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THE EXISTENCE AND CONSTRUCTION OF LAX LIMITS

by John W. GRAY *)

0. INTRODUCTION.

Lax limits in a 2-category (such as *Cat*) are objects which satisfy a universal mapping property similar to that satisfied by ordinary limits except that the relevant cones, instead of consisting of commutative triangles, consist of triangles with specified 2-cells (e.g., natural transformations in *Cat*) in them. There is a clear analogy with the notion of homotopy limits in Topology which were invented at about the same time as lax limits. The connections between these two kinds of limits are discussed in [11 and 18].

In this paper we show that if a 2-category is (finitely) complete it has (finite) lax limits. This was first announced in [9], though no notion of finiteness was considered there. (Cf. also, [8], page 289 and [10], page 188). This first proof was generalized by D. Bourn in [3]. Later R. Street in [16] gave a completely different proof which motivated me to rethink my original construction. I found that it led to a simple existence proof together with an algorithm for constructing a lax limit over a 2-category $\underline{I}$ in terms of an ordinary limit over a (functorially) associated category $Prol\,\underline{I}$. A first, partially erroneous version was circulated in 1978. In the meantime, A. and C. Ehresmann had independently developed a much more general theory in their series of four papers, «Multiple functors», *Cahiers de Topo. et Géom. Diff.* XV, 215-290; XIX, 295-333; XIX, 387-443; XX, 59-104. In particular, they prove (in different notations) Theorems 2.2, 2.3, 5.1.1 and 6.6, but they do not consider $Prol\,\underline{I}$ for a 2-category $\underline{I}$, or discuss its naturality. They also point out that 1.3 is essentially in Appelgate-Tierney, «Iterated cotriples», *Lecture Notes in Math.* 137, 56-99.

*) Some of these results were obtained while the author was a Fullbright-Hays fellow at the University of Sydney in 1975. Later work was partially supported by NSF grant n° MCS 77-01974.

Section 1 is concerned with some preliminary notions about presentations and completeness. Section 2 shows that the existence of lax limits reduces to that of ordinary limits (e.g., products and equalizers) plus the existence of the lax limit of the diagram consisting of a single arrow. As part of this reduction, the ingredients are constructed for what turns out to be essentially a lax functor from a diagram 2-category $\underline{I}$ to the bicategory $Spans\,\underline{A}$ of spans in the 2-category of interest, $\underline{A}$. Section 3 describes such lax functors and shows how to convert such a thing into an ordinary functor from $Prol\,\underline{I}$ to $\underline{A}$. Section 4 shows how a 2-functor $I: \underline{I} \to \underline{A}$ (where $\underline{A}$ is finitely complete) determines a 2-functor $PR(I): Prol\,\underline{I} \to \underline{A}$ and proves the main theorem that the lax limit of I is isomorphic to the ordinary limit of $PR(I)$. Section 5 discusses extensions of this theorem and simplifications of $Prol\,\underline{I}$. Section 6 consists of examples showing how particular lax limits are given explicitely as limits.

Concerning notation, in general the terminology of Kelly-Street [12] will be followed in this paper rather than that of [10]. Thus we write lax natural rather than quasi-natural, lax limits rather than Cartesian q-limits, lax functors rather than pseudo-functors, etc. *Sets* denotes the category of (small) sets, *Cat* the 2-category of (small) categories, and *2-Cat* the 2-category of small 2-categories. Size considerations are consistently ignored, but two or three universes should suffice. If $\underline{A}$ is a 2-category, $\underline{A}_0$ denotes the underlying category. A 2-cell a between 1-cells f and g which have common 0-cells A and B as domain and codomain is denoted by

$$a: f \Rightarrow g: A \to B.$$

The values of the composition functor $\underline{A}(A, B) \times \underline{A}(B, C) \to \underline{A}(A, C)$ are denoted by juxtaposition (in reverse order). This takes precedence over composition within the categories $\underline{A}(A, B)$ which is denoted by $a \cdot \beta$. 2-categories which are regarded as index categories are denoted by $\underline{I}$, with cells typically written $\phi: s \Rightarrow s': i \to j$.

Examples of 2-categories which are complete and hence by the results of this paper lax complete, or cocomplete and hence lax cocomplete, are as follows:

i) *Cat* is complete and cocomplete.

ii) If $\underline{V}$ is a closed category which is complete (resp., cocomplete) then the 2-category of $\underline{V}$-categories is complete (resp., cocomplete); e.g. additive categories, or (k-space)-categories.

iii) If $\underline{A}$ is a category with pullbacks, then the 2-category $Cat(\underline{A})$ of category-objects in $\underline{A}$ is well-defined. If $\underline{A}$ is complete, then $Cat(\underline{A})$ is complete and if $\underline{A}$ is cocomplete with universal, disjoint, monomorphic coproducts, then $Cat(\underline{A})$ is cocomplete. Thus $Cat^{\underline{C}^{op}}$, which is isomorphic to the 2-category of fibred categories over $\underline{C}$, is complete and cocomplete. Similarly, given any Grothendieck topology on $\underline{C}$, the 2-category of category-valued sheaves on $\underline{C}$ is complete and cocomplete.

1. PRESENTATIONS.

Let $\underline{1}$, $\underline{2}$, and $\underline{3}$ denote the categories described in [10], pages 3 and 4. In general, $\underline{n}$ denotes the ordered category with n objects. Besides these categories regarded as locally discrete 2-categories, there are 2-categories based on ordered categories whose hom objects are themselves ordered categories. We shall write $\underline{n}\langle\underline{m}\rangle$ for the 2-category with objects $0, \ldots, (n-1)$, whose hom objects are given by

$$\underline{n}\langle\underline{m}\rangle(i,j) = \underline{m}^{j-i} \text{ if } i \leq j \text{ and } \emptyset \text{ otherwise,}$$

and in which composition is the isomorphism $\underline{m}^{j-i} \times \underline{m}^{k-j} \xrightarrow{\sim} \underline{m}^{k-i}$. It is easyly established that the full subcategory (resp., sub-2-category) of *Cat* (resp., *2-Cat*) generated by $\underline{3}$ (resp., $\underline{3}\langle\underline{3}\rangle$) is dense and hence every small category (resp., 2-category) is a coequalizer of a pair of maps between coproducts over suitable index sets of copies of $\underline{3}$ (resp. $\underline{3}\langle\underline{3}\rangle$).

These presentations are canonical and usually are much bigger than is necessary. In general, if $\underline{A}$ is a subcategory of a category $\underline{B}$, let $\underline{A}_1$ denote the closure of $\underline{A}$ under finite colimits in $\underline{B}$, let

$$\underline{A}_n = (\underline{A}_{n-1})_1 \text{ and let } \underline{A}_\infty = \varinjlim \underline{A}_n .$$

1.1. DEFINITION. Let $(\underline{2})$ (resp., $(\underline{2}\langle\underline{2}\rangle)$) denote the full subcategory of *Cat* (resp., *2-Cat*) generated by $\underline{2}$ (resp. $\underline{2}\langle\underline{2}\rangle$). A small category (resp., 2-category) is called *finitely presented* if it belongs to $(\underline{2})_\infty$ (resp. $(\underline{2}<\underline{2}>)_\infty$).

Actually, $(\underline{2})_2 = (\underline{2})_\infty$ in *Cat* and $(\underline{2}<\underline{2}>)_3 = (\underline{2}<\underline{2}>)_\infty$, and these notions coincide with the Gabriel-Ulmer notions of finitely presentable objects in *Cat* and *2-Cat* respectively.

In order to utilize presentations in the construction of lax limits, we need some properties of complete 2-categories. Recall that a 2-category is a category enriched in the cartesian closed category *Cat*, so completeness for a 2-category $\underline{A}$ means that $\underline{A}$ has limits preserved by the *Cat*-valued representable functors and that cotensors exist; i.e., if $A \in \underline{A}$ and $\underline{V} \in Cat$, then there is an object $\underline{V} \pitchfork A \in \underline{A}$ and a *Cat*-natural isomorphism

$$\underline{A}(\underline{V} \pitchfork A, X) \approx Cat(\underline{V}, \underline{A}(A, X)) = \underline{A}(A, X)^{\underline{V}}$$

(cf., Day and Kelly [6]). We call $\underline{A}$ *finitely complete* if it has finite limits and cotensors with finitely presentable categories.

1.2. PROPOSITION. *A 2-category $\underline{A}$ is (finitely) complete iff it has (finite) limits and $\underline{2} \pitchfork$- exists.*

PROOF. Note that $\underline{V} \pitchfork A$ turns colimits in the first variable into limits. Since $\underline{3}$ is the colimit in *Cat* of the diagram

$$\underline{2} \xleftarrow{\ 0\ } \underline{1} \xrightarrow{\ 1\ } \underline{2},$$

this determines $\underline{3} \pitchfork A$ from $\underline{2} \pitchfork A$. Since any $\underline{V}$ has a presentation in terms of $\underline{3}$, this in turn determines $\underline{V} \pitchfork A$. Clearly, if $\underline{V}$ is finitely presentable then only finite limits are needed to construct $\underline{V} \pitchfork A$.

We need one more property of complete 2-categories. Let $\underline{A}$ be a fixed 2-category and let $\underline{A}/adj/2\text{-}Cat$ denote the category whose objects are 2-functors $F: \underline{A} \to \underline{B}$ where $\underline{B}$ is any 2-category and where F has a *Cat*-enriched right adjoint. A morphism from F_1 to F_2 is a 2-functor

$$H: \underline{B}_1 \to \underline{B}_2 \quad \text{such that} \quad H F_1 = F_2.$$

1.3. PROPOSITION. *If $\underline{A}$ is complete, then so is $\underline{A}/adj/2\text{-}Cat$.*

PROOF. Let $\underline{I}$ be a small category with objects and morphisms denoted by

$p_{ji}: i \to j$. Let $I: \underline{I} \to \underline{A}/adj/2\text{-}Cat$ be a functor with values denoted by

$$I(i) = F_i : \underline{A} \to \underline{B}_i \quad \text{and} \quad I(p_{ji}) = H_{ji} , \text{ where } H_{ji} F_i = F_j .$$

Let $\underline{B} = lim\, \underline{B}_i$, the inverse limit being taken in *2-Cat* with respect to the functors H_{ji}, and let $F: \underline{A} \to \underline{B}$ be the functor induced by the F_i's. For each i , choose a right adjoint U_i to F_i and let $\theta_{ji}: U_i \to U_j H_{ji}$ be the (enriched) natural transformation which is transpose to the identity morphism $H_{ji} F_i = F_j$. It follows from [10], Theorem I, 6.8, that the θ_{ji}'s compose properly so the maps

$$\theta_{ji} H_i : U_i H_i \to U_j H_{ji} H_i = U_j H_j$$

(here $H_i : \underline{B} \to \underline{B}_i$ is the limit-cone) determine a diagram in $\underline{A}^{\underline{B}}$ whose limit $U = lim\, U_i H_i$ is right adjoint to F since

$$\begin{aligned} \underline{A}(A, UB) = \underline{A}(A, lim\, U_i H_i(B)) &\approx lim\, \underline{A}(A, U_i H_i(B)) \\ \approx lim\, \underline{B}_i(F_i A, H_i(B)) &\approx lim\, \underline{B}_i(H_i F A, H_i B) = \underline{B}(FA, B). \end{aligned}$$

Hence F is the limit of I in $\underline{A}/adj/2\text{-}Cat$.

2. EXISTENCE OF LAX LIMITS.

If $\underline{A}$ and $\underline{B}$ are 2-categories, let $Fun(\underline{A}, \underline{B})$ denote the 2-category whose objects are 2-functors from $\underline{A}$ to $\underline{B}$, whose 1-cells are lax natural transformations (= quasi-natural transformations in [10]) and whose 2-cells are modifications (see [10], page 28). For any pair of 2-categories, $\underline{I}$ and $\underline{A}$, there is a constant imbedding $\Delta_I : \underline{A} \to Fun(\underline{I}, \underline{A})$ and a *Cat*-enriched right adjoint to Δ_I is called a *lax limit* functor. It is denoted by $llim_{\underline{I}}: Fun(\underline{I}, \underline{A}) \to \underline{A}$. (In [10], *llim* is written *Cart q-lim* . The dual notion is often abbreviated to Q in calculations there.)

The functorial behavior of $llim_{\underline{I}}$ in the variable $\underline{I}$ is discussed in [10], pages 189-197, in the greatest possible generality. Here we can use a simpler result since we can work in the category $(2\text{-}Cat)_0$. Let $[\underline{A}]$ denote the constant 2-functor equal to $\underline{A}$, it and $Fun(-, \underline{A})$ being regarded as contravariant functors from $(2\text{-}Cat)_0$ to *2-Cat*.

2.1. PROPOSITION. *The 2-functors* $llim_{\underline{I}}$ *are the 1-cell components of a* *lax natural transformation from* $Fun(-, \underline{A})$ *to* $[\underline{A}]$.

PROOF. If $F: \underline{I} \to \underline{I}'$ is a 2-functor, then $Fun(F, \underline{A})\Delta_{\underline{I}'} = \Delta_{\underline{I}}$ so there is a transpose natural transformation denoted by

$$llim_F : llim_{\underline{I}'} \to llim_{\underline{I}} \circ Fun(F, \underline{A}).$$

The point of Theorem I, 6.8 in [10], is that if $llim_F$ is defined this way then the equations of lax naturality are satisfied provided 2-cells are not allowed from an F to an F'.

We are now in a position to reduce the calculation of lax limits over a colimit of 2-categories to the lax limits over the factors. Let

$$D: \underline{D} \to (2\text{-}Cat)_0$$

be a functor from a small category $\underline{D}$ whose objects and morphisms are denoted by $\phi_{ji}: i \to j$. Write

$$D(i) = \underline{I}_i \quad \text{and} \quad D(\phi_{ji}) = G_{ji}.$$

Let $I = \varinjlim \underline{I}_i$ be the colimit in $2\text{-}Cat$ with structure maps $G_i : \underline{I}_i \to \underline{I}$.

2.2. THEOREM. *If* $\underline{A}$ *is a complete 2-category, then for any* $F: \underline{I} \to \underline{A}$, *one has* $llim_{\underline{I}} F = lim_{\underline{D}}(llim_{\underline{I}_i}(F G_i))$.

PROOF. Recall from [10], I, 4.4, iii or 4.14, that $Fun(-, \underline{A})$ turns colimits in the first variable into limits. Hence $Fun(\underline{I}, \underline{A}) = lim_{\underline{D}}(Fun(\underline{I}_i, \underline{A}))$. This is the situation of 1.3 where F_i there is $\Delta_{\underline{I}_i}$ here. The induced functor F there is $\Delta_{\underline{I}}$ here and the right adjoint U_i there is $llim_{\underline{I}_i}$ here. The transpose natural transformations are the $llim_{G_{ji}}$ from 2.1 and hence the right adjoint to $\Delta_{\underline{I}}$ is given by the indicated formula. This formula can also be derived from Street [16].

2.3. THEOREM. *Let* $\underline{A}$ *be a 2-category. If* $\underline{A}$ *is (finitely) complete then* $\underline{A}$ *has lax limits of type* $\underline{I}$ *for all (finitely presentable) small 2-categories* $\underline{I}$.

PROOF. Let $\underline{I}$ be a small 2-category. By 2.2, $llim_{\underline{I}}$ can be computed as a limit of lax limits over the constituents of any presentation of $\underline{I}$. Hence by Section 1, it is sufficient to show that $llim_{\underline{3}<\underline{3}>}$ exists together with the required maps for endofunctors of $\underline{3}<\underline{3}>$. Actually, we shall show that $llim_{\underline{I}}$ exists for any object in $\{\underline{3}<\underline{3}>\}$ (the full subcategory of *2-Cat* determined by the $\underline{n}<\underline{m}>$'s with $1 \leq n,\ m \leq 3$). The induced maps between lax limits corresponding to functors in this subcategory exist, because $Fun(-, \underline{A})$ is a functor in the first variable, but some of these will be described explicitely because they are needed later.

Step 1. Locally discrete 2-categories. If $\underline{I} = \underline{1}$ then the lax limit of a functor from $\underline{1}$ to $\underline{A}$ is just the value of the functor. The case $\underline{I} = \underline{2}$ is the most important and determines everything else. Write $\underline{2}$ as $0 \xrightarrow{t} 1$. The two functors $\hat{0}, \hat{1} : \underline{1} \to \underline{2}$ and the natural transformation with component t from the first to the second determine, for an object $B \in \underline{A}$, a diagram

$$\begin{tikzcd} \underline{2}\pitchfork B \arrow[rr, bend left, "(d_0)_B"] \arrow[rr, bend right, "(d_1)_B"'] \arrow[rr, phantom, "\mu_B \Downarrow"] & & B \end{tikzcd} \qquad (1)$$

(where, e.g., $(d_0)_B = \hat{0}\pitchfork B$, etc) which is universal for such diagrams in the sense that given any 2-cell $a : f \Rightarrow g : A \to B$, then there is a unique 1-cell $a^* : A \to \underline{2}\pitchfork B$ such that $\mu_B a^* = a$. Let

$$\underline{I} = \hat{f} : \underline{2} \to \underline{A}, \quad \text{i.e.,} \quad I(t) = f : A \to B \text{ in } \underline{A}.$$

Then the lax limit of $\underline{I}$ is easily seen to be the pullback

$$\begin{tikzcd} llim_{\underline{2}}\hat{f} \arrow[r, "q_f"] \arrow[d, "0_f"'] & \underline{2}\pitchfork B \arrow[r, "(d_1)_B"] \arrow[d, "(d_0)_B"] & B \\ A \arrow[r, "f"] & B & \end{tikzcd} \qquad (2)$$

and the universal lax cone is the diagram

$$\begin{tikzcd} & llim_2\hat{f} \arrow[dl, "0_f"'] \arrow[dr, "1_f"] \arrow[d, phantom, "\theta_f \Uparrow"] & \\ A \arrow[rr, "f"'] & {} & B \end{tikzcd} \qquad (3)$$

where $1_f = (d_1)_B\, q_f$ and $\theta_f = \mu_B\, q_f$. Note that there is the map

$$\lambda_f : A \to llim_{\underline{2}} \hat{f} \tag{4}$$

characterized as the unique map such that $\theta_f \lambda_f = id_f$. In particular, the unique functor $\tau : \underline{2} \to \underline{1}$ determines the natural transformation with components $i_A = \lambda_{id_A} : A \to llim_{\underline{2}}(\widehat{id_A})$. In Cat, where $\underline{2} \pitchfork \underline{B} = \underline{B}^2$, the pullback (2) describes the comma category $f/\underline{B}$, so we shall adopt this as a general notation for $llim_{\underline{2}} \hat{f}$ although it suggests an asymmetry that is not present and could be avoided by the clumsier notation $A/f/B$. We shall frequently regard (3) as a span in A (cf., [10], page 46)

$$S_f : A \xleftarrow{\;0_f\;} f/B \xrightarrow{\;1_f\;} B \tag{5}$$

from A to B. In this notation, the map i_A above can be regarded as a map of spans

$$\begin{tikzcd} 1_A \arrow[dd, "c_A"'] & & & A \arrow[dll, "id_A"'] \arrow[dd, "i_A"] \arrow[drr, "id_A"] & & \\ & : & A & & & A \\ S_{id_A} & & & id_A/A \arrow[ul, "(d_0)_A"] \arrow[urr, "(d_1)_A"'] & & \end{tikzcd} \tag{6}$$

Now consider the case $\underline{I} = \underline{3}$. Since $\underline{3}$ is the pushout of two copies of $\underline{2}$, functors from $\underline{I}$ to $\underline{A}$ are of the form

$$I = \langle \hat{f}, \hat{g} \rangle \quad \text{where } f : A \to B \text{ and } g : B \to C \text{ in } \underline{A}.$$

By 2.2, $llim_{\underline{3}}\langle \hat{f}, \hat{g}\rangle$ is the indicated pullback

$$\begin{tikzcd} & & llim_{\underline{3}}\langle \hat{f}, \hat{g}\rangle \arrow[dl, "llim_\alpha"'] \arrow[dr, "llim_\beta"] & & \\ & f/B \arrow[dl, "0_f"'] \arrow[dr, "1_f"] \arrow[d, phantom, "\theta_f \Nearrow"] & & g/C \arrow[dl, "0_g"'] \arrow[dr, "1_g"] \arrow[d, phantom, "\theta_g \Nearrow"] & \\ A \arrow[rr, dashed, "f"'] & {} & B \arrow[rr, dashed, "g"'] & {} & C \end{tikzcd} \tag{7}$$

which is drawn as the composed span $S_g S_f$ in $\underline{A}$. Not indicated in the diagram is the important 1-cell

$$c_{f,g} : llim_{\underline{3}} < \hat{f}, \hat{g} > \;\rightarrow\; gf/C \tag{8}$$

which corresponds to the «composition» map $\gamma : \underline{2} \rightarrow \underline{3}$. It is the unique 1-cell satisfying the equation

$$\theta_g(llim_{\beta}) \cdot g\, \theta_f(llim_{\alpha}) = \theta_{gf} c_{f,g} \tag{9}$$

and is to be regarded as a map of spans from $S_g S_f$ to S_{gf}. We note, for future reference that if $\gamma : A \underset{A}{\times} f/B \rightarrow (f\, id_A)/B$ and

$$\lambda : llim_{\underline{3}} < \hat{f}, \hat{g} > \underset{C}{\times} h/D \;\rightarrow\; f/B \underset{B}{\times} llim_{\underline{3}} < \hat{g}, \hat{h} >$$

are the canonical isomorphisms, then

a) $c_{id_A, f}(i_A \underset{A}{\times} id_{f/B}) = \gamma$ (with a similar equation for c_{f, id_A}),

b) $c_{gf,h}(c_{f,g} \underset{C}{\times} h/D) = c_{f,hg}(f/B \underset{B}{\times} c_{g,h})\, \lambda$.

Once lax limits over $\underline{3}$ are known to exist, it follows by the density of $\underline{3}$ in *Cat* that lax limits exist over all small categories (resp., finitely generated categories).

Step 2. Arbitrary 2-categories. It is easily verified directly from the definitions that if a 2-category $\underline{I}$ has the property that each $\underline{I}(i,j)$ has a terminal object and if these terminal objects form a subcategory $\underline{I}'$, then lax limits over $\underline{I}$ are isomorphic to lax limits over the category $\underline{I}'$. Clearly, the 2-categories $\underline{n} < \underline{m} >$ satisfy this, the subcategory being isomorphic to $\underline{n}$. Hence lax limits over $\underline{3} < \underline{3} >$ exist, which completes the proof of the theorem. We note for future reference that if $\alpha : f \Rightarrow f' : A \rightarrow B$ is a 2-cell in $\underline{A}$, then there is a uniquely determined 1-cell

$$\alpha/B : f'/B \rightarrow f/B \quad \text{such that} \quad \theta_f(\alpha/B) = \theta_{f'} \cdot \alpha\, 0_{f'}$$

and this determines a map of spans

$$\begin{tikzcd} S_{f'} \arrow[d, "S_\alpha"'] \\ S_f \end{tikzcd} : \qquad \begin{tikzcd} & f'/B \arrow[dl, "0_{f'}"'] \arrow[dr, "1_{f'}"] \arrow[dd, "\alpha/B"] & \\ A & & B \\ & f/B \arrow[ul, "0_f"] \arrow[ur, "1_f"'] & \end{tikzcd} \tag{10}$$

which satisfies obvious compatibility conditions with the maps c_A and $c_{f,g}$ in (6) and (8).

2.4. DISCUSSION. In this construction of lax limits via presentations, the idea is to replace each map f in the original diagram by the corresponding span S_f. For each composition gf, there is a composed span with a map $c_{f,g}: S_g S_f \to S_{gf}$, for each object there is a map $c_A : A \to S_{id_A}$ and for each 2-cell $\alpha : f \Rightarrow f'$, there is a map $S_\alpha : S_{f'} \to S_f$. Regarding all of these spans and maps between them as a bigger diagram, then the lax limit of the original diagram is just the ordinary limit of this bigger diagram. The problem is to describe this bigger diagram in some systematic fashion.

3. THE PROLONGATION OF A 2-CATEGORY.

Recall that a *lax functor* as described in Street [15] (or a pseudo-functor as in [10], page 40) between 2-categories $\underline{A}$ and $\underline{B}$ assigns to each object $A \in \underline{A}$ an object $F(A) \in \underline{B}$, to each pair of objects, $A, B \in \underline{A}$, a functor $F: \underline{A}(A,B) \to \underline{B}(FA, FB)$, to each composable pair of 1-cells (f,g), a 2-cell $c_{f,g}: F(g)F(f) \to F(gf)$ and to each object $A \in \underline{A}$, a 2-cell $c_A : id_{FA} \to F(id_A)$ such that the usual coherence conditions are satisfied. If $\underline{A}$ and $\underline{B}$ are only bicategories then the equations are more complicated (see [10]). An *op-lax* functor has the 2-cells $c_{f,g}$ and c_A going the opposite direction. If, for any 2-category $\underline{A}$, we write ${}^{op}\underline{A}$ for the weak dual in which only the direction of 2-cells is reversed, i.e.,

$$({}^{op}\underline{A})(A,B) = \underline{A}(A,B)^{op},$$

then clearly $F: \underline{A} \to \underline{B}$ is a lax functor iff ${}^{op}F: {}^{op}\underline{A} \to {}^{op}\underline{B}$ is an op-lax functor, where ${}^{op}F$ has the same values on n-cells, $n = 0, 1, 2$, as F. If the 2-cells $c_{f,g}$ and c_A are isomorphisms for all A, f, g, then F is called a *pseudo*-functor. By taking inverses, pseudo and oppseudo can always be converted into the opposite case. By a *lax natural transformation* between lax functors we mean a left lax transformation as in Street op. cit., or a quasi-natural transformation as in [10], page 43.

For any category $\underline{A}$ with pullbacks, let *Spans*$(\underline{A})$ denote the bicategory of spans in $\underline{A}$ (cf. [10], page 46, [12] or [2]). Lindner [13] shows that the classifying category of $Spans(\underline{A})$ (see [2]) represents Mackey-functors, which identifies functors with this as domain. We are concerned here with op-lax functors with codomain ${}^{op}Spans(\underline{A})$. In a certain sense, prolongation will be a «left adjoint» to ${}^{op}Spans(-)$.

3.1. DEFINITION. Let $\sigma : F \Rightarrow G : \underline{A} \to {}^{op}Spans(\underline{B})$ be a lax natural transformation between lax functors. σ is called *special* if for every $A \in \underline{A}$, σ_A is given by a span of the form

$$FA \xleftarrow{1_{FA}} FA \xrightarrow{\sigma'_A} GA \ .$$

NOTE. The description of lax naturality in this case reduces to giving maps σ_f so that the diagrams

$$\begin{tikzcd} FA \arrow[d, "\sigma'_A"'] & F(f) \arrow[l] \arrow[r] \arrow[d, "\sigma_f"] & FB \arrow[d, "\sigma'_B"] \\ GA & G(f) \arrow[l] \arrow[r] & GB \end{tikzcd} \qquad (1)$$

commute for every $f : A \to B$ in $\underline{A}$. The composition of these reduces to composing the σ'_A's and the σ_f's.

3.2. DEFINITION. i) *Lax* (resp., *op-Lax*) denotes the category whose objects are (small) bicategories and whose morphisms are lax (resp., op-lax) functors. (For composition, see [10], page 42.) If $\underline{B}$ and $\underline{B}'$ are bicategories, then $op\text{-}Lax(\underline{B}, \underline{B}')$ denotes the bicategory whose objects are op-lax functors from $\underline{B}$ to $\underline{B}'$, whose morphisms are lax natural transformations and whose 2-cells are modifications. (Cf. $Pseud(\underline{B}, \underline{B}')$ in [10] page 45.)

ii) *Lex* denotes the 2-category of (small) categories, left exact functors (i.e., those finite limits that exist in the domain category are preserved) and natural transformations.

3.3. THEOREM. *There is a functor $Prol : 2\text{-}Cat_0 \to Lex$ and for any 2-category $\underline{A}$ with pullbacks, a functor*

$$R_{\underline{I}} : op\text{-}Lax(\underline{I}, {}^{op}Spans(\underline{A}))_s \to Lex(Prol\,\underline{I}, \underline{A})$$

which is natural in $\underline{I}$. (The s means only special lax natural transformations.)

PROOF. *Step 1.* If $\underline{I}$ is a category, then $Prol\underline{I}$ is the total category of the discrete 0-fibration corresponding to the functor

$$|Cat(-,\underline{I})|\,|\{\underline{3}\}^{op}:\{\underline{3}\}^{op}\to Sets.$$

Here $\{\underline{3}\}$ is the full subcategory of Cat_0 determined by $\underline{1}$, $\underline{2}$ and $\underline{3}$, and the indicated functor is regarded as covariant (cf. [10], page 210 f where this is denoted by $[\mathit{1}, F_{\underline{I}}]$). The fibres of $Prol\underline{I}$ over $\underline{1}$, $\underline{2}$ and $\underline{3}$ respectively are the sets of the (names of the) objects, morphisms and commutative triangles of $\underline{I}$. Since these fibres are discrete, the only limits in $Prol\underline{I}$ are those which come from colimits in $\{\underline{3}\}$, the only non-trivial one being $\underline{3}$ as a pushout of two copies of $\underline{2}$. If $s: i\to j$ and $t: j\to k$ in $\underline{I}$, then the pullback of

$$\bar{s}\longrightarrow\bar{j}\longleftarrow\bar{t}$$

in $Prol\underline{I}$ is denoted by $<\bar{s},\bar{t}>$. (The bars denote the objects in $Prol\underline{I}$ corresponding to data in $\underline{I}$.) If $F:\underline{I}\to\underline{I}'$ is a functor, it determines a natural transformation

$$Cat(-,F): Cat(-,\underline{I})\to Cat(-,\underline{I}')$$

and hence a (cartesian) functor $Prol\,F: Prol\underline{I}\to Prol\underline{I}'$ between the associated 0-fibrations which is clearly left exact. This gives a functor

$$Prol: Cat_0\to Lex.$$

If $\underline{I}$ is a 2-category then corresponding to the non-identity 2-cells in $\underline{I}$, we adjoin maps in $Prol\underline{I}_0$ subject to equations, as follows: let

$$\phi: s\Rightarrow s': i\to j,\quad \phi': s'\Rightarrow s'': i\to j,$$
$$\psi: t\Rightarrow t': j\to k,\quad \psi': t'\Rightarrow t'': j\to k$$

be 2-cells in $\underline{I}$. The following maps are added to $Prol\underline{I}_0$:

$$\bar{\phi}: \bar{s}'\to\bar{s}\quad\text{and}\quad <\bar{\phi},\bar{\psi}>: <\bar{s}',\bar{t}'>\to<\bar{s},\bar{t}>$$

satisfying the relations

$$\bar{\phi}\bar{\phi}'=\overline{\phi'.\phi}\quad\text{and}\quad <\bar{\phi},\bar{\psi}><\bar{\phi}',\bar{\psi}'>=<\overline{\phi'.\phi},\ \overline{\psi'.\psi}>$$

and the commutativity of the diagrams

$$\begin{tikzcd}[column sep=small]
 & \bar{s}' \arrow[dl] \arrow[dr] \arrow[dd, "\bar{\phi}"] & \\
\bar{i} & & \bar{j} \\
 & \bar{s} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
\bar{s}' \arrow[d, "\bar{\phi}"'] & <\bar{s}', \bar{t}'> \arrow[l] \arrow[r] \arrow[d, "<\bar{\phi}, \bar{\psi}>"] & \bar{t}' \arrow[d, "\bar{\psi}"] \\
\bar{s} & <\bar{s}, \bar{t}> \arrow[l] \arrow[r] & \bar{t}
\end{tikzcd}$$

$$\begin{tikzcd}
<\bar{s}', \bar{t}'> \arrow[r] \arrow[d, "<\bar{\phi}, \bar{\psi}>"'] & \overline{t's'} \arrow[d, "\overline{\psi\phi}"] \\
<\bar{s}, \bar{t}> \arrow[r] & \overline{ts}
\end{tikzcd}$$

Step 2. Let $K: \underline{I} \to {}^{op}Spans\,\underline{A}$ be an op-lax functor, where $\underline{I}$ and $\underline{A}$ are 2-categories. Let $R_I(K) = K^*: Prol\,\underline{I} \to \underline{A}$ be the left exact functor given by $K^*(\bar{i}) = K(i)$ and if $K(i) \longleftarrow K(s) \longrightarrow K(j)$ denotes the span which K assigns to $s: i \to j$ in $\underline{I}$, then K^* takes the diagram on the left below to the indicated maps in A :

$$\begin{tikzcd}
\overline{ts} \arrow[drrr] \arrow[dddr] & & & \\
 & <\bar{s}, \bar{t}> \arrow[ul] \arrow[r] \arrow[d] & \bar{t} \arrow[r] \arrow[d] & \bar{k} \\
 & \bar{s} \arrow[r] \arrow[d] & \bar{j} & \\
 & \bar{i} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
K(ts) \arrow[drrr] \arrow[dddr] & & & \\
 & P \arrow[ul, "c_{s,t}"'] \arrow[r] \arrow[d] & K(t) \arrow[r] \arrow[d] & K(k) \\
 & K(s) \arrow[r] \arrow[d] & K(j) & \\
 & K(i) & &
\end{tikzcd}$$

where P is the pullback. If ϕ and ψ are 2-cells as above then

$$K^*(\bar{\phi}) = K(\phi): K(s') \to K(s)$$

while $K^*(<\bar{\phi}, \bar{\psi}>)$ is the induced map between pullbacks. By construction K^* is a left exact functor. Clearly, a special lax natural $K_1 \to K_2$ determines a natural transformation $K_1^* \to K_2^*$ and there are no non-identity modifications between special lax natural transformations.

Step 3. $Lex(Prol\text{-}, \underline{A})$ is clearly a contravariant functor from $2\text{-}Cat_0$ to

2-Cat. Furthermore, from the description of the composition of special lax natural transformations, it follows that $op\text{-}Lax(\underline{I}, {}^{op}Spans\,\underline{A})_s$ is a locally discrete 2-category and $R_{\underline{I}}$ is a functor. It is easily checked that if $M: \underline{I}' \to \underline{I}$ is a 2-functor then $(KM)^* = K^* \circ Prol\,M$ so that $R_{(-)}$ is a natural transformation.

REMARK. If $Prol\,\underline{I}$ is defined over $\{\underline{4}\}^{op}$ instead of $\{\underline{3}\}^{op}$ then the definition of $R_{\underline{I}}$ extends to an equivalence of categories and then the identity functor on $Prol\,\underline{I}$ corresponds to an op-lax functor $\eta_{\underline{I}}$ satisfying:

$$\begin{tikzcd}
\underline{I} \arrow[r, "\eta_{\underline{I}}"] \arrow[dr, "K"'] & {}^{op}Spans(Prol\,\underline{I}) \arrow[d, "{}^{op}Spans(K^*)"] \\
 & {}^{op}Spans(\underline{A})
\end{tikzcd}$$

given an op-lax functor K, then there is a left exact functor K^* making the diagram commute up to a special lax natural equivalence and K^* is unique up to a unique natural equivalence. This property will not be used here.

One can clearly recapture the 2-category $\underline{I}$ from the fibred category $Prol\,\underline{I}$. An aspect of this will be needed in the proof of the main theorem. Recall that $L\pi_0\,\underline{I}$ denotes the category constructed from the 2-category $\underline{I}$ by replacing each category $\underline{I}(i,j)$ by its set $\pi_0\,\underline{I}(i,j)$ of path components (cf. [10], page 10).

3.4. PROPOSITION. *There is a natural transformation* $P_{\underline{I}}: Prol\,\underline{I} \to L\pi_0\,\underline{I}$.

PROOF. Let $P_{\underline{I}}(\bar{i}) = i$ and if $s: i \to j$ in $\underline{I}$, then

$$P_{\underline{I}}(\bar{i} \longleftarrow \bar{s} \longrightarrow \bar{j}) = (i \xleftarrow{id_i} i \xrightarrow{[s]} j)$$

where $[s]$ is the equivalence class of s in $L\pi_0\underline{I}$. If $\phi: s \Rightarrow s'$ is a 2-cell, then $P_{\underline{I}}(\bar{\phi}) = id_i$ and if $t: j \to k$ then

$$P_{\underline{I}}(\bar{s} \longleftarrow \langle \bar{s}, \bar{t} \rangle \longrightarrow \bar{t}) = (i \xleftarrow{id_i} i \xrightarrow{[s]} j),$$

$$P_{\underline{I}}(\langle \bar{s}, \bar{t} \rangle \longrightarrow \overline{ts}) = i \xrightarrow{id_i} i.$$

See also [10], page 210. By construction, $P_{\underline{I}}$ is natural in $\underline{I}$.

4. THE PROLONGATION OF A FUNCTOR AND LAX LIMITS.

In all of this section $\underline{A}$ is a finitely complete 2-category.

4.1. PROPOSITION. *There is an op-lax functor $\Phi_{\underline{A}} : \underline{A} \to {}^{op}Spans(\underline{A})$ which is « op-lax natural » in $\underline{A}$.*

PROOF. On objects, $\Phi_{\underline{A}}(A) = A$. If $f: A \to B$ is a 1-cell in $\underline{A}$, then

$$\Phi_{\underline{A}}(f) = S_f : A \longleftarrow f/B \longrightarrow B$$

(cf., 2.3 (5)). If $\alpha : f \Rightarrow f'$ is a 2-cell, then

$$\Phi_{\underline{A}}(\alpha) = S_\alpha : S_{f'} \Rightarrow S_f$$

as in 2.3 (10). The structural maps c_A and $c_{f,g}$ are described in 2.3 (6) and (8). Since $\Phi_{\underline{A}}(\alpha)$ is contravariant, we must view this operation as taking values in ${}^{op}Spans(\underline{A})$, where c_A and $c_{f,g}$ therefore go the other way. As noted in 2.3 this data satisfies the definition of an op-lax functor.

We regard ${}^{op}Spans(-)$ as a functor from the category of finitely complete 2-categories to the category of bicategories and op-lax functors. This does not underly any 2-category or bicategory so, strictly speaking, $\Phi_{\underline{A}}$ cannot be op-lax natural. Nevertheless, if $F: \underline{A} \to \underline{B}$ is a 2-functor, then there is an op-lax natural transformation

$$\Phi_F : \Phi_{\underline{B}} \circ F \Rightarrow {}^{op}Spans(F) \circ \Phi_{\underline{A}}$$

whose components are $(\Phi_F)_A = id_{FA}$ and, if $f: A \to B$ is a 1-cell in $\underline{A}$, then $(\Phi_F)_f : F(S_f) \to S_{Ff}$ is the unique 1-cell such that

$$\theta_{F(f)}(\Phi_F)_f = F(\theta_f).$$

Note that $(\Phi_F)_f$ goes the other way in ${}^{op}Spans(\underline{B})$. Uniqueness implies that Φ_F is op-lax natural and, if GF is defined, that $\Phi_{GF} = \Phi_G \;\square\; \Phi_F$.

4.2. PROPOSITION. *Composition with $\Phi_{\underline{A}}$ determines a natural transformation*

$$\Phi_{\underline{A}} \circ (-) : Fun(-, \underline{A}) \to op\text{-}Lax(-, {}^{op}Spans(\underline{A}))_s$$

between contravariant functors from $2\text{-}Cat_0$ to Cat_0.

PROOF. A 2-functor $I: \underline{I} \to \underline{A}$ is taken to the op-lax functor

$$\Phi_{\underline{A}} \circ I : \underline{I} \to {}^{op}Spans(\underline{A})$$

and a lax natural transformation $\sigma : I \to I'$ goes to the special op-lax natural transformation $\Phi_{\underline{A}} \circ \sigma$ whose components are

$$(\Phi_{\underline{A}} \circ \sigma)_i = \sigma_i \quad \text{and} \quad (\Phi_{\underline{A}} \circ \sigma)_s = \bar{\sigma}_s$$

where $\bar{\sigma}_s : I(s)/I(j) \to I'(s)/I'(j)$ is the unique 1-cell such that

$$\theta_{I'(s)}\bar{\sigma}_s = \sigma_j \theta_{I(s)} \cdot \sigma_s 0_{I(s)} .$$

It is easily checked that $\Phi_{\underline{A}} \circ (-)$ is natural in $\underline{I}$.

4.3. DEFINITION. Let $PR_{\underline{I}}(-)$ denote the composed functor

$$Fun(\underline{I}, \underline{A}) \xrightarrow{\Phi_{\underline{A}} \circ (-)} op\text{-}Lax(\underline{I}, {}^{op}Spans(\underline{A}))_s \xrightarrow{R_{\underline{I}}} Lex(Prol\,\underline{I}, \underline{A})$$

from 4.2 and 3.3. Thus,

$$\text{for } I : \underline{I} \to \underline{A}, \quad PR(I) = (\Phi_{\underline{A}} \circ I)^* ;$$

so, e. g., $PR(I)(\bar{i}) = I(i)$ and if $s : i \to j$, then $PR(i)(\bar{s}) = I(s)/I(j)$, etc.

4.4. THEOREM. *There is an isomorphism*

$$llim_{\underline{I}} I \approx lim_{Prol\,\underline{I}} PR(I).$$

PROOF. In order to carry out the proof we must make $PR(I)$ functorial in I in a way that carries some information about limits. We first describe the appropriate categories.

i) Let $(Cat \leftrightarrows \underline{A}_0)$ denote the category whose objects are functors $I : \underline{I} \to \underline{A}_0$, where $\underline{I}$ is a small category, and whose morphisms are pairs $(M, m) : I \to I'$, where $M : \underline{I}' \to \underline{I}$ is a functor and $m : IM \to I'$ is a natural transformation; i. e.,

$$(Cat \leftrightarrows \underline{A}_0) = [Cat, \underline{A}_0^{op}]_0^{op}$$

in the notation of [10]. The name functor $N : \underline{A}_0 \to (Cat \leftrightarrows \underline{A}_0)$ takes an object $A \in \underline{A}_0$ to its name $\hat{A} : \underline{1} \to \underline{A}_0$ and a morphism $f : A \to B$ to the map $(id, f) : \hat{A} \to \hat{B}$. The category $\underline{A}_0$ is complete iff N has a right adjoint, lim. $(Cat \leftrightarrows \underline{A}_0)$ is the underlying category of a 2-category $(Cat \leftrightarrows \underline{A})$, in which a 2-cell is a pair

$$(n,\lambda): (M,m) \Rightarrow (M',m'): I \to I',$$

where $n: M \to M'$ is natural and $\lambda: m \to m'.In$ is a modification. Limits in $\underline{A}$ are Cat-enriched iff lim is a Cat-enriched right adjoint to N, where N is extended to 2-cells by the rule $N(\sigma) = (1,\sigma)$.

ii) Similarly, let $(2\text{-}Cat \leftrightarrows_l \underline{A})_0$ denote the category whose objects are 2-functors $I: \underline{I} \to \underline{A}$ where $\underline{I}$ is a small 2-category and whose morphisms are pairs $(M,m): I \to I'$ where $M: \underline{I}' \to \underline{I}$ is a 2-functor and $m: IM \to I'$ is a lax natural transformation. There is a 2-category $(2\text{-}Cat \leftrightarrows \underline{A})$ in which a 2-cell is a pair (n,λ) exactly as above except that n is a Cat-enriched natural transformation. As above, there is a name 2-functor

$$N': \underline{A} \to (2\text{-}Cat \leftrightarrows_l \underline{A})$$

and $\underline{A}$ has all (small) lax limits iff N' has a right adjoint $llim$. In [10], page 189, a larger enrichment is considered in which n is only required to be lax natural. It is proved there (in dual form) that $llim$ (there called $Cart\ q\text{-}lim$) is the enriched right adjoint to N'.

Let

$$J: (Cat \leftrightarrows \underline{A}) \to (2\text{-}Cat \leftrightarrows_l \underline{A})$$

be the inclusion and extend the definition of PR to give a functor

$$PR: (2\text{-}Cat \leftrightarrows_l \underline{A})_0 \to (Cat \leftrightarrows \underline{A}_0)$$

as follows: on objects $I: \underline{I} \to \underline{A}$, we take $PR(I): Prol\,\underline{I} \to \underline{A}_0$ as before. If $(M,m): \underline{I} \to \underline{I}'$ is a morphism, then by the naturality of $\Phi_{\underline{A}} \circ (-)$ and $R_{\underline{I}}$, $PR(IM) = PR(I) \circ Prol\,M$ so we may take

$$PR(M,m) = (Prol\,M,\ PR(m)).$$

Since $Prol$ and PR are functors, the extension of PR is also a functor. We have now that $N \dashv lim$ and $N' \dashv llim$. Since $JN = N'$, if we also had $J \dashv PR$, then the result would follow. Unfortunately PR is not right adjoint to J, but it is close enough as the following lemma indicates.

4.5. LEMMA. *There is a natural transformation*

$$\Lambda_{I,I'}: (2\text{-}Cat \leftrightarrows_l \underline{A})(J(I), I') \to (Cat \leftrightarrows \underline{A})(I, PR(I'))$$

between Cat-valued functors, which is an isomorphism when $\underline{I}$ *lies in the* *image of* N.

Using the lemma, one completes the proof of 4.4 by the usual sequence of isomorphisms.

$$\underline{A}(X,\ llim_{\underline{I}} I) \approx (2\text{-}Cat \leftrightarrows_l \underline{A})(N'X, I) \approx (2\text{-}Cat \leftrightarrows_l \underline{A})(JNX, I)$$
$$\approx (Cat \leftrightarrows \underline{A})(NX, PR(I)) \approx \underline{A}(X,\ lim_{Prol\,\underline{I}} PR(I)),$$

where the third isomorphism comes from the lemma.

PROOF OF THE LEMMA. Using the functors $P_{\underline{I}}$ from 3.4, we construct a candidate for an adjunction morphism $(P, \lambda): Id \to PR \circ J$. If $\underline{I}$ is a category, and $I: \underline{I} \to \underline{A}_0$, then $(P_{\underline{I}}, \lambda_I): I \to PR(J(I))$ is the morphism in $(Cat \leftrightarrows \underline{A})$ in which $\lambda_I: I \circ P_{\underline{I}} \to PR(J(I))$ is the natural transformation with components $(\lambda_I)_{\bar{i}} = id_{I(i)}$ and if $s: i \to j$ in $\underline{I}$, then $(\lambda_I)_{\bar{s}} = \lambda_{I(s)}$. (Cf. 2.3 (4).) Since $PR(J(I))$ is left exact, there is a uniquely determined value for $(\lambda_I)_{<\bar{s},\bar{t}>}$ making λ_I natural. It follows easily from the uniqueness in the definition of $\lambda_{I(s)}$ and $\Phi \circ \sigma$ that λ is natural in I. The natural transformation Λ is then given by composition with $(P_{\underline{I}}, \lambda_I)$; i.e.,

$$\Lambda_{I,I'}(M, m) = (P_{\underline{I}} \circ Prol\, M,\ PR(m) \,.\, (\lambda_I Prol\, M))$$

and $\Lambda_{I,I'}(n, \delta) = (\tilde{n}, \tilde{\delta})$ where

$$\tilde{n}: P_{\underline{I}} \circ Prol\, M \ \to\ P_{\underline{I}} \circ Prol\, M'$$

is equal to $n P_{\underline{I}'}: M P_{\underline{I}'} \to M' P_{\underline{I}'}$, using the naturality of $P_{\underline{I}}$, and

$$\tilde{\delta}: PR(m). \lambda_I Prol\, M \ \to\ (PR(m'). \lambda_I Prol\, M'). I\tilde{n}$$

has components given by

$$\tilde{\delta}_{\bar{i}} = \tilde{\delta}_{\bar{s}} = \tilde{\delta}_{<\bar{s},\bar{t}>} = \delta_i \quad \text{for} \quad \bar{s}: \bar{i} \to \bar{j},\ \bar{t}: \bar{j} \to \bar{k} \ \text{in}\ Prol\,\underline{I}'.$$

The commutativity conditions satisfied by δ imply those for $\tilde{\delta}$. Clearly, $\Lambda_{I,I'}$ is a functor.

In general there is no inverse to Λ, but if I is in the image of N, i.e., $I = \hat{X}: \underline{1} \to \underline{A}$, then Λ is a bijection since any map $(R, r): \hat{X} \to PR(I')$ is uniquely of the form $(R, r) = \Lambda(M, m)$ where $M: \underline{I}' \to \underline{1}$ is the unique such functor and $m: \hat{X}M \to I'$ is the lax natural transformation (i.e., lax cone from $\hat{X}$ to I') with components $m_i = r_{\bar{i}}$ and $m_s = \theta_{I'(s)} r_{\bar{s}}$.

4.6. COROLLARY. *If U is a 2-functor which preserves limits and cotensors with $\underline{2}$, then U preserves lax limits.*

5. EXTENSIONS AND SIMPLIFICATIONS.

5.1. *Less lax limits.* There are numerous examples of «lax limits» which are universal for lax cones in which some (or all) of the 2-cells in the cone are required to be either isomorphisms or identities. If all are isomorphisms then the corresponding «lax limit» is called a pseudo-limit and if all are identities then we get an extension of the idea of an ordinary limit to what we shall call an *ordinary limit of a 2-functor.*

Let *Triplet* denote the 2-category whose objects are triplets $(\underline{I};\underline{I}_1,\underline{I}_2)$ consisting of a 2-category $\underline{I}$ and two 2-subcategories $\underline{I}_1$ and $\underline{I}_2$ of $\underline{I}$, and whose morphisms are 2-functors $F:\underline{I}\to\underline{I}'$ such that $F(\underline{I}_i)\subset\underline{I}'_i$. The 2-cells are lax natural transformations $m: F\Rightarrow G$ such that if $s: i\to j$ is a 1-cell in $\underline{I}_i$ then m_s is a 2-cell in $\underline{I}_i$, for $i = 1, 2$. There is an imbedding $Tr: 2\text{-}Cat\to Triplet$ taking a 2-category $\underline{A}$ to the triplet $(\underline{A}, iso\,\underline{A}, id\,\underline{A})$, where $iso\,\underline{A}$ (resp., $id\,\underline{A}$) is the 2-subcategory of all isomorphic (resp., identity) 2-cells in $\underline{A}$. A lax natural transformation

$$m: F\Rightarrow G: (\underline{I};\underline{I}_1,\underline{I}_2)\to Tr\,\underline{A},$$

where $\underline{I}_1$ and $\underline{I}_2$ have no non-identity 2-cells, is an ordinary lax natural transformation whose restriction to $\underline{I}_1$ is pseudo-natural and whose restriction to $\underline{I}_2$ is natural.

As in the absolute case, let $(Triplet\leftrightarrows_l Tr\,\underline{A})$ denote the 2-category whose objects are triplet functors $I:(\underline{I};\underline{I}_1,\underline{I}_2)\to Tr\,\underline{A}$ and whose morphisms are pairs $(M,m): I\to I'$ where $M:(\underline{I}';\underline{I}'_1,\underline{I}'_2)\to(\underline{I};\underline{I}_1,\underline{I}_2)$ is a triplet functor and $m: IM\to I'$ is a triplet lax natural transformation. 2-cells are pairs $(n,\lambda):(M,m)\Rightarrow(M',m')$ where $n: M\Rightarrow M'$ is a *Cat*-enriched triplet natural transformation and $\lambda: m\to m'.\,In$ is a modification. As before there is a name 2-functor $N': \underline{A}\to(Triplet\leftrightarrows_l Tr\,\underline{A})$ taking $A\in\underline{A}$ to $\hat{A}:(\underline{1};\underline{1},\underline{1})\to Tr\,\underline{A}$. The right adjoint to N', if it exists, is still denoted by *llim*, its value on I being written $llim_{(\underline{I};\underline{I}_1,\underline{I}_2)}I$ and called a *relative* lax limit of I.

5.1.1. THEOREM. *There is a functor* $PR:(Triplet\rightleftarrows_l \underline{A}) \to (Cat\rightleftarrows\underline{A})$ *such that*

$$llim_{(\underline{I};\underline{I}_1,\underline{I}_2)} I \approx Lim_{Prol(\underline{I};\underline{I}_1,\underline{I}_2)} PR(I).$$

PROOF. The category $Prol(\underline{I};\underline{I}_1,\underline{I}_2)$ is constructed from $Prol\,\underline{I}$ by adjoining a span $(\bar{i} \longleftarrow \tilde{s} \longrightarrow \bar{j})$ and a map of spans from this to $(\bar{i} \longleftarrow \bar{s} \longrightarrow \bar{j})$ for each 1-cell s in $\underline{I}_1$, and an arrow $t_\lambda : \bar{i} \to \bar{t}$ whose composition with the (unique) map $\bar{t} \to \bar{i}$ is the identity on $\bar{i}$ for each 1-cell t in $\underline{I}_2$. All other compositions are free.

Given $I:(\underline{I};\underline{I}_1,\underline{I}_2) \to Tr\,\underline{A}$, then $PR(I): Prol(\underline{I};\underline{I}_1,\underline{I}_2) \to \underline{A}_0$ will be the functor whose restriction to $Prol\,\underline{I}$ is as before and whose value on t_λ for a 1-cell t in $\underline{I}_2$ is $\lambda_{I(t)}$. (See 2.3 (4).) To describe the values of $PR(I)$ on the spans added to $Prol(\underline{I})$ for the 1-cells in $\underline{I}_1$, let $\underline{E}$ denote the category with two objects 0, 1 and two maps

$$m: 1 \to 0,\quad n: 0 \to 1 \quad \text{such that} \quad mn = nm = id.$$

Define f/iB analogously to the definition of $f/B = llim_{\underline{2}} f$ in 2.3 (2) with $\underline{2}$ replaced by $\underline{E}$. This determines a span $A \longleftarrow f/iB \longrightarrow B$ and a 2-cell $\tilde{\theta}_f$ as in 2.3 (3) which is the universal pseudo-cone over f. In particular there is a unique map

$$\gamma_f : f/iB \to f/B \quad \text{satisfying} \quad \theta_f \gamma_f = \tilde{\theta}_f .$$

For each 1-cell s in $\underline{I}_1$, let $PR(I)(\tilde{s}) = I(s)/iI(j)$ and $PR(I)$ takes the map of spans from $\tilde{s}$ to $\bar{s}$ to the map $\gamma_{I(s)}$. These constructions determine a pullback preserving functor $PR(I)$ as before. With obvious modifications, the proof of 4.4 proceeds as before.

5.1.2. COROLLARY. *If* $\underline{A}$ *is a complete 2-category then* $\underline{A}$ *has relative lax limits. In particular, it has pseudo-limits and ordinary limits of 2-functors.*

5.2. *Laxer limits.* It can happen that a 2-category is not finitely complete but does admit weaker notions of limits (an example is the category of Grothendieck topoi) corresponding to more general kinds of adjoints to constant embeddings. In the terminology of [10], page 168, these are strict, i-weak, i-quasi-adjunctions. Following Grothendieck [7] and Cole [5], this notion will be denoted by a capital letter. A Limit looks like a pseudo-limit as in 5.1 except that the universal mapping property is only satisfied up to isomorphic 2-cells. Explicitely, let $Fun_p(\underline{I}, \underline{A})$ be the 2-subcategory of $Fun(\underline{I}, \underline{A})$ consisting of pseudo-natural transformations (all 2-cells are isomorphisms) and let $\Delta: \underline{A} \to Fun_p(\underline{I}, \underline{A})$ be the constant imbedding. By Proposition I, 7.8.2 of [10], a strict, i-weak, i-quasi-right adjoint to Δ, called $Lim_{\underline{I}}: Fun_p(\underline{I}, \underline{A}) \to \underline{A}$, is determined by giving for each $I: \underline{I} \to \underline{A}$ a pseudo-natural transformation $\epsilon_I : \Delta\, Lim_{\underline{I}} I \to I$ which is Terminal in the 2-comma category $[\Delta, I]$; i. e., if $h: \Delta X \to I$ is any pseudo-natural transformation then there is a map $h': \Delta X \to Lim_{\underline{I}} I$ and an isomorphic modification

$$\eta : h \to \epsilon_I(\Delta h')$$

such that, given any other

$$r: X \to Lim_{\underline{I}} I \quad \text{and} \quad \rho: h \to \epsilon_I(\Delta r),$$

then there is a unique isomorphism

$$\sigma: h' \Rightarrow r \quad \text{such that} \quad \epsilon_I(\Delta\sigma).\eta = \rho .$$

This same description can be used even if I is only a lax functor. However such a lax functor can always be replaced by a strict 2-functor $\tilde{I}: \tilde{\underline{I}} \to \underline{A}$ using the construction in I, 4.23 of [10] since one easily demonstrates the following result.

5.2.1. PROPOSITION. *If $\underline{A}$ has small Limits and if $I: \underline{I} \to \underline{A}$ is a lax functor, then $Lim_{\underline{I}} I \approx Lim_{\tilde{\underline{I}}} \tilde{I}$.*

Limits of arbitrary 2-functors will not be required in the following but only Limits of 2-functors $I: \underline{I} \to \underline{A}$ such that for every 2-cell σ in I, $I(\sigma)$ is invertible in $\underline{A}$. Call such 2-functors *special*. It is immediate from the construction of $\tilde{I}$ that if I is a special pseudo-functor, then $\tilde{I}$ is a special 2-functor.

What we are interested in here is the corresponding weakened notion of lax limit, to be denoted by $Llim$; i. e., a strict, i-weak, i-quasi-right adjoint to $\Delta: \underline{A} \to Fun(\underline{I}, \underline{A})$. The difference is that ϵ_I and h are lax natural transformations, but η and σ are still isomorphisms.

The other necessary ingredient is the existence of Cotensors ; i. e., the representable functors $\underline{A}(-, B): \underline{A}^{op} \to Cat$ should have strict, i-weak, i-quasi-left adjoints denoted by $-\pitchfork B: Cat \to \underline{A}^{op}$. In general, these pseudo-functors need only be defined for finite categories; in fact, as before, it suffices that $\underline{2}\pitchfork$- be defined, provided $\underline{A}$ has Pullbacks. This reduces to the following elementary condition (by applying I, 7.8.1 of [10]): for each B there is a functor $\eta_B: \underline{2} \to \underline{A}(\underline{2}\pitchfork B, B)$, such that, for any $h: \underline{2} \to \underline{A}(A, B)$ there is an $h': A \to \underline{2}\pitchfork B$ and a modification

$$\lambda_h : \underline{A}(h', B)\eta_B \Rightarrow h$$

which is unique in the sense that given any other

$$g: A \to \underline{2}\pitchfork B \quad \text{and} \quad \gamma: \underline{A}(g, B)\eta_B \Rightarrow h,$$

then there is a unique isomorphic 2-cell

$$\tau: g \Rightarrow h' \quad \text{such that} \quad \gamma = \lambda_h \cdot \underline{A}(\tau, B).$$

An analogous condition holds for 2-cells.

5.2.2. THEOREM. *If $\underline{A}$ has Cotensors with $\underline{2}$ and Limits of special 2-functors, then $\underline{A}$ has Lax Limits and $Llim_{\underline{I}} I \approx Lim_{Prol\,\underline{I}} PR(I)$.*

PROOF. $Prol\,\underline{I}$ is the same category as in Section 3 and $PR(I)$ is composition with a modified functor $\Phi_{\underline{A}}: \underline{A} \to {}^{op}Spans(\underline{A})$ (cf., 4.1) constructed as before except $\underline{2}\pitchfork B$ and pullbacks are replaced by $\underline{2}\pitchfork B$ and Pullbacks. The resulting operation $PR(I)$ is only a pseudo-functor since the diagrams that used to commute, by the pullback and cotensor properties, now all have specified isomorphisms in them. However, the proof of 4.4 goes through as before by replacing the sequence of isomorphisms at the end of the proof by a corresponding sequence of equivalences. This combined with 5.2.1 gives the result.

These notions can also be combined with those in 5.1 leading to an isomorphism:

$$Llim_{(\underline{I};\underline{I}_1,\underline{I}_2)} I \approx Lim_{Prol(\underline{I};\underline{I}_1,\underline{I}_2)} PR(I).$$

5.3. *Simplifications.* In calculating $llim_{\underline{I}} I$ from $llim_{Prol\,\underline{I}} PR(I)$ one may of course restrict attention to any initial subcategory of $Prol\,\underline{I}$. In particular $PR(I)$ is pullback preserving so objects of the form $<\bar{s}, \bar{t}>$ are taken to pullbacks and hence can frequently be eliminated.

i) Objects of the form $<\bar{s}, \bar{t}>$ where at least one entry corresponds to an identity morphism can be eliminated.

ii) Call $ss' = tt'$ a trivial equation for ss' if either

a) one of t or t' is an identity,

or b) tt' arises from ss' by rebracketing using the associative law.

If ss' satisfies only trivial equations and there are no non-identity 2-cells with domain or codomain ss', then $<\bar{s}, \bar{s}'>$ can be eliminated.

iii) A presentation of $\underline{I}$ in terms of Street's computads [16] can be used to simplify $Prol\,\underline{I}$.

6. EXAMPLES.

We recall here a number of standard examples and then show in more detail how algebras for a monad fit this scheme. Finally we discuss lax ends and indexed limits.

6.1. *Comma objects.* Let $\underline{I}$ be the category with three objects 0, 1, 2 and two non-identity morphisms $d_i : i \to 0$, $i = 1, 2$. Let $\underline{I}_1 = \emptyset$ and $\underline{I}_2 = \{ d_2 \}$. If $I : \underline{I} \to \underline{A}$ is given by $I(d_i) = f_i : A_i \to B$, then

$$llim_{(\underline{I};\, \emptyset, \underline{I}_2)} I = (f_1, f_2)$$

is the comma object of f_1 and f_2. Using 5.3, one sees that it is the pullback of the diagram

$$f_1/B \xrightarrow{1_{f_1}} B \xleftarrow{f_2} A$$

Also $Lim_{(\underline{I};\emptyset, \underline{I}_2)} I$ is called a Pullback and $Llim_{(\underline{I};\, \emptyset, \underline{I}_2)} I$ is a Comma object.

6.2. *Inserters, Identifiers and Inverters.* These names were suggested by R. Street for the following notions.

i) Let $\underline{I}$ be the category with two objects, 0 and 1, and two non-identity morphisms s_1 and s_2 from 0 to 1. Let $\underline{I}_1 = \emptyset$ and $\underline{I}_2 = \{ s_2 \}$. If $I : \underline{I} \to \underline{A}$ is given by $I(s_i) = f_i : A \to B$, then

$$llim_{(\underline{I};\, \emptyset, \underline{I}_2)} I = Ins(f_1, f_2)$$

is the universal solution to the problem of giving a map $g: X \to A$ together with a 2-cell $\mu: f_1 g \Rightarrow f_2 g$; i.e., of *inserting* a 2-cell from f_1 to f_2. By 5.3, it is the equalizer of

$$1_{f_1}: f_1/B \to B \quad \text{and} \quad 1_{f_2}\lambda_{f_2}0_{f_1}: f_1/B \to B.$$

ii) Let $\underline{I} = \underline{2}<\underline{2}>$ be represented as the 2-cell $\phi: s_1 \Rightarrow s_2: 0 \to 1$, let $\underline{I}_1 = \emptyset$ and $\underline{I}_2 = \underline{I}$. If $I: \underline{I} \to \underline{A}$ is given by $I(\phi) = a: f_1 \Rightarrow f_2: A \to B$,

$$llim_{(\underline{I};\, \emptyset,\, \underline{I})} I = lim_{\underline{I}} I = Id(a)$$

is the universal solution to the problem of giving a map $g: X \to A$ such that $ag = id$; i.e., of making a an identity 2-cell. By 5.3, it is the equalizer of the two maps λ_{f_1} and $(a/B)\lambda_{f_2}$ from A to f_1/B.

iii) Let $I: \underline{2}<\underline{2}> \to \underline{A}$ be as in ii and let $\underline{I}_i = \{s_i\}$ for $i = 1, 2$.

$$llim_{(\underline{I};\, \underline{I}_1,\, \underline{I}_2)} I = Inv(a)$$

is the universal solution to the problem of giving a map $g: X \to A$ with ag invertible ; i.e., of inverting a. By 5.3, it is the equalizer of the two maps γ_{f_1} and $(a/B)\lambda_{f_2}0_{f_1}$ from f_1/iB to f_1/B. A coinverter is a localization.

6.3. *Some different examples.* i) Let $I = \underline{2} \times \underline{2}$ and let $I: \underline{2} \times \underline{2} \to \underline{A}$ have as image the commutative square given by the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, "m"] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

Then $llim_{\underline{I}} I$ is the pullback of the diagram

$$f/B \underset{B}{\times} g/D \longrightarrow m/D \longleftarrow h/C \underset{C}{\times} k/D.$$

ii) *Constant functors.* Let $\Delta A: \underline{I} \to \underline{A}$ denote the constant 2-functor whose value is the object A of $\underline{A}$. Since $\underline{A}(B, llim_{\underline{I}}\Delta A)$ is isomorphic to the category of lax cones from B to ΔA, it follows that

$$llim_{\underline{I}} \Delta A \approx (L\pi_0 \underline{I}) \pitchfork A .$$

In particular, for a category $\underline{I}$, the value is $\underline{I} \pitchfork A$. Thus the existence of lax limits implies the existence of cotensors. Writing

$$\underline{I} \pitchfork A = lim_{Prol\, \underline{I}} PR(\Delta A)$$

gives another proof that if $\underline{A}$ has limits and cotensors with $\underline{2}$, then $\underline{A}$ has all cotensors.

iii) *Functors with codomain Cat*. Lax limits and lax colimits of 2-functors $\underline{I} \to Cat$ are explicitely known (cf., [10], pages 201 and 219). It is an amusing exercise to verify that these are the appropriate limits and colimits of the associated functors (which of course differ, since for colimits $\underline{2} \otimes A$ replaces $\underline{2} \pitchfork A$) from $Prol\, \underline{I}$ to Cat.

6.4. *Algebras for a monad.* We shall analyze lax limits over the sequence of 2-categories with one object $*$, $\underline{N} \subset \tilde{\underline{N}} \subset \Delta$, where $\underline{N}$ is the semi-group of natural numbers regarded as a locally discrete 2-category with one object, $\tilde{\underline{N}}$ has $\underline{N}$ as underlying category and strictly monotone maps as 2-cells, and Δ (the usual simplicial category) is the same as $\tilde{\underline{N}}$ except it has all monotone maps as 2-cells. In all of these, 0 represents the identity 1-cell of $*$ and 1 generates the 1-cells since $n = 1^n$ for all n. The 2-cells of $\tilde{\underline{N}}$ are generated (by horizontal composition with 1) by the single 2-cell $\rho: 0 \Rightarrow 1$ and the 2-cells of Δ are generated by ρ and the 2-cell $\sigma: 2 \Rightarrow 1$ subject to the monad identities

$$\sigma . 1\rho = \sigma . \rho 1 = id \quad \text{and} \quad \sigma . \sigma 1 = \sigma . 1 \sigma \ .$$

i) *Algebras for an endomorphism.* A functor $I: \underline{N} \to \underline{A}$ is completely determined by the object $A = I(*)$ and the endomorphism $t = I(1): A \to A$. Similarly a lax cone $\nu: X \to I$ is determined by a single 1-cell $x = \nu_*: X \to A$ and a single 2-cell $\xi = \nu_1: tx \Rightarrow x$ since the equations for lax naturality require that $\nu_2 = \nu_1 . t\nu_1$, etc. If $\underline{A} = Cat$, then the universal solution to this problem is called the category of t-dynamics (cf., [14]). By 5.3, it follows that $Dyn(t) = llim_{\underline{N}} I$ is the equalizer of the two maps 0_t and 1_t from t/A to A. We denote the equalizer map here by $d_t: Dyn(t) \to t/A$.

ii) *Algebras for an endomorphism with a unit.* A 2-functor $I: \tilde{\underline{N}} \to \underline{A}$ is determined by the preceding data together with a 2-cell $I(\rho) = \eta: id_A \Rightarrow t$. A lax cone is as above except that the equations for lax naturality require that $\xi . \eta x = id_x$. Some calculations and simplifications show that

$$A^{(t,\eta)} = llim_{\tilde{\underline{N}}} I$$

is the equalizer of the two maps $(\eta/A)d_t$ and $i_A 1_t d_t$ from $Dyn(t)$ to id_A/A. Let $e_t\colon A^{(t,\eta)} \to Dyn(t)$ be the equalizer.

iii) *Algebras for a monad.* A 2-functor $I\colon \Delta \to \underline{A}$ is determined by the preceding data together with a 2-cell $\mu = I(\sigma)\colon tt \to t$ satisfying the usual monad equations. A lax cone satisfies one more equation, $\xi \cdot t\xi = \xi \cdot \mu x$. Thus $llim_{\Delta} I = A^T$ is the object of Eilenberg-Moore algebras for the monad $T = (t,\eta,\mu)$ on A. The presence of the 2-cell σ means that $Prol\,\Delta$ does not simplify as far as before. However, by 5.3, it follows that A^T is the limit of the diagram

$$\begin{tikzcd}
t/A \times_A t/A \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[dr, "c_{t,t}"'] & t/A \arrow[r, shift left, "0_t"] \arrow[r, shift right, "1_t"'] \arrow[d, "\mu/A"] \arrow[dr, "\eta A"] & A \arrow[d, "i_A"] \\
& tt/A & id_A/A
\end{tikzcd}$$

where a and b describe the pullback of 0_t and 1_t. If $f_t\colon Dyn(t) \to t/A \times_A t/A$ is the induced map, one can show that A^T is the equalizer of the two maps $(\mu/A)d_t e_t$ and $c_{t,t} f_t e_t$.

iv) *Algebras for a distibutive law.* Given monads (t,η,μ) and (t',η',μ') on A a distibutive law is a 2-cell $\sigma\colon tt' \Rightarrow t't$ satisfying appropriate identities (cf., Beck [1]). This corresponds to a suitable 2-functor $I\colon \underline{I} \to \underline{A}$ such that $llim_{\underline{I}} I$ is the object $A^{(\sigma)}$ of algebras for the distributive law. One can show that it can be calculated as the equalizer of two maps from $A^T \times_A A^{T'}$ to tt'/A.

6.5. *Lax ends.* By analogy with the terminology of [10], these are called cartesian ends in [4]. Let $T\colon \underline{I}^{op} \times \underline{I} \to \underline{A}$ be a 2-functor. Denoting the lax end by $l\int_i T(i,i)$, we wish to show that

$$l\int_i T(i,i) \approx llim_{(\underline{J}\,;\,\underline{I}_1,\underline{I}_2)} \tilde{T}$$

for a suitable 2-functor $\tilde{T}\colon \underline{J} \to \underline{A}$. To construct $\underline{J}$, start with $Prol\,\underline{I}_0$ and add 1-cells $\bar{\phi}$ and 2-cells ϕ_0, ϕ_1 as illustrated

$$\begin{tikzcd}
& \bar{s} \arrow[dl, "0_s"'] \arrow[dr, "1_s"] \arrow[d, "\bar{\phi}"] \arrow[dl, phantom, shift right=6, "\phi_0 \Downarrow"] \arrow[dr, phantom, shift left=6, "\Downarrow \phi_1"] & \\
\bar{i} & \bar{s}' \arrow[l, "0_{s'}"] \arrow[r, "1_{s'}"'] & \bar{j}
\end{tikzcd} \qquad (1)$$

for each 2-cell $\phi : s \Rightarrow s' : i \to j$ in $\underline{I}$ as well as 1-cells $<\bar{\phi}, \bar{\psi}>$ and 2-cells $<\phi, \psi>_0$, $<\phi, \psi>_1$ as illustrated

$$\begin{tikzcd}
\bar{s} \arrow[d, "\bar{\phi}"'] \arrow[dr, phantom, "\searrow <\phi,\psi>_0"] & <\bar{s}, \bar{t}> \arrow[l, "\bar{a}_{s,t}"'] \arrow[r, "\bar{b}_{s,t}"] \arrow[d, "<\bar{\phi}, \bar{\psi}>"] \arrow[dr, phantom, "<\phi,\psi>_1 \swarrow"] & \bar{t} \arrow[d, "\bar{\psi}"] \\
\bar{s}' & <\bar{s}', \bar{t}'> \arrow[l] \arrow[r] & \bar{t}'
\end{tikzcd} \qquad (2)$$

for each composable pair where $\psi : t \Rightarrow t' : j \to k$, subject to the relations

i) $\bar{\phi}'\bar{\phi} = \overline{\phi' . \phi}$, $<\bar{\phi}', \bar{\psi}'><\bar{\phi}, \bar{\psi}> = <\overline{\phi' . \phi}, \overline{\psi' . \psi}>$,

ii) $\phi_n' \bar{\phi} = (\phi' . \phi)_n$, $n = 0, 1$,

ii) $<\phi', \psi'>_0 <\bar{\phi}, \bar{\psi}> . \bar{\phi}' <\phi, \psi>_0 = <\phi' . \phi, \psi' . \psi>_0$,

$<\phi', \psi'>_1 <\bar{\phi}, \bar{\psi}> . \bar{\psi}' <\phi, \psi>_1 = <\phi' . \phi, \psi' . \psi>_1$.

Call this category $\underline{J}'$ and let $\underline{J} = (\underline{J}')^{op}$. Let $\tilde{T} : \underline{J} \to \underline{A}$ be the 2-functor taking (1) to

$$\begin{tikzcd}
 & T(i,s) \arrow[r] \arrow[d, phantom, "T(i,\phi) \Downarrow"] & T(i,j) & T(s,j) \arrow[l] \arrow[d, phantom, "\Downarrow T(\phi,j)"] & \\
T(i,i) \arrow[ur] \arrow[dr] & & & & T(j,j) \arrow[ul] \arrow[dl] \\
 & T(i,s') \arrow[r] & T(i,j) \arrow[uu, "id"'] & T(s',j) \arrow[l] &
\end{tikzcd}$$

In addition,

$\tilde{T}(\bar{a}_{s,t}) = T(i,t)$, $\tilde{T}(\bar{b}_{s,t}) = T(s,k)$, $\tilde{T}(\bar{c}_{s,t}) = id$, $\tilde{T}(<\bar{\phi}, \bar{\psi}>) = id$,
$\tilde{T}(\bar{c}_i) = id$ and $\tilde{T}(<\phi, \psi>_0) = T(i, \psi)$, $\tilde{T}(<\phi, \psi>_1) = T(\phi, k)$.

Let $\underline{J}_1 = \emptyset$ and $\underline{J}_2$ be the subcategory consisting of all 1-cells except the 0_s 's for $s \neq id$ and the $\bar{a}_{s,t}$'s for $t \neq id$. Then $l\int_i T(i,i)$ is the relative lax limit of $\tilde{T}$.

6.6. *Indexed limits.* Let $J : \underline{I} \to Cat$. In [16] Street constructs a 2-category $El\,J$ and a 2-functor $P : El\,J \to \underline{I}$ together with a 2-subcategory $el\,J$ of $El\,J$ such that for any $I : \underline{I} \to \underline{A}$, the indexed limit $lim(J, I)$ is isomorphic to the relative lax limit $llim_{(El\,J\,;\,\emptyset,\,el\,J\,)} IP$. Hence

$$lim(J, I) \approx lim_{Prol(El\,J\,;\,\emptyset,\,el\,J\,)} PR(IP).$$

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