20.

C. A. J. COLEMAN, Induced and subduced representations, in: «*Group Theory and its Applications*», edited by M. Loebl, Acad. Press, New York, 1968.

D. E. DUBUC, Kan extensions in enriched category theory, *Lecture Notes in Math.* 145, Springer (1970).

F-K. A. FREI & H. KLEISLI, A question in categorical shape theory: «When is a shape-invariant functor a Kan extension?», *Lecture Notes in Math.* 719, Springer (1979).

Institut de Mathématiques
Faculté des Sciences
FRIBOURG. SUISSE.