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ON MEASURES IN FIBRE SPACES

by Anthony Karel SEDA

INTRODUCTION.

Let S and X be locally compact Hausdorff spaces and let $p: S \to X$ be a continuous surjective function, hereinafter referred to as a *fibre space with projection* p, *total space* S *and base space* X. Such spaces are commonly regarded as broad generalizations of product spaces $X \times Y$ fibred over X by the projection on the first factor. However, in practice this level of generality is too great and one places compatibility conditions on the fibres of S such as : the fibres of S are all to be homeomorphic ; p is to be a fibration or étale map; S is to be locally trivial, and so on. In this paper fibre spaces will be viewed as generalized transformation groups, and the specific compatibility requirement will be that S is provided with a category or groupoid G of operators. This approach is by no means new and was used early on in efforts, especially by Charles Ehresmann and his school, to give coordinate-free definitions of fibre bundles.

Certain features of the point of view adopted here are worthy of comment. Firstly, one is free to place a variety of conditions on G and on its action on S. At one extreme we may simply require that G be an untopologised category, in which case we are not really placing any conditions on S. At the other extreme G may be required to be a locally trivial topological groupoid whose action on S is continuous, then S will itself automatically be a locally trivial fibre bundle. Indeed, charts are most easily introduced by introducing them into G. In between these extremes lie a variety of interesting examples and, as shown in Section 4, Cartan's construction [4] is one such and need not be locally trivial. Another feature of this approach is that one may form the space S/G of orbits provided with the canonical quotient map $q: S \to S/G$. Thus, S really does look like a product now fibred over its two projections. In fact S/G coincides with Y and q with the projection on the second factor (with the obvious G) if S is a product $X \times Y$. This fact enables us to give a general form of Fubini's Theorem concerning iterated integrals, Theorem 3.4 and Section 3.5.

Thinking of $X \times Y$ as a product bundle over X with projection p, consider positive measures μ defined on X and ν defined on Y and let $m = \mu \times \nu$ be their product. Then each fibre $\{x\} \times Y$ of S supports a copy μ_x of ν and the assignment $x \mapsto \mu_x$ is a disintegration of m with respect to p and μ. Equivalently, m is the integral $m = \int_X \mu_x \, d\mu(x)$ of the family $\{\mu_x\}$ with respect to μ. This example leads one to propose generally that in studying measures m on S one might reasonably consider measures which are the integral with respect to a measure μ on X of a family of measures $\{ \mu_x \mid x \in X \}$, where μ_x is supported by the fibre $S_x = p^{-1}(x)$ over x. In other words, we are studying μ-scalarly essentially integrable mappings from X into the cone $\mathfrak{M}_+(S)$ of positive measures on S as treated by Bourbaki [2], and called simply *μ-integrable families of measures* here. However, our aims are more specific and geometrical than those of Bourbaki. We have in mind applications of the results here to the construction of convolution algebras which can be functorially assigned to manifolds and foliated manifolds, see [21] for a preliminary announcement of this programme. Moreover, in paying regard to our compatibility condition it is entirely natural to require that the family $\{\mu_x\}$ be *invariant* with respect to the action of G, as defined in Section 1. Invariance in this sense implies associativity of the convolution product alluded to above. For reasons such as these we have considerable information about the measures μ_x, and we will be seeking conditions under which the family $\{\mu_x\}$ is μ-integrable for a given μ. When such is the case, the measure m defined by $m = \int_X \mu_x \, d\mu(x)$ will be called *G-invariant.* Such a measure clearly generalises the idea of an invariant measure for a transformation group, and it is the purpose of this paper to develop some of their properties. For technical reasons discussed in Section 1 all measures used here will be Baire measures, and we note that Baire measures are general enough for all geometrical applications. Actually, certain geometric examples partially motivated this study and will be examined as we proceed. In this context, another natural and desirable property of m is that m should be positive on non-empty measurable open sets, and this is discussed in Theorem 3.4.

Other authors have investigated the problem (and associated problems) of constructing measures in locally trivial fibre bundles [7, 8, 9, 12], and some of their work is related to ours in Sections 4 and 5. In particular, the results of [9] are used to obtain a classification theorem, Theorem 5.1, for G-invariant measures in locally trivial fibre bundles. The reader is referred to [12] for some remarks concerning applications of measures in fibre bundles to integral geometry, and to [7, 8] for a treatment of the problem of integrating sections of a vector bundle.

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1. G-INVARIANT MEASURES.

Let $p: S \to X$ be a fibre space, suppose G is a category with object set X and suppose G acts on S with respect to p, see [18, 19, 20] for definitions and terminology used here. We employ synonymously the terms: *G acts on S; G is a category of operators on S; G is a transformation category of S; S is a G-space.*

Given $a \in G(x,y)$, we denote by ϕ_a the induced map $S_x \to S_y$ defined by $\phi_a(s) = a.s$. Note that the fibre S_x of S over x is closed in S and hence is locally compact.

The following standing assumption will be made at all times:

1.1. ASSUMPTION. *For all* $a \in G$, *the map* ϕ_a *is continuous.*

Often, G will be a groupoid, in which case ϕ_a is a homeomorphism. Often, too, G will be a topological groupoid, and its action on S will be continuous, in which case 1.1 holds automatically.

Next a few remarks and conventions concerning measure Theory. All topological spaces will be assumed to be Hausdorff and locally compact, unless otherwise stated, and our ultimate concern will be with integrating continuous functions having compact support. We let $\mathcal{k}(Y)$ denote the set of real valued continuous functions having compact support defined on a topological space Y. For such purposes Baire measures are both adequate and natural; they are the measures which are defined on the σ-ring (of Baire sets) generated by the compact G_δ's, which give finite measure to each compact G_δ [1]. Baire measures are automatically regular and integration with respect to such a measure yields a measure in the sense of Bourbaki [2]. Moreover, they behave technically better than regular Borel measures and can always be uniquely extended to regular Borel measures if so desired. Thus, Baire measures will be employed throughout and the term measure will mean Baire measure unless stated otherwise. Finally a measure μ on Y will be called *non-trivial* if there is a Baire set E in Y with $\mu(E) > 0$, and we will denote by $B(Y)$ the class of Baire sets in Y. Our first main definition may now be stated.

1.2. DEFINITION. For each $x \in X$ let μ_x be a Baire measure defined on S_x. The indexed collection $\{\mu_x \mid x \in X\}$, or more briefly $\{\mu_x\}$, will be referred to as a *family of measures on* S. If each μ_x is non-trivial, then $\{\mu_x\}$ will be called a *non-trivial family*. A family $\{\mu_x\}$ will be called *G-invariant* if ϕ_a is measurable and measure preserving for each $a \in G$. Thus for all $x, y \in X$ and $a \in G(x,y)$ we have the relation

$$\mu_x(\phi_a^{-1}(E)) = \mu_y(E) \text{ for each Baire set } E \text{ in } S_y.$$

Notice that a continuous function need not be Baire measurability preserving and so this assumption must be stated in our definition. However if G is a groupoid, the measurability statement in Definition 1.2 holds automatically, in view of 1.1, and the equality there may be cast in the form

$$(1.2)' \quad \mu_x(E) = \mu_y(\alpha . E) \quad \text{for each Baire subset } E \text{ of } S_x .$$

Notationally, one may write $\mu_y = \phi_\alpha(\mu_x)$ here, for μ_y is the image of μ_x under ϕ_α.

The reformulation given in (1.2)' is valid, too, if G is a category in which each operation ϕ_α is injective and each S_x is compact metric, except that one replaces μ_y by its restriction $\mu_y|\phi_\alpha(S_x)$ to $\phi_\alpha(S_x)$, see Fxample 1.7.

In the case of a groupoid G, the construction of any (non-trivial) G-invariant family can be achieved as follows. Let μ_z be a (non-trivial) measure defined on S_z and invariant under the action of the group $G\{z\}$. For each $x \in X$ with $G(z,x)$ non empty, choose $\tau_x \in G(z,x)$ and let μ_x be the image $\phi_{\tau_x}(\mu_z)$ of μ_z. If $\alpha \in G(x,y)$, then we may write

$$\alpha = \tau_y \beta \tau_x^{-1} \quad \text{for a unique choice of } \beta \in G\{z\},$$

and it follows that

$$\mu_y(\alpha . E) = \mu_x(E) \quad \text{for any Baire set } E \text{ in } S_x .$$

It follows also from this fact that the definition of μ_x is independent of the choice of τ_x in $G(z,x)$. Working in this way over the transitive components of G one obtains the (non-trivial) G-invariant family $\{\mu_x\}$. Clearly, any such family arises thus, and the existence of a non-trivial family is equivalent to the existence of non-trivial $G\{x\}$-invariant measures μ_x on S_x for each $x \in X$. This latter requirement implies conditions on S and G for in general such an invariant measure on S_x need not exist. We will make the implicit assumption usually that non-trivial $G\{x\}$-invariant measures do exist on S_x, for each $x \in X$, since trivial families are of no particular interest.

1.3. Let $\{\mu_x\}$ be a family of measures on S. Given any Baire set E in

S, the set $E_x = E \cap S_x$ is readily seen to be a Baire set in S_x. However, the σ-ring $\{E \cap S_x \mid E \in B(S)\}$ can be properly contained in the σ-ring $B(S_x)$ of Baire sets of S_x, unless for example S_x is actually a G_δ in S. Nevertheless, we shall regard each measure μ_x as a Baire measure defined on S, with support contained in S_x, in accordance with the relation $\mu_x(E) = \mu_x(E \cap S_x)$. This leads us to our second main definition.

1.4. DEFINITION. Let $\{\mu_x\}$ be a family of Baire measures on S and let μ be a non-trivial Baire measure on X. The family $\{\mu_x\}$ will be called *μ-integrable* if the function M

$$\mathrm{M}: X \to R \quad \text{defined by} \quad \mathrm{M}(x) = \mu_x(E)$$

is μ-measurable for each Baire set E in S and the set function m on S defined by the expression

$$m(E) = \int_X \mu_x(E)\, d\mu(x)$$

is a Baire measure. If, further, $\{\mu_x\}$ is G-invariant, then m will be called a *G-invariant measure on S.*

A few remarks are in order at this juncture. Firstly, m will be denoted by $m = \int_X \mu_x\, d\mu(x)$ and the expression for m in Definition 1.4 always determines a measure in the general sense, provided the integrand is μ-measurable for each E. It does not, however, determine necessarily a Baire measure, that is, one giving finite measure to each compact G_δ, even if $\{\mu_x\}$ is G-invariant. Thus, we include this condition in our definition. Secondly, given $\{\mu_x\}$ arbitrarily, non-trivial Baire measures μ always exist on X with the property that $\{\mu_x\}$ is μ-integrable; for example, μ may be chosen to be atomic with finitely many atoms. It follows from this remark, by placing unit mass at the point $x \in X$, and 1.3 that each measure μ_x in a G-invariant family is actually G-invariant. It is of interest to know when M is universally integrable, that is, when μ can be chosen arbitrarily here and sufficient conditions for this to hold are given in Section 3.

It will be convenient to elaborate a little upon 1.3. Suppose A is a closed subset of a locally compact Hausdorff space Y and μ is a Baire measure defined on the Baire sets of A, that is, defined on the σ-ring of subsets of A generated by the compact G_δ's of the topological subspace A of Y. Then one may regard μ as a Baire measure defined on Y by means of the expression

$$\mu(E) = \mu(E \cap A) \quad \text{for each Baire set } E \text{ in } Y.$$

This means that questions involving regularity arguments can be approached by consideration of sets in A or in Y as desired. An example of this is the proof of the following proposition:

1.5. PROPOSITION. *Let A be a closed subset of a locally compact Hausdorff space Y. Suppose that μ_1 and μ_2 are Baire measures defined on A and extended to Y in accordance with the definition above. If*

$$\mu_1(E \cap A) = \mu_2(E \cap A) \quad \textit{for all Baire sets } E \textit{ of } Y,$$

then $\mu_1(F) = \mu_2(F)$ for all Baire sets F of A.

PROOF. Suppose that the conclusion is false. Then there exists a Baire set in A, and hence a compact G_δ, C, in A such that $\mu_1(C) \neq \mu_2(C)$. Suppose $\mu_1(C) > \mu_2(C)$ and let $\eta = \mu_1(C) - \mu_2(C)$. By regularity of μ_1 and μ_2 applied in the first instance to their regular Borel extensions there exists an open Baire set V in Y such that $C \subset V$ and simultaneously

$$\mu_1(C) < \mu_1(V) < \mu_1(C) + \frac{\eta}{4}, \quad \mu_2(C) < \mu_2(V) < \mu_2(C) + \frac{\eta}{4}$$

Since

$$\mu_i(V) = \mu_i(A \cap V), \quad i = 1, 2,$$

these inequalities yield

$$\mu_1(A \cap V) - \mu_2(A \cap V) > \mu_1(C) - \mu_2(C) - \frac{\eta}{4} = \frac{3\eta}{4} > 0.$$

This gives a contradiction since we have

$$\mu_1(A \cap V) \neq \mu_2(A \cap V) \quad \text{for a Baire set } V \text{ in } Y.$$

This proposition has an obvious, equivalent statement in terms of integrals and permits us to reformulate the notion of a G-invariant measure in terms of integrals as follows, for some details see [19].

1.6. THEOREM. *For a Baire measure m on S the following statements are* *equivalent:*

a) m is G-invariant.

b) For any non-negative Baire measurable function f on S:

(i) $\int_S f\, dm = \int_X \int_{S_x} f\, d\mu_x\, d\mu$.

(ii) $\int_{S_x} (f \circ \phi_\alpha)\, d\mu_x = \int_{S_y} f\, d\mu_y$ *for all* $x, y \in X$ *and* $\alpha \in G(x,y)$.

If, further, G is a groupoid, then a and b are each equivalent to:

c) For any non-negative function $f \in k(S)$:

(i) $\int_S f\, dm = \int_X \int_{S_x} f\, d\mu_x\, d\mu$,

(ii) $\int_{S_x} (f \circ \phi_\alpha)\, d\mu_x = \int_{S_y} f\, d\mu_y$ *for all* $x, y \in X$, $\alpha \in G(x,y)$.

This theorem is analogous to the classical theorem of Tonelli and asserts that if f is m-integrable, then for μ-almost all $x \in X$, $\int_{S_x} f\, d\mu_x$ is finite and the equality (i) in b is valid.

This section will be concluded with a discussion of an example.

1.7. EXAMPLE (Transverse measures on foliations).

Ruelle and Sullivan [16] introduced the notions of transverse measure on a (partial) foliation of a manifold, and that of geometric current. Following the exposition of [5], let M be a smooth m-dimensional manifold smoothly foliated by l-dimensional leaves, where $m = k + l$. Suppose that $\{W_\sigma \times R^l\}_{\sigma \in \Sigma}$ is a locally finite collection of foliation charts for M whose interiors cover M, where each W_σ is compact. The W_σ's are called transversals and are assumed to be smoothly embedded in M. Suppose further that each W_σ has defined on it a Baire measure μ_σ. Then [5] the collection $\{\mu_\sigma \mid \sigma \in \Sigma\}$ is called *translation invariant* provided that «for each Baire subset $F \subset W_\sigma$ and for each leaf invariant homeomorphism $h: M \to M$ such that $h|_F$ embeds F into another transversal W_τ (with $\tau = \sigma$ possibly), we have $\mu_\tau(h(F)) = \mu_\sigma(F)$.» This definition may be formalised within the terms of this work as follows. Let H be the group of all leaf preserving homeomorphisms of M and X be the set

$$X = \{(\sigma, F) \mid \sigma \in \Sigma \text{ and } F \in B(W_\sigma)\}.$$

For $x = (\sigma, F)$ and $y = (\tau, F') \in X$ define

$$G(x,y) = \{ (\sigma, \tau, h|_F) \mid h \in H, \; h(F) \subset F' \},$$

and let G be the union of the sets $G(x,y)$, x, $y \in X$. Then G is a category over X with composition defined by

$$(\tau, \delta, h'|_{F'}) \circ (\sigma, \tau, h|_F) = (\sigma, \delta, h'h|_F).$$

Now let $S_x = \sigma \times F$, where $x = (\sigma, F)$, put $S = \bigcup_{x \in X} S_x$ and let p be the obvious projection $S \to X$; X is regarded as a discrete space and S as a disjoint union. The category G acts on S by evaluation, thus

$$(\sigma, \tau, h|_F).(\sigma, f) = (\tau, h(f)).$$

Finally, each S_x supports a Baire measure μ_x defined by $\mu_{(\sigma, F)} = \mu_\sigma|_F$. It is easily seen that translation invariance of $\{\mu_\sigma\}$ in the sense of [5] is equivalent to G-invariance of $\{\mu_x\}$ as in (1.2)'.

Ruelle and Sullivan have used geometric currents to obtain, in certain cases, non trivial real homology classes by means of integrating differential forms. The technique uses partitions of unity and the demonstration of independence of the choice of the partition of unity makes essential use of the translation invariance of $\{\mu_\sigma\}$. In [5] it is shown that important translation invariant measures can be constructed by «counting intersections with a compact leaf» and then passing to a limit.

These ideas can be taken further and we may consider «cocycles» and «measures associated with a cocycle» due to Ruelle [15]. Let G be a category acting on S. We define a *cocycle* associated with the pair (S,G) to be a family $\{f_a \mid a \in G\}$ of functions such that

a) $f_a : S_{\pi'(a)} \to R$ is continuous and strictly positive, where $\pi'(a)$ denotes the final point of a.

b) If a, β are invertible elements of G such that βa is defined, then $f_{\beta a} = f_\beta . (f_a \circ \phi_\beta^{-1})$.

We define a family $\{\mu_x \mid x \in X\}$ of measures on S to be a *measure associated with the cocycle* $\{f_a\}$ if we have the relation

$$\mu_{\pi(a)} = \phi_a^{-1}(f_a \cdot \mu_{\pi'(a)} |_{Im(\phi_a)})$$

for each $a \in G$, where $\pi(a)$ denotes the initial point of a and $f_a \cdot \mu_{\pi'(a)}$ the scalar product of f_a with the measure $\mu_{\pi'(a)}$.

Ruelle's original definitions may be recovered as above by taking for the set of W_σ the set of all transverse open k-dimensional submanifolds of M.

REMARK. I am indebted to M^{me} Fhresmann for drawing my attention to the fact that the category G discussed here and the idea of a cocycle are but special cases, respectively, of the «structure transversale d'un feuilletage» given by Charles Ehresmann, «Structures feuilletées», *Proc. Fifth Canad. Math. Congress, Montréal* (1961), 129-131, and of the crossed homomorphism associated (by Ehresmann) to a fibration, *Catégories et Structures*, Chapitre II, Dunod, Paris, 1965.

2. MORPHISMS AND G-INVARIANT MEASURES.

This section will be devoted to studying morphisms $f: S \to S'$ of G-spaces which preserve G-invariant measures m and m'. Specifically if

$$m = \int_X \mu_x \, d\mu(x) \quad \text{and} \quad m' = \int_X \mu'_x \, d\mu(x)$$

and f preserves m and m', we want to investigate the effect of the induced map $f_x: S_x \to S'_x$ on μ_x and μ'_x, and vice versa. To do this it is necessary to examine certain uniqueness questions involving m, $\{\mu_x\}$ and μ, and we consider these next. Under the topological hypotheses adopted here concerning S and X, G will not play a significant role and therefore the results will be valid for an arbitrary fibre space S. Such a space can always be regarded as a G-space where G is, for example, a disjoint union of groups. For this reason and because of the ultimate applications we will continue to use the terminology G-invariant measures, etc... In later sections, the same sort of results will be obtained under certain conditions on G which will play an important role.

Let $m = \int_X \mu_x \, d\mu(x)$ be a G-invariant measure on S. It is clear that m is uniquely determined by μ and $\{\mu_x\}$. It is also easy to see that we can have

$$m = \int_X \mu_x \, d\mu(x) = \int_X \mu'_x \, d\mu'(x) \quad \text{with} \quad \mu \neq \mu' \ \text{and} \ \{\mu_x\} \neq \{\mu'_x\} .$$

Indeed this can happen with $\{\mu'_x\}$ not even G-invariant, nor even μ'-almost everywhere G-invariant in an obvious sense. Nevertheless, the next theorem shows, under quite mild restrictions, that m and μ determine $\{\mu_x\}$ and that m and $\{\mu_x\}$ determine μ.

2.1. THEOREM. *Let S and X be locally compact Polish spaces.*

a) Suppose $\{\mu_x\}$ and $\{\mu'_x\}$ *are two μ-integrable families of measures on S with the property that*

$$\int_X \mu_x \, d\mu(x) = \int_X \mu'_x \, d\mu(x).$$

Then $\mu_x = \mu'_x$ for μ-almost all x.

b) If $\{\mu_x\}$ is a non-trivial family which is both μ- and μ'-integrable, and

$$\int_X \mu_x \, d\mu(x) = \int_X \mu_x \, d\mu'(x),$$

then $\mu = \mu'$.

PROOF. a) Since S is the union of a sequence of compact sets, p is Baire measurability preserving. Also, if m is the common value of the two integrals, then μ is a pseudo-image of m by p, see proof of Part b. Thus, $\{\mu_x\}$ and $\{\mu'_x\}$ are two disintegrations of m and by [2], Chapter 6, Section 3, n° 3, Theorem 2, $\mu_x = \mu'_x$ μ-almost everywhere.

b) We prove first that μ and μ' are absolutely continuous each with respect to the other. To do this, suppose $A \subset X$ is μ-null and denote by E the set $p^{-1}(A)$. Then

$$m(E) = \int_X \mu_x(E) \, d\mu(x) = \int_A \mu_x(E) \, d\mu_x(x) = 0 .$$

Hence

$$\int_X \mu_x(E) \, d\mu'(x) = 0 \quad \text{and so} \quad \int_X \chi_A \, \mu_x(E) \, d\mu'(x) = 0 ,$$

where χ_A denotes the characteristic function of A. Hence, $\chi_A \mu_x(E) = 0$ μ'-almost everywhere. But $\mu_x(E) > 0$ for all x since $\{\mu_x\}$ is non-trivial, and it follows therefore that $\mu'(A) = 0$, that is, A is μ'-null. The reverse implication follows in the same way.

Let f be the Radon-Nikodym derivative $d\mu'/d\mu$. It follows from the Radon-Nikodym theorem that

$$\int_X g\, d\mu' = \int_X gf\, d\mu$$

for every measurable function g on X. Thus, if E is any Baire set in S, we can take $g(x) = \mu_x(E)$ to conclude that

$$\int_X \mu_x(E)\, d\mu'(x) = \int_X \mu_x(E) f(x)\, d\mu(x) = \int_X \mu_x(E)\, d\mu(x).$$

Consequently, we obtain

$$\int_X f(x)\mu_x\, d\mu(x) = \int_X \mu_x\, d\mu(x),$$

and so by Part a it follows that $f(x)\mu_x = \mu_x$ for μ-almost all x. Hence, $f(x) = 1$ for μ-almost all x and so $\mu = \mu'$ as required.

Part b of this theorem is true in general, that is, without the restriction that S and X be Polish spaces, but the proof given here has been included because of its important consequences in Remark 2.2 below. However, Part a is not in general true. There are no invariance conditions placed on $\{\mu_x\}$ and $\{\mu_x'\}$ here, but if such conditions are suitably imposed, then the required conclusion can be drawn under no metrizability conditions, see Theorem 3.6. The same comment applies to Theorem 2.4, see Corollary 3.7. I am indebted to MM. Cartier, Choquet, Mokobodzki and Wright for valuable conversations concerning these questions.

2.2. REMARK. Suppose $m = \int_X \mu_x\, d\mu(x)$ is G-invariant and $f: X \to R$ is a non negative μ-essentially bounded μ-measurable function. By setting $m' = \int_X f(x)\mu_x d\mu(x)$, we can obtain significant Baire measures on S associated with m, depending on the choice of f. However, the proof of Theorem 2.1 b shows that $m' = \int_X \mu_x\, d\mu'(x)$ and is, hence, actually G-invariant, where μ' is defined on X by

$$\mu'(A) = \int_A f\, d\mu \quad \text{for each Baire set } A \subset X.$$

For an example of the use of this remark, see Section 3.3.

2.3. DEFINITION (see [18, 20]). Let G be a category acting on two fibre spaces $p: S \to X$ and $p': S' \to X$. A fibre preserving continuous function $f: S \to S'$ is called *G-equivariant* or a *morphism* if $f(a.s) = a.f(s)$ for all $a \in G$, $s \in S$ such that $a.s$ is defined.

Such mappings are the appropriate morphisms in the category of G-spaces over X, where G is fixed, but they are not quite the right ones for measure theoretic questions: One needs to assume in addition (or prove in appropriate contexts) that f is Baire measurability preserving. We let f_x denote the restriction of f to S_x. The main result of this section is the following theorem:

2.4. THEOREM. *Suppose S' and X are locally compact Polish spaces and $f: S \to S'$ is fibre preserving, continuous and a proper map. Let $\{\mu_x\}$ and $\{\mu_x'\}$ be μ-integrable families of Baire measures on S and S' respectively and let m and m' be the corresponding measures on S and S'. Then f is m, m' measure preserving iff f_x is μ_x, μ_x' measure preserving for μ-almost all x.*

PROOF. Since f is proper, the preimage $f^{-1}(C)$ is a compact G_δ in S whenever C is one in S', whether or not S' is a Polish space. Likewise, the same is true of f_x, for each x, and so f and each f_x are Baire measurability preserving.

Suppose f_x is μ_x, μ_x' measure preserving for μ-almost all x. Let $E' \subset S'$ be any Baire set, then

$$m'(E') = \int_X \mu_x'(E_x')d\mu(x) = \int_X \mu_x(f_x^{-1}(E')_x)d\mu(x) = m(f^{-1}(E'))$$

and f preserves m and m'.

For the converse, define for each $x \in X$ a Baire measure μ_x'' on S_x' by

$$\mu_x''(E') = \mu_x(f_x^{-1}(E')) \quad \text{for each Baire set } E' \text{ in } S_x'.$$

The function $x \mapsto \mu_x''(E')$ coincides with the function $x \mapsto \mu_x(f^{-1}(E'))$ and is therefore μ-integrable for each Baire set E' in S'. Hence, we may set $m'' = \int_X \mu_x'' d\mu(x)$. Since f preserves m and m', we have

$$m'(E') = \int_X \mu_x'(E')d\mu(x) = m(f^{-1}(E')) =$$

$$= \int_X \mu_x(f_x^{-1}(E'))d\mu(x) = \int_X \mu_x''(E')d\mu(x) = m''(E').$$

Hence, by Theorem 2.1 a, $\mu_x' = \mu_x''$ for μ-almost all x or, in other words f_x preserves μ_x, μ_x' for μ-almost all x as required.

3. A CONTINUITY THEOREM AND FUBINI'S THEOREM.

Relatively few conditions have been placed on G so far or on its action on S. Here and in subsequent sections we will investigate the effect of imposing successively more conditions both on G and on its action. The following standing hypothesis will be made for the whole of this section:

3.1. *G denotes a locally compact, locally transitive Hausdorff topological groupoid; for each $x \in X$ the final map $\pi': \pi^{-1}(x) \to X$ is assumed to be an open map* *); *the action of G on S is continuous.*

Here, π and π' denote respectively the initial and final maps of G and the condition concerning π' is always valid if G has compact transitive components, or if G is locally trivial. Another case when these conditions are fulfilled is discussed in Section 4.

Under the conditions of 3.1 we have the following result, see [17] :

3.2. THEOREM. *Let $\{\mu_x\}$ be a G-invariant family of Baire measures on S and let $f: S \to R$ be any continuous function having compact support. Then the function θ defined on X by $\theta(x) = \int_S f\, d\mu_x$ is continuous and has compact support.*

It is shown in [17] that this theorem has the consequence that for any Baire set E in S, the function M, where $M(x) = \mu_x(E)$, is Baire measurable. Thus, we may choose *any* Baire measure μ on X and set $m = \int_X \mu_x\, d\mu(x)$. Moreover, m is a Baire measure for any choice of μ. Thus, Theorem 3.2 provides quite natural conditions under which M is universally integrable, and we will now illustrate this theorem with a simple example.

*) This condition actually implies that G is locally transitive.

3.3. EXAMPLE (Surfaces and volumes of revolution).

Let $X = [x_0, x_1]$ be an interval in R with $x_0 < x_1$, and suppose $f: X \to R$ is continuous and such that $f(x) > 0$ for all $x \in X$. Let S be the region of the plane bounded by the x-axis, the curve $y = f(x)$ and ordinates at $x = x_0$, $x = x_1$, and let $p: S \to X$ be the projection on the first factor. Given $x, y \in X$, let $\tau_{xy}: S_x \to S_y$ be the map obtained by linear projection, thus

$$\tau_{xy}(x, s) = (y, \frac{f(y)}{f(x)} . s).$$

Let $G = \{\tau_{xy} \mid x, y \in X\}$ and identify G with $X \times X$. Then G satisfies 3.1. Moreover, evaluation determines an action of G on S which is continuous because f is continuous. Let μ_x be the measure defined on S_x by $\mu_x = (1/f(x))\lambda_x$, where λ_x is Lebesgue measure on S_x. Then $\{\mu_x\}$ is G-invariant. Thus by Theorem 3.2 and Section 2.2, we may define a G-invariant measure m on S by $m = \int_X f(x)\mu_x \, d\mu(x)$, where μ is the Lebesgue measure on X. The total mass of m, $m(S)$, is $\int_X f(x) d\mu(x)$ and is the «area under the curve $y = f(x)$». If we rotate the curve $y = f(x)$ about X, we obtain a surface $\hat{S}$ (respectively volume) fibred by circles (respectively discs). A trivial modification of the foregoing yields G-invariant measures on $\hat{S}$ which include the classical notions of «surface area of revolution» and «volume of revolution» of elementary calculus. In particular, if f is continuously differentiable, then

$$m = 2\pi \int_X f(x)\sqrt{(1 + f'(x)^2)}\, \mu_x \, d\mu(x)$$

is G-invariant and $m(\hat{S})$ is the classical surface area of revolution.

3.4. THEOREM *(Fubini). Let S be a locally compact Polish space, suppose that the transitive components of G are compact and let $q: S \to Y$ be the canonical map of S onto the quotient space $Y = S/G$ of orbits of G.*

a) Let m be a G-invariant measure on S and let ν be a pseudo-image on Y of m by q, then there is a corresponding family $\{\lambda_y \mid y \in Y\}$ of Baire measures on S with the properties:

1° The support of λ_y is contained in the orbit y.

2° $\{\lambda_y \mid y \in Y\}$ *is uniquely determined* ν*-almost everywhere.*

3°
$$\int_S f\, dm = \int_X \int_{S_x} f\, d\mu_x\, d\mu = \int_Y \int_y f\, d\lambda_y\, d\nu$$

for each m-integrable Baire measurable function f *on* S.

b) There exist G-invariant measures m *on* S *with the property that* $m(O) > 0$ *for all non-empty open sets* $O \subset S$.

PROOF. a) Let d be a metric on S compatible with the topology and let d_x denote the restriction of d to S_x. Then $\{d_x \mid x \in X\}$ will constitute a metric family and by [20], Theorem 1, Y is metrisable and q is an open map. Thus, Y is a Polish space and a pseudo-image ν of m exists on Y. Apply now Bourbaki's theorem, as in the proof of Theorem 2.1, to obtain $\{\lambda_y \mid y \in Y\}$ with the properties 1 and 2. The final property follows from the definition of a disintegration and Theorem 1.6.

b) We first show that X is metrisable. To see this we note that the object set of each transitive component of G is compact and both open and closed in X, and hence X is paracompact. Now each orbit of G is compact and metrisable, and each object set of each transitive component is the p-image of an orbit and is, hence, metrisable, since X is Hausdorff and p is continuous. Therefore, X is paracompact and locally metrisable and it follows that X is metrisable by Smirnoff's metrisability theorem.

Secondly, X is separable since S is separable and we may choose a countable dense set $\{x_n \mid n = 1, 2, \ldots\}$ in X. By placing unit mass δ_n at x_n and setting

$$\mu = \sum_{n=1}^{\infty} \frac{1}{n^2} \delta_n$$

we obtain a Baire measure on X which is positive on open sets. The results of [22] show that the measures μ_x on S_x can be chosen to be $G\{x\}$-invariant and positive on open sets. By Theorem 3.2 we can use μ as defined above to obtain a G-invariant measure m, where $m = \int_X \mu_x\, d\mu(x)$, and we claim that $m(O) > 0$ for each non-empty open set $O \subset S$. To prove this, let $s \in O$ and let $x = p(s)$. Then $p(O)$ is a neighborhood of x since p is open, see [20], Theorem 1. The action of G restricts to a continuous function $(.) : \pi^{-1}(x) \times S_x \to S$, so on using local compactness of G and of S, there are compact neighborhoods U of the identity, I_x, in $\pi^{-1}(x)$ and B of s in S_x such that, if $W = \pi'(U)$, then

$$U.B = (.)(U \times B) \subset O \quad \text{and} \quad W \subset p(O).$$

Moreover, W is a neighborhood of x and $U.B$ is compact. Therefore, $U.B$ is a Baire set since S is metrisable. Now choose $r_z \in G(x,z)$ for each $z \in W$ in such a way that $r_z \in U$ and, in particular, $r_x = I_x$. Then the set $V = \bigcup_{z \in W} r_z . B$ is contained in $U.B$ and has the property $\mu_z(V) = \mu_x(B)$ for all $z \in W$. Therefore,

$$m(O) \geq m(U.B) = \int_X \mu_z(U.B)\, d\mu(z)$$
$$\geq \int_X \mu_x(B)\, d\mu(z) = \mu(W)\, \mu_x(B) > 0,$$

since W and B have non-empty interiors. Thus, m has positive value on non-empty open sets and the proof is complete.

3.5. This theorem actually contains Fubini's theorem for the product of two locally compact Polish spaces X and Y as we will now briefly demonstrate. In fact, we may suppose that X and Y are compact by use of Bourbaki's localisation principle [2]. Thus let μ be a Baire measure on X and ν be a Baire measure on Y. Let S be $X \times Y$ and p the projection on the first factor, and take for G the product $X \times X$, thus

$$G(x,x') = \{(x,x')\}$$

with the obvious law of composition. Then G acts on S in accordance with

$$(x,x').(x,y) = (x',y).$$

Thus, if $\alpha = (x,x')$, then $\phi_\alpha : \{x\} \times Y \to \{x'\} \times Y$ behaves as the identity on the factor Y. For each $x \in X$ let μ_x be the image of ν by the map $y \mapsto (x,y)$ and set $m = \int_X \mu_x\, d\mu(x)$. Then m is G-invariant and

$$m(A \times B) = \mu(A)\,\nu(B) \quad \text{for Baire sets } A \subset X,\ B \subset Y.$$

Therefore, $m = \mu \times \nu$.

Now if $(x,y) \in S$, then the orbit $G.(x,y)$ coincides with the set $X \times \{y\}$ and can be identified with X. This means that, as a set, S/G can be identified with Y and q with the projection on the second factor. However, q is open and so is a quotient map which implies that $S/G = Y$ as topological spaces. It is obvious that ν on Y is a pseudo-image of m by q. Applying Theorem 3.4, we obtain $\{\lambda_y \mid y \in Y\}$ and by a simple argument using the uniqueness there, we conclude that for ν-almost all y, $\lambda_y = \mu$ under the identification of each orbit with X. The conclusion of Theorem 3.4 now asserts that

$$\int_{X\times Y} f\, dm = \int_X \int_Y f\, d\nu\, d\mu = \int_Y \int_X f\, d\mu\, d\nu$$

for each m-integrable function f on $X \times Y$, which is Fubini's Theorem.

Theorems generalising Fubini's can be found in [9] and [12]. They are more like our Theorem 1.6 and do not involve interchange of the order of integration. Indeed, it is not clear what Y should be if one does not consider G and its action on S.

The next theorem is complementary to Theorem 2.1 a. Here the metrizability restrictions on S and X are removed and exchanged for the conditions on G of 3.1. This has an immediate corollary in that Theorem 2.4 remains valid under the same exchange.

3.6. THEOREM. *Let G and S satisfy 3.1, where S and X are supposed only to be locally compact Hausdorff spaces. Suppose $\{\mu_x\}$ and $\{\mu'_x\}$ are two G-invariant families on S and*

$$m = \int_X \mu_x\, d\mu(x) = \int_X \mu'_x\, d\mu(x)$$

for some measure μ on X. Then $\mu_x = \mu'_x$ for all x in the object set of any transitive component on which μ is non-trivial.

PROOF. It suffices to consider the case when G is transitive.

Suppose $\mu_x \neq \mu'_x$ for some x and let C be a compact G_δ in S_x with $\mu_x(C) \neq \mu'_x(C)$. Assume

$$\mu_x(C) > \mu'_x(C) \text{ and let } \eta = \mu_x(C) - \mu'_x(C).$$

By regularity there is an open Baire set U in S such that $C \subset U$ and

$$\mu_x(U) < \mu_x(C) + \frac{\eta}{2}, \quad \mu'_x(U) < \mu'_x(C) + \frac{\eta}{2}.$$

There exists a compact neighborhood V of C in S such that $C \subset V \subset U$

and by Urysohn's Lemma a function $f \in \mathfrak{k}(S)$ such that

$$f \equiv 1 \text{ on } C \quad\text{and}\quad f \equiv 0 \text{ on } S \setminus V.$$

Then we have

$$0 < \mu_x(C) \leq \int_S f\, d\mu_x \leq \mu_x(U) < \mu_x(C) + \frac{\eta}{2}$$

and

$$0 \leq \mu_x'(C) \leq \int_S f\, d\mu_x' \leq \mu_x'(U) < \mu_x'(C) + \frac{\eta}{2}.$$

Therefore

$$\Phi(x) = \int_S f\, d\mu_x - \int_S f\, d\mu_x' \geq \frac{\eta}{2}.$$

But Φ is continuous by Theorem 3.2 and so there is a neighborhood W of x in X, which may be taken to be a Baire set, such that for all $y \in W$ we have

$$\Phi(y) = \int_S f\, d\mu_y - \int_S f\, d\mu_y' \geq \frac{\eta}{4}.$$

Consequently,

$$\int_W \Phi(y)\, d\mu(y) \geq \frac{\eta}{4}\mu(W).$$

Therefore,

$$\int_W \int_{S_y} f\, d\mu_y\, d\mu(y) \geq \int_W \int_{S_y} f\, d\mu_y'\, d\mu(y) + \frac{\eta}{4}\mu(W).$$

That is,

$$\int_{p^{-1}(W)} f\, dm \geq \int_{p^{-1}(W)} f\, dm + \frac{\eta}{4}\mu(W),$$

using Theorem 1.6. If y is any other point of X, we may choose $a \in G(x,y)$ and, by using the invariance of $\{\mu_x\}$ and $\{\mu_x'\}$, we can apply the above argument to each point $y \in X$. In this way we obtain, for each $y \in X$, a neighborhood W_y of y and a function $f^y \in \mathfrak{k}(S)$ such that

$$\int_{p^{-1}(W_y)} f^y\, dm \geq \int_{p^{-1}(W_y)} f^y\, dm + \frac{\eta}{4}\mu(W_y).$$

But at least one W_y has positive measure, since μ is always supposed non-trivial, and for such W_y the inequality above gives a contradiction. This completes the proof.

3.7. COROLLARY. *Suppose G, S and S' satisfy the condition of 3.1. Let $f: S \to S'$ be G-equivariant and a proper map. Suppose finally that $\{\mu_x\}$ and $\{\mu_x'\}$ are G-invariant families on S and S' respectively. Let μ be a measure on X which is non-trivial on each transitive component in X and* *let m and m' be the corresponding G-invariant measures. Then f is m, m' measure preserving iff f_x is μ_x, μ'_x measure preserving for all $x \in X$.*

PROOF. The proof is exactly the same as the proof of Theorem 2.4 noticing only that the measures μ''_x as defined there form a G-invariant family since f is equivariant, and use is made of Theorem 3.6 instead of Theorem 2.1 a.

4. FIBRE BUNDLES.

In this section we will interpret the construction of fibre bundles, due to Cartan [4], within the framework of transformation groupoids *). The results of the earlier sections will then be applied to construct measures in their total space, and certain of the results of [9] will be recovered by imposing the condition of local triviality.

Suppose E is a principal H-space, where H is a topological group acting continuously on the right of E, and let $X = E/H$ be the orbit space endowed with the quotient topology of the orbit map $\rho: E \to X$. Contrary to the earlier sections, we are not at present supposing that E, H and X are locally compact Hausdorff spaces. Then ρ is both continuous and open and E is a principal bundle over X with projection ρ, see [4] or [11]. Let E_x denote the fibre over x. We make the following definition.

4.1. DEFINITION. Let $x, y \in X$. A map $\omega: E_x \to E_y$ will be called *admissible* if ω is an H-map, that is,

$$\omega(e.h) = \omega(e).h \quad \text{for all } e \in E_x \text{ and } h \in H.$$

An admissible map is a homeomorphism and we denote by $\mathcal{G}(E)$ the groupoid of all admissible maps between the fibres of E. We have the following elementary fact.

4.2. PROPOSITION. *Let $x, y \in X$ and $e' \in E_x$, $e'' \in E_y$. Then there exists a unique admissible map $\omega: E_x \to E_y$ such that $\omega(e') = e''$.*

*) See «Catégories topologiques et catégories différentiables» by Charles Ehresmann, *Coll. Géo. Diff. Glo. Bruxelles* (1959), 137-150, for similar results in the case where local triviality is assumed.

Thus $\mathcal{G}(E)$ is transitive and an admissible map ω is uniquely determined by any pair (e', e'') such that $\omega(e') = e''$. Let H act on the right of $E \times E$ according to

$$(e', e'').h = (e'.h, e''.h).$$

Then $E \times E$ is a principal H-space and we may form the space $\hat{E} = (E \times E)/H$ of orbits of this action endowed with the quotient topology of the orbit map $\hat{\rho}: E \times E \to \hat{E}$. Then $\hat{E}$ is a topological groupoid over X. Proposition 4.2 implies that there is an identification $\Gamma: \hat{E} \to \mathcal{G}(E)$ defined by setting $\Gamma([e', e''])$ to be the unique admissible map ω such that $\omega(e') = e''$, where $[e', e'']$ denotes the orbit of (e', e''). Thus $\mathcal{G}(E)$ becomes a topological groupoid over X and, moreover, $\pi': \pi^{-1}(x) \to X$ is an open map for each $x \in X$. There is, too, an action of $\mathcal{G}(E)$ on E defined by evaluation, that is $\omega . e = \omega(e)$. This action is continuous and we will prove a more general result concerning fibre bundles later. The facts stated here lead to a natural equivalence between certain categories of topological groupoids and of principal bundles. In particular one has the following

4.3. PROPOSITION. *Two principal H-bundles E and E' are isomorphic iff $\mathcal{G}(E)$ and $\mathcal{G}(E')$ are isomorphic topological groupoids.*

Next suppose H acts continuously on the left of a space F, though the action is not supposed here to be effective or faithful, etc, and F need not be locally compact Hausdorff. Then H acts on the right of $E \times F$ according to

$$(e, f).h = (e.h, h^{-1}.f).$$

Let S be the orbit space $(E \times F)/H$ of this action endowed with the quotient topology of $E \times F$ by the orbit map $\delta: E \times F \to S$. Finally, define

$$p: S \to X \quad \text{by} \quad p([e, f]) = \rho(e),$$

where $[e, f]$ is the orbit of (e, f) in $E \times F$. Then p is continuous and open, and S is the fibre bundle over X with fibre F, group H and associated principal bundle $\rho: E \to X$ as defined by Cartan [4].

We are going to define an action of $\mathcal{G}(E)$ on S. First, however, it is necessary to recall the concept of the translation function $r: E^* \to H$, where

$$E^* = \{(e, e.h) \mid e \in E, h \in H\} \subset E \times E.$$

The function r is defined by $r((e, e.h)) = h$ and is continuous. Here, $\mathcal{G}(E) \times_X S$ is the set

$$\{(\omega, [e, f]) \mid \pi(\omega) = p([e, f]) = \rho(e)\},$$

and the desired action is defined by

$$\omega.[e, f] = [e', e''].[e, f] = [e'', r(e', e).f],$$

where we identify ω with $[e', e'']$ using Γ. One readily shows that $(.)$ is a well defined action and continuity follows from the following commutative diagram of fibred products :

$$\begin{array}{ccc} (E \times E) \times_X (E \times F) & \xrightarrow{\quad \theta \quad} & E \times F \\ {\scriptstyle \hat{\rho} \times \delta} \downarrow & & \downarrow {\scriptstyle \delta} \\ \hat{E} \times_X S & \xrightarrow[\quad (.) \quad]{} & S \end{array}$$

Here

$$\theta((e', e''), (e, f)) = (e'', r(e', e).f)$$

and is continuous, $\hat{\rho} \times \delta$ is open and so therefore is its restriction to the $(\hat{\rho} \times \delta)$-saturated set $(E \times E) \times_X (E \times F)$. This means that the map $\hat{\rho} \times \delta$ in the diagram is a quotient map, and so $(.)$ is continuous by the universal property of quotients.

4.4. REMARK. The constructions made here do not suppose any form of local triviality and the introduction of charts and atlases will shortly be considered. However, it is worth noting that it is shown in [4] that any fibre space $p: S \to X$ which is locally trivial in the most general sense, and has locally compact fibre F, may in fact be thought of as a fibre bundle with group H as above. One takes for H the group of all homeomorphisms of F with a topology derived from the compact-open topology, due to Arens. The associated principal bundle E can be constructed directly and so S can be viewed as a $\mathcal{G}(E)$-space.

4.5. In order to apply the results of Section 3 it is necessary to assume that E, F and H are locally compact Hausdorff spaces and that H preserves a non-trivial Baire measure ν on F. It follows then that X, $\mathcal{G}(E)$ and S are all locally compact, being open continuous images of locally compact spaces. They need not be Hausdorff however. We will therefore make the assumption that they are Hausdorff. This will be the case, for example, if H is compact, or if E is locally trivial and also under suitable restrictions on the action of H, see [3].

Each fibre S_x of S may be identified with F by choosing $e \in E_x$ and defining

$$i: F \to S_x \quad \text{by} \quad i(f) = [e, f].$$

Changing e to e' - say - has the effect of operating on F by the element $r(e,e')$ of H, that is, $i'(f) = i(r(e,e').f)$. Thus, ν is carried to a well defined measure μ_x on S_x and $\{\mu_x\}$ is $\mathcal{G}(E)$-invariant. Choosing μ on X and applying Theorem 3.2, we obtain a $\mathcal{G}(E)$-invariant measure

$$m = \int_X \mu_x \, d\mu(x).$$

It follows that m is uniquely determined by and uniquely determines ν for a fixed μ.

A more intuitive description of m is available if S is locally trivial. In fact, if $\{V_j, \phi_j\}$ is an atlas for S, then as shown in [17], m has the property

$$m(\phi_j(A)) = (\mu \times \nu)(A)$$

for each j and each Baire set $A \subset V_j \times F$. Thus, m is «the product of μ and ν in the fibre bundle S» as defined in [9] by Goetz, and called a *local product measure* here.

4.6. EXAMPLE (Foliated bundles).

Let $\xi = (p, E, M)$ be a C^1-bundle over M as in [10] or [13] and suppose $\mathcal{F}$ is a foliation of ξ, thus the leaves of $\mathcal{F}$ foliate E and are transverse to the fibres with complementary dimension. Suppose E has compact fibres and let V denote the representative fibre of E. Let $y \in E$, then, as shown by Ehresmann, the leaf $\mathcal{F}_y$ through y has a unique topology making $p: \mathfrak{F}_y \to M$ a covering space. Let $\lambda : [0, 1] \to M$ be a path in M from a to b and define

$$h(\lambda): E_a \to E_b \qquad \text{by} \qquad h(\lambda)(y) = \lambda_y(1),$$

where λ_y denotes the unique path in $\mathfrak{F}_y$ starting at y which covers λ. Then $h(\lambda)$ is a homeomorphism and by the covering homotopy property homotopic paths in M from a to b yield the same homeomorphism $E_a \to E_b$; we obtain in this way a homomorphism of groupoids $h: \Pi_1(M) \to H(E)$, where $\Pi_1(M)$ denotes the fundamental groupoid of M and $H(E)$ denotes the groupoid of all homeomorphisms between the fibres of E.

Fix a base point $a \in M$, identify V with E_a, let Ω be the image under h of the group $\Pi_1(M, a)$ and give Ω the discrete topology. Then [10] E is a bundle with fibre V and group Ω acting effectively. Let $\dot{E}$ be the associated principal bundle. Then the groupoid $\mathcal{G}(\dot{E})$ is a locally trivial topological groupoid acting continuously on E and is the image $h(\Pi_1(M))$ in $H(E)$ and called the *holonomy groupoid of* $(\xi, \mathfrak{F})$.

REMARK. $\Pi_1(M)$ can be identified with the groupoid $\mathcal{G}(\hat{M})$, where $\hat{M}$ is the universal covering space of M, and is hence a topological groupoid. Moreover, the action of $\Pi_1(M)$ induced by h on E is then continuous.

An important hypothesis in [10, 13] is that Ω preserves a non-trivial Baire measure ν on V, in which case certain homomological and cohomological conclusions are made. This hypothesis is equivalent to the existence of $\mathcal{G}(\dot{E})$-invariant measures m on E which are locally the product of a measure μ on M and ν. Moreover, μ can be chosen arbitrarily and m depends only on μ and ν.

Now form the leaf space $Q = E/\mathfrak{F}$ with projection $\omega: E \to Q$ and suppose Q is standard as a Borel space and that ω is Borel measurable. This is the case, for example, if Q is Hausdorff and each leaf is compact, see [5]. Applying Rohlin's theorem [14] we can disintegrate m relative to $\omega(m)$ on Q and obtain measures ν_L on each leaf $L \in \mathfrak{F}$. It would be interesting to relate m and $\{\nu_L \mid L \in \mathfrak{F}\}$ to the families of measures on the leaves arising from leaf diffusion processes and Markov processes [6].

5. TWO CLASSIFICATION THEOREMS.

Given $p: S \to X$ and a category or groupoid G of operators on S, the question arises of giving usable criteria to determine when an arbitrarily given measure m in S is G-invariant. A more specific related problem is the following: given a non-trivial G-invariant family $\{\mu_x\}$ on S and non-trivial measure m on S, find criteria which imply the existence of μ on X such that $m = \int_X \mu_x \, d\mu(x)$. This question will be settled, next, in an important case.

CONDITION C *(due to Goetz [9]). For any two Baire sets E and E' in S, if $\mu_x(E) = k\mu_x(E')$ for all $x \in X$, then $m(E) = km(E')$, where k is any positive real number.*

5.1. THEOREM. *Suppose G is a locally trivial topological groupoid acting continuously on S. Then $m = \int_X \mu_x \, d\mu(x)$, and hence is G-invariant, for some μ on X iff m and $\{\mu_x\}$ satisfy Condition C.*

PROOF. The necessity is clear.

Conversely, let $\{U_j, \lambda_j\}$ be a local trivialisation for G. We may suppose G is transitive and choose a base point $z \in X$. Thus, $\lambda_j: U_j \to G$ is continuous for each j and $\lambda_j(x) \in G(z, x)$ for each $x \in U_j$. For each index j define

$$\phi_j : U_j \times S_z \to p^{-1}(U_j) \quad \text{by} \quad \phi_j(x, s) = \lambda_j(x).s.$$

Then the collection $\{\phi_j, U_j\}$ gives an atlas for a locally trivial fibre bundle structure on S with structure group $G\{z\}$ and transition functions

$$g_{ji}(x) = \lambda_j(x)^{-1} \lambda_i(x) \quad \text{on} \quad U_i \cap U_j.$$

It is easy to see that a measure m on S is G-invariant iff it is the product in S of μ_z and some measure μ on X. Hence, if m and $\{\mu_x\}$ satisfy Condition C, then [9], Theorem 2, shows that there exists μ on X such that $m = \int_X \mu_x \, d\mu(x)$. Indeed, μ is uniquely determined as follows: for any Baire set $A \subset U_j$,

$$\mu(A) = \frac{m(\phi_j(A \times E))}{\mu_z(E)}$$

for any Baire set $E \subset S_z$ such that $\mu_z(E)$ is positive and finite.

5.2. REMARKS.

1° In [9] it is shown that Condition C is quite accessible and can be interpreted in the special case of the bundle $B \to B/H$, where B is a group and H a closed subgroup, as Weil's necessary and sufficient condition $\Delta(g) = \delta(g)$ for B-invariant measures in the coset space B/H.

2° Pepe's work [12] does not shed any more light on G-invariant measures than does Theorem 5.1 and it is assumed there that S is locally trivial throughout. Let $p: S \to X$ be a locally trivial fibre bundle with fibre F. Pepe associates with S a bundle $\hat{p}: \mathfrak{M} \to X$ of Borel regular measures whose fibre consists of the set $\mathfrak{M}(F)$ of Borel regular measures on F. This construction uses known results to be found in [23], Section 3. The basic idea in [12] is that a section $\sigma: X \to \mathfrak{M}$ associates to each $x \in X$ a measure $\sigma(x)$ supported on $p^{-1}(x)$, and that evaluation on such a section describes a wide class of measures on S. There is some hope of extending these ideas to the setting of C*-algebra bundles as described elsewhere by Dauns, Hofmann and Fell. In so doing, one aims to obtain integral representations of «functionals» defined there, and it is hoped to pursue this elsewhere.

5.3. DEFINITION. Suppose H is a topological group acting continuously on S and on X. Then $p: S \to X$ is called an *H-fibre space* if p is equivariant, that is,

$$p(h.s) = h.p(s) \quad \text{for} \quad h \in H \quad \text{and} \quad s \in S.$$

For example, an H-vector bundle as defined in K-theory.

Let $G = H \times X$ be the associated topological groupoid over X, thus

$$G(x,y) = \{ (h,x) \mid h.x = y \}.$$

Actions of G on S via p correspond in 1-1 fashion with actions of H on S and X such that p is equivariant according to the relations

$$h.s = (h, p(s)).s, \quad (h,x).s = h.s.$$

The final result relates G-invariant measures on S to H-invariant measures.

5.4. THEOREM. *Suppose* $\{\mu_x\}$ *is a G-invariant family on S which is μ-integrable, where μ is an H-invariant measure on X. Then* $m = \int_X \mu_x \, d\mu(x)$ *is an H-invariant measure on S. Conversely, suppose that H is separable, S and X are Polish spaces and p is a proper map. Given an H-invariant measure m on S, there exists a G-invariant family* $\{\mu_x \mid x \in X\}$ *on S and an H-invariant measure μ on X such that* $m = \int_X \mu_x \, d\mu(x)$*, and therefore m is G-invariant.*

PROOF. For the first part, a direct calculation shows that $m(h.E) = m(E)$ for each $h \in H$ and each Baire subset E of S, that is, m is H-invariant.

For the converse, let $\mu = p(m)$, thus $\mu(A) = m(p^{-1}(A))$ for each Baire set A in X and it is clear that μ is H-invariant. Applying Bourbaki's disintegration theorem [2], Chapter 6, Section 3 n°1 Theorem 1, once more, we can write $m = \int_X \mu_x \, d\mu(x)$ for a μ-almost everywhere uniquely determined family $\{\mu_x \mid x \in X\}$. Let $\{h_n\}$ be a sequence of elements in H. Then $m(h_n . E) = m(E)$ for each Baire set E and each positive integer n, and so

$$\int_X \mu_x(h_n . E) \, d\mu(x) = \int_X \mu_x(E) \, d\mu(x).$$

For each n define Baire measure μ_x^n on S_x by

$$\mu_x^n(E) = \mu_y(h_n . E), \quad \text{where} \quad y = h_n . x.$$

Then we have

$$m(h_n . E) = \int_X \mu_y(h_n . E) d\mu(y) = \int_X \mu_x^n(E) \, d\mu(x)$$

on using the H-invariance of μ, and therefore

$$\int_X \mu_x^n \, d\mu(x) = \int_X \mu_x \, d\mu(x).$$

So by essential uniqueness of the disintegration of m, there is a μ-null set $N(n) \subset X$ such that $\mu_x^n = \mu_x$ whenever $x \in X \setminus N(n)$, or in other words,

$$\phi_{(h_n, x)}(\mu_x) = \mu_{h_n . x},$$

where $\phi_{(h,x)}$ denotes the operation of the element $g = (h, x) \in G$. Consequently, this last equality holds for all n and all x in the complement of the μ-null set $\bigcup_{n=1}^{\infty} N(n) = N$. But H is separable and so $\{h_n\}$ may be chosen to be dense in H, in which case a straightforward continuity argument implies $\phi_{(h,x)}(\mu_x) = \mu_{h.x}$ for all $h \in H$ and all x in the complement of N. Thus, we can assume this equality for all x and then $\{\mu_x\}$ is G-invariant, which completes the proof.

REMARK. Let Γ be a pseudogroup of local isomorphisms of a fibre space $p: S \to X$, and let G be the sheaf of germs of elements of Γ. Then G is a topological groupoid which acts continuously on S. By techniques similar to those used in proving Theorem 5.4, one might hope to relate Γ-invariant measures in S to G-invariant ones and indeed some partial results can be obtained thus. It is hoped to treat these questions more fully elsewhere, particularly those concerned with invariant measures for holonomy pseudogroups of foliations.

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