Marta Bunge

Robert Pare

Stacks and equivalence of indexed categories

<http://www.numdam.org/item?id=CTGDC_1979__20_4_373_0>

STACKS AND EQUIVALENCE OF INDEXED CATEGORIES

by Marta BUNGE and Robert PARE [1]

0. INTRODUCTION.

The purpose of this paper is to study the various notions of equivalence of indexed categories and how they relate to Giraud's stacks [4 and 5].

This research was motivated by the study of Morita equivalence for category objects in a topos and its relationship with stack completions (the results of which appear in [2]). First of all, the stack completion of an internal category may not be internal, and we are forced to consider indexed categories. Secondly, various notions of equivalence of categories appear (because of the absence of the axiom of choice) and are closely related to the notion of stack. In order to properly understand stacks it is first necessary to sort out the properties of equivalence of categories. This is done in Section 1.

In Section 2 we then give the basic properties of stacks and their relationship with equivalence of categories. Some examples are given in Section 3.

The results of this paper are necessary for [2], but we believe they are also of independent interest.

Giraud's notion of stack is given relative to a base site. This notion is not intrinsic to the topos of sheaves on the site, in the sense that two sites may have equivalent categories of sheaves but non-equivalent categories of stacks. To make the notion independent of the site, we consider only stacks relative to the regular epimorphism topology on the topos itself. Although most applications are for toposes, we only need the base category to be a regular category [1]. Since this seems to be the right level of generality and the proofs are no more difficult, we work in this context.

1. WEAK AND LOCAL EQUIVALENCE.

Let $\underline{S}$ be a finitely complete regular category in the sense of [1], i.e., $\underline{S}$ has finite limits and regular coimage factorizations that are stable under pullback. Let $\underline{A}$ and $\underline{B}$ be $\underline{S}$-indexed categories (see [7]).

(1.1) DEFINITION. An $\underline{S}$-indexed functor $F: \underline{A} \to \underline{B}$ will be called a *weak equivalence functor* if

(i) for each $I \epsilon \,|\, \underline{S}|$, $F^I: \underline{A}^I \to |\underline{B}^I|$ is fully faithful, (ii) for each $I \epsilon \,|\, \underline{S}|$ and $b \epsilon \,|\, \underline{B}^I|$, there exist a regular epi $\alpha: J \twoheadrightarrow I$ in $\underline{S}$, $a \epsilon \,|\, \underline{A}^J|$, and an isomorphism $\theta: F^J a \to \alpha^* b$ in $\underline{B}^J$.

By the *2-pullback* of the following diagram of categories

$$
\begin{array}{ccc}
 & & \underline{B}\,' \\
 & & \downarrow G \\
\underline{A} & \xrightarrow{\ F\ } & \underline{B}
\end{array}
$$

we mean the category $\underline{A}\,'$ whose objects are triples (a, b', θ) where

$$a \epsilon \,|\, \underline{A}|, \quad b' \epsilon \,|\, \underline{B}\,'| \quad \text{and} \quad \theta: F a \to G b' \text{ is an isomorphism in } \underline{B},$$

and whose morphisms are pairs of morphisms making the obvious squares commute. We have functors $F': \underline{A}\,' \to \underline{B}\,'$ and $G': \underline{A}\,' \to \underline{A}$ defined by

$$F'(a, b', \theta) = b' \quad \text{and} \quad G'(a, b', \theta) = a,$$

and a natural isomorphism

$$t: F G' \to G F' \quad \text{defined by} \quad t(a, b', \theta) = \theta.$$

There is an obvious universal property, which need not concern us here. This whole discussion extends easily to the case of indexed categories.

(1.2) LEMMA. *Weak equivalences are closed under composition, and stable under 2-pullbacks.*

PROOF. If $F: \underline{A} \to \underline{B}$ and $G: \underline{B} \to \underline{C}$ are weak equivalence functors, then so is $G F: \underline{A} \to \underline{C}$. Firstly, since F^I and G^I are fully faithful for each I, so is $G^I F^I$. Secondly, given I and $c \epsilon \,|\, \underline{C}^I|$, there exist

$$\alpha: J \twoheadrightarrow I \text{ and } b \epsilon \,|\, \underline{B}^J| \text{ with } \theta: G^J b \to \alpha^* c \text{ an isomorphism,}$$
$$\beta: K \twoheadrightarrow J, \; a \epsilon \,|\, \underline{A}^K| \text{ with } \psi: F^K a \to \beta^* b \text{ an isomorphism.}$$

Hence, we have $\alpha\beta : K \twoheadrightarrow I$ and $a \in |\underline{A}^K|$, with

$$G^K F^K a \xrightarrow{\ G^K \psi\ } G^K \beta^* b \xrightarrow{\ \approx\ } \beta^* G^J b \xrightarrow{\ \beta^* \theta\ } \beta^* \alpha^* c \xrightarrow{\ \approx\ } (\alpha\beta)^* c$$

an isomorphism, where the unlabeled arrows are canonical isos.

Assume that

is a 2-pullback of $\underline{S}$-indexed categories and that F is a weak equivalence functor. Let (a, b', θ) and $(\hat{a}, \hat{b}', \hat{\theta})$ be any two objects in $\underline{A}^{'I}$ where I is some object of $\underline{S}$. For any $v : b' \to \hat{b}'$ there exists a unique w making

commute, since $\hat{\theta}$ is an isomorphism and F^I is full and faithful. Thus $F^{'I}$ is full and faithful. Also, for any $b' \in |B^{'I}|$ there exist

$$\alpha : J \twoheadrightarrow I \quad \text{and} \quad a \in |\underline{A}^J| \quad \text{and an isomorphism } \theta : F^J a \to \alpha^* G^I b'.$$

This gives an object

$$(a, \ \alpha^* b', \ \theta' : F^J a \xrightarrow{\ \theta\ } \alpha^* G^I b' \approx G^J \alpha^* b')$$

of $\underline{A}^{'J}$ and $F^{'J}(a, \alpha^* b', \theta') = \alpha^* b'$. Thus F' is a weak equivalence functor. $\square$

(1.3) DEFINITION. Given $\underline{S}$-indexed categories $\underline{A}$, $\underline{B}$, say that they are *weakly equivalent* (and write $\underline{A} \underset{w}{\equiv} \underline{B}$) if there exist weak equivalence functors as in

$$\underline{A} \xleftarrow{\ F\ } \underline{E} \xrightarrow{\ G\ } \underline{B} .$$
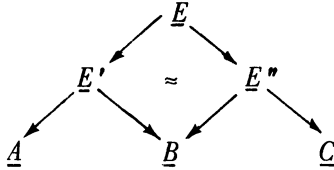
for some $\underline{S}$-indexed category $\underline{E}$.

(1.4) PROPOSITION. *Weak equivalence is an equivalence relation.*

PROOF. If $\underline{A} \underset{w}{\equiv} \underline{B}$ and $\underline{B} \underset{w}{\equiv} \underline{C}$ then we have weak equivalence functors

$$\underline{A} \longleftarrow \underline{E}' \longrightarrow \underline{B} \quad \text{and} \quad \underline{B} \longleftarrow \underline{E}'' \longrightarrow \underline{C}.$$

Forming the 2-pullback and composing in

gives transitivity. The relation is trivially reflexive and symmetric. □

If C is an internal category in $\underline{S}$, its externalization [C] is an $\underline{S}$-indexed category. For such special categories, we wish to know the internal meaning of weak equivalence functors. For this, we need the following construction: For any internal category C we have the object of isomorphisms of C :

$$Iso\,(C) \; \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} \; C_0$$

with the domain and codomain morphisms. This is easily constructed as a finite limit from the data of C.

(1.5) PROPOSITION. *Let* C *and* D *be internal categories in* $\underline{S}$, *and let* $F : C \to D$ *be an internal functor. Then the corresponding* $\underline{S}$-*indexed functor* $[F] : [C] \to [D]$ *is a weak equivalence iff*

(i)

$$\begin{array}{ccc}
C_1 & \overset{F_1}{\longrightarrow} & D_1 \\
{\scriptstyle \langle \partial_0, \partial_1 \rangle} \downarrow & & \downarrow {\scriptstyle \langle \partial_0, \partial_1 \rangle} \\
C_0 \times C_0 & \overset{F_0 \times F_0}{\longrightarrow} & D_0 \times D_0
\end{array}$$

is a pullback, i. e., F *is internally full and faithful, and*

(ii) in

$$\begin{array}{ccccc}
C_0 \underset{D_0}{\times} Iso\,(D) & \overset{p_1}{\longrightarrow} & Iso\,(D) & \overset{\partial_1}{\longrightarrow} & D_0 \\
\downarrow & & \downarrow {\scriptstyle \partial_0} & & \\
C_0 & \overset{F_0}{\longrightarrow} & D_0 & &
\end{array}$$

$\partial_1 p_1$ *is a regular epimorphism.*

PROOF. (i) The diagram in (i) is a pullback iff for each I in S,

$$\begin{array}{ccc} [I, C_1] & \longrightarrow & [I, D_1] \\ \downarrow & & \downarrow \\ [I, C_0] \times [I, C_0] & \longrightarrow & [I, D_0] \times [I, D_0] \end{array}$$

is a pullback. This means exactly that for every pair of objects c_1, c_2 in $[C]^I$ and every morphism $\delta: [F]^I(c_1) \to [F]^I(c_2)$ in $[D]^I$, there exists a unique morphism

$$\gamma: c_1 \to c_2 \quad \text{in} \quad [C]^I \quad \text{such that} \quad [F]^I(\gamma) = \delta,$$

that is to say $[F]^I$ is full and faithful.

(ii) The morphism $\partial_1 p_1$ is a regular epimorphism iff for every $d: I \to D_0$ there exist a regular epi $a: J \twoheadrightarrow I$ and $J \to C_0 \underset{D_0}{\times} Iso(D)$ such that

$$\begin{array}{ccc} J & \xrightarrow{\quad a \quad} & I \\ \downarrow & & \downarrow{d} \\ C_0 \underset{D_0}{\times} Iso(D) & \xrightarrow{\partial_1 p_1} & D_0 \end{array}$$

commutes (if $\partial_1 p_1$ is a regular epi, take $a = d^*(\partial_1 p_1)$; if the property holds, take $d = 1_{D_0}$ to see that $\partial_1 p_1$ is a regular epi). Thus, $\partial_1 p_1$ is a regular epimorphism iff for every object d of $[D]^I$ there exist a regular epi $a: J \twoheadrightarrow I$, an object c of $[C]^J$ and an iso $\theta: [F]^J c \to a^* d$. □

Note that according to our definition, internal categories C and D are considered weakly equivalent whenever there exist an indexed category $\underline{E}$ and weak equivalence functors

$$[C] \longleftarrow \underline{E} \longrightarrow [D],$$

where $\underline{E}$ need not be small.

(1.6) PROPOSITION. *With notation above, if* C *and* D *are weakly equivalent, then the* $\underline{E}$ *may be chosen to be small.*

PROOF. Let $F: \underline{E} \to [C]$ and $G: \underline{E} \to [D]$ be weak equivalence functors.

Since F is full and faithful and $[C]$ has small homs, $\underline{E}$ also has small homs. Let $g \in |[C]^{C_0}|$ be the generic family of objects of C (i.e., $id: C_0 \to C_0$). Then there exist a regular epi $\gamma: C' \twoheadrightarrow C_0$, an object $e_0 \in |\underline{E}^{C'}|$ and an iso $\theta: F^{C'}e_0 \to \gamma^*g$. Consider $Full(e_0)$, the full subcategory of $\underline{E}$ generated by the family e_0 (see [7], Section 2.1). By Corollary 3.11.2 of [7], $Full(e_0)$ is small.

The inclusion $Full(e_0) \to \underline{E}$ is full and faithful by construction. We wish to show that it is a weak equivalence functor. Let $e \in |\underline{E}^I|$, then $F^I e \in [C]^I$ and so there exists a unique $f: I \to C_0$ such that $f^*(g) = F^I e$. Taking the pullback

$$\begin{array}{ccc}
J & \xrightarrow{\quad a \quad} & I \\
f' \downarrow & & \downarrow f \\
C' & \xrightarrow{\quad \gamma \quad} & C_0
\end{array}$$

we get a regular epi $a: J \twoheadrightarrow I$. Then

$$F^J f'^* e_0 \approx f'^* F^{C'} e_0 \xrightarrow{f'^*(\theta)} f'^* \gamma^*(g) \approx a^* f^*(g) \approx a^* F^I e \approx F^J a^* e$$

is an isomorphism, and since F^J is full and faithful there exists an iso in $\underline{E}^J$, $f'^* e_0 \to a^* e$. Since $f'^* e_0$ is in $Full(e_0)^J$, the inclusion is a weak equivalence functor. $\square$

In [7], all categories were considered as having a specified subgroupoid of isomorphisms, called canonical, and the isomorphisms appearing in the definitions of indexed category and indexed functor were required to be canonical. The motivation for this was that for « large categories » such as $\underline{S}$ and categories constructed from $\underline{S}$, the substitution functors are only defined up to isomorphism while for small categories (i.e. of the form $[C]$) they are defined «up to equality», and to recover C from $[C]$ we must remember this information. Since we are only interested in equivalence of categories, we are now content to recover C up to strong equivalence (see (1.8)). Thus we deviate from the conventions of [7] and assume here that all isomorphisms are canonical. In practice this means that the isomorphisms in I, 1.1 and I, 1.2 of [7] can be arbitrary, even for categories of the form $[C]$. As a consequence, we no longer have a bijection between indexed

functors $[C] \to [D]$ and internal functors $C \to D$. However, we have the following:

(1.7) PROPOSITION. *The functor which takes an internal functor* $F : C \to D$ *to its externalization* $[F] : [C] \to [D]$ *is an (ordinary) equivalence of the category of internal functors* $C \to D$ *with the category of indexed functors* $[C] \to [D]$.

PROOF. That this functor is full and faithful follows from the fact that indexed natural transformations (see Definition (I, 1.3) in [7]) are not equipped with «structural isomorphisms» as indexed categories and functors are, and so we can apply a Yoneda Lemma argument.

In general, if G is an indexed functor with coherent isomorphisms $c_\alpha : G^J a^* \xrightarrow{\approx} a^* G^I$, and if H^I are any functors with natural isomorphisms $t^I : H^I \xrightarrow{\approx} G^I$, then H becomes an indexed functor if we let

$$c'_\alpha = (H^J a^* \xrightarrow{t^J a^*} G^J a^* \xrightarrow{c_\alpha} a^* G^I \xrightarrow{(a^* t^I)^{-1}} a^* H^I),$$

and t^I is automatically an indexed natural isomorphism. Now, if we have $G : [C] \to [D]$, define H^I as follows: for any object $[f]$ of $[C]^I$ (i.e., $f : I \to C_0$), $H^I([f]) = f^* G^{C_\bullet}([1_{C_\bullet}])$. Then H^I extends uniquely to a functor such that

$$t^I[f] = (H^I([f]) = f^* G^{C_\bullet}([1_{C_\bullet}]) \xrightarrow[c_f]{\approx} G^I(f^*[1_{C_\bullet}]) = G^I([f]))$$

is a natural isomorphism $H^I \xrightarrow{\approx} G^I$. Then by the above definition

$$c'_\alpha[f] = a^* c_f[1_{C_\bullet}] \cdot c_\alpha[f] \cdot (c_{fa}[1_{C_\bullet}])^{-1}$$
$$= a^* c_f[1_{C_\bullet}] \cdot c_\alpha f^*[1_{C_\bullet}] \cdot (c_{fa}[1_{C_\bullet}])^{-1}$$
$$= c_{fa}[1_{C_\bullet}] \cdot (c_{fa}[1_{C_\bullet}])^{-1} = 1,$$

where the third equality follows from the coherence conditions which the c_α satisfy. Since the c'_α are all identities, the Yoneda Lemma tells us that $H = [F]$ for some internal functor $F : C \to D$. □

(1.8) DEFINITION. Two indexed categories $\underline{A}$ and $\underline{B}$ are said to be *equivalent* (or *strongly equivalent*, for emphasis) if there exist indexed functors

$$A \underset{G}{\overset{F}{\rightleftarrows}} B$$

and indexed natural isomorphisms $GF \approx id_{\underline{A}}$ and $FG \approx id_{\underline{B}}$. We shall write $\underline{A} \equiv \underline{B}$ to indicate that $\underline{A}$ and $\underline{B}$ are equivalent.

This definition is equivalent to saying that there is an indexed functor $F: \underline{A} \to \underline{B}$ such that for every I, $F^I: \underline{A}^I \to \underline{B}^I$ is an ordinary equivalence functor (since we are assuming the axiom of choice in the meta-language, we can choose «inverses» $G^I: \underline{B}^I \to \underline{A}^I$, and the coherence conditions are automatic).

If $\underline{S}$ satisfies the axiom of choice (i.e., regular epis split), a weak equivalence functor is a strong equivalence functor. Indeed, if $b \epsilon |\underline{B}^I|$ there exist $\alpha: J \twoheadrightarrow I$, $a \epsilon |\underline{A}^J|$, and an isomorphism $\theta: F^J a \xrightarrow{\approx} \alpha^* b$. If $s: I \to J$ is a splitting for α, then

$$F^I s^* a \approx s^* F^J a \approx s^* \alpha^* b \approx (\alpha s)^* b \approx b,$$

and since F^I is fully faithful it is an equivalence functor. The converse also holds: if every weak equivalence functor between $\underline{S}$-indexed categories is an equivalence, then $\underline{S}$ satisfies the axiom of choice. Given a regular epi $e: B \twoheadrightarrow A$, consider the weak equivalence functor $F_e: B_e \to A$ as in the proof of Proposition (1.12) later on.

There is yet another notion of equivalence, not agreeing in general, with the notion of strong equivalence, but equivalent to it if $\underline{S}$ satisfies the axiom of choice. This is the notion of *local equivalence*, introduced by G. Wraith in [8] for $\underline{S}$-toposes and internal categories, and further studied in [9].

(1.9) DEFINITION. Two $\underline{S}$-indexed categories $\underline{A}$ and $\underline{B}$ are said to be *locally equivalent* if there exists an object U with global support (i.e., $U \twoheadrightarrow 1$ is a regular epi), such that the localizations (see [7] page 16) $\underline{A}/U$ and $\underline{B}/U$ are strongly equivalent as $\underline{S}/U$-indexed categories. We write $A \underset{l}{\equiv} B$ to indicate that $\underline{A}$ and $\underline{B}$ are locally equivalent.

If $\underline{A}/U$ is equivalent to $\underline{B}/U$, and if V is a refinement of U (i.e. there is a regular epi $V \twoheadrightarrow U$) then $\underline{A}/V$ is easily seen to be equivalent to

$\underline{B}/V$. It follows that local equivalence is an equivalence relation (for transitivity, take a common refinement of both covers). For internal categories C and D in $\underline{S}$, $[C] \underset{\overline{l}}{\equiv} [D]$ means that U^*C and U^*D are equivalent category objects of $\underline{S}/U$ for some U with global support. When $\underline{A}$ and $\underline{B}$ are $\underline{S}$-topoi, with their canonical $\underline{S}$-indexing, this definition also agrees with that of [9].

(1.10) EXAMPLE. Let I and J be objects of $\underline{S}$, and consider the corresponding discrete category objects I and J

$$(I = I \rightrightarrows I \leftleftarrows I$$

all morphisms identities). I is strongly equivalent to J iff $I \approx J$. Thus, $I \underset{\overline{l}}{\equiv} J$ means that I and J are locally isomorphic (e.g., in $Sh(S^1)$ the helix over S^1 is locally isomorphic to the constant sheaf ΔZ). For I and J to be weakly equivalent means that there exist an internal category C and weak equivalence functors

$$J \xleftarrow{\;G\;} C \xrightarrow{\;F\;} I$$

(by Proposition (1.6)). Since I is discrete, $Iso(I) = I$ and condition (ii) of Proposition (1.5) says that F_0 must be a regular epi. Condition (i) says that

$$
\begin{array}{ccc}
C_1 & \longrightarrow & I \\
{\scriptstyle <\partial_0,\partial_1>}\downarrow & & \downarrow{\scriptstyle \Delta} \\
C_0 \times C_0 & \xrightarrow{\;F_0 \times F_0\;} & I \times I
\end{array}
$$

is a pullback, and so

$$C_1 \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} C_0$$

is the kernel-pair of F_0. Therefore $I = coeq(\partial_0,\partial_1)$. Similarly

$$J = coeq(\partial_0,\partial_1) \quad \text{and so} \quad I \approx J.$$

This shows that

$$I \underset{w}{\equiv} J \qquad \text{iff} \qquad I \approx J .$$

So local equivalence does not imply weak equivalence even for internal categories. We do have the following result, however.

(1.11) PROPOSITION. *The following conditions on a regular category $\underline{S}$ are equivalent:*

    (i) Every object of $\underline{S}$ with full support has a global section.

    (ii) $\underline{A} \underset{l}{\equiv} \underline{B}$ implies $\underline{A} \equiv \underline{B}$.

PROOF. (i) $\Rightarrow$ (ii). Let U have full support and assume that $F : \underline{A}/U \to \underline{B}/U$ is an $\underline{S}/U$-indexed equivalence functor, with inverse $G : \underline{B}/U \to \underline{A}/U$. Let $u : 1 \to U$ be the element given by (i). Define an $\underline{S}$-indexed functor

$$u^*F : \underline{A} \to \underline{B} \quad \text{by} \quad (u^*F)^I(A) = F^{ul}(A),$$

where uI denotes the morphism

$$I \xrightarrow{\ \ l\ \ } 1 \xrightarrow{\ \ u\ \ } U .$$

It is easily seen that u^*F is an $\underline{S}$-indexed equivalence with inverse u^*G.

    (ii) $\Rightarrow$ (i). Assume that U has full support and let

$$U \times U \times U \rightrightarrows U \times U \leftleftarrows U$$

be the indiscrete category on U, denoted U_{ind}. Then $U_{ind} \underset{l}{\equiv} 1$ by the functors

$$U^*U_{ind} \underset{\delta}{\overset{p_1}{\rightleftarrows}} U^*1$$

given by

$$U \times U \underset{\delta}{\overset{p_1}{\rightleftarrows}} U$$

with p_1 and 1_U into U.

Thus, by (ii), $U_{ind} \equiv 1$, and so there must exist $1 \to U$. $\quad \square$

    Say that a regular category satisfies the internal axiom of choice

if for every regular epi $a : J \twoheadrightarrow I$ there exists a U with full support such that U^*a splits (i.e., regular epis split locally). We then have :

(1.12) PROPOSITION. *The following conditions on* S *are equivalent :*

(i) S satisfies the internal axiom of choice.

(ii) for every pair of internal categories C , D *in* S, *every weak equivalence functor* $F : C \to D$ *is a local equivalence functor.*

PROOF. (i) $\Rightarrow$ (ii). Let $F : C \to D$ be a weak equivalence functor. Let $I = D_0$, and let $d : I \to D_0$ be the identity. There exist

$$a : J \twoheadrightarrow I \quad \text{and} \quad c : J \to C_0$$

as well as an iso θ with

By the internal axiom of choice, $a : J \twoheadrightarrow I$ is locally split, i.e., there exists $U \twoheadrightarrow I$ with

Hence, for $U \twoheadrightarrow I$,

and so, $U \times F_0$ has a left quasi-inverse $(U \times c) \sigma$. Since $U \times F$ is internally fully faithful (as F is), it gives an equivalence $U^*C \to U^*D$.

(ii) $\Rightarrow$ (i). Let $e : B \twoheadrightarrow A$ be a regular epi in $\underline{S}$. Let B_e be the internal category in $\underline{S}$ given by the complex

383

where

$$B \underset{A}{\times} B \xrightarrow{\;\;\pi_0\;\;} B$$

$$\pi_1 \Big\downarrow \qquad PB \qquad \Big\downarrow e$$

$$B \xrightarrow[\;\;e\;\;]{} A$$

Let A be the discrete internal category on A, i.e., given by

$$A \rightrightarrows A \leftleftarrows A$$

all identity morphisms. Then, $e: B \twoheadrightarrow A$ is the object part of an internal functor $F_e : B_e \to A$, clearly a weak equivalence functor. By assumption, there exists $U \twoheadrightarrow 1$ in S, with

$$U^* B_e \underset{G_0}{\overset{U \times e}{\rightleftarrows}} U^* A$$

and G_0 «inverse» to $U \times e$. Since $U^* A$ is discrete, e splits locally. $\square$

REMARK. There is also the concept of local weak equivalence $\underline{A} \underset{wl}{\equiv} \underline{B}$, which means that there exists an object U with full support such that: $\underline{A}/U \underset{w}{\equiv} \underline{B}/U$. We shall not use this concept here.

REMARK. The following diagram illustrates the inter-relationship between the various notions of equivalence with the various axioms of $\underline{S}$, where (AC) and (IAC) denote, respectively, the axiom of choice and the internal axiom of choice, whereas ($\exists \Gamma$) denotes the axiom which states that every object of S with full support has a global section.
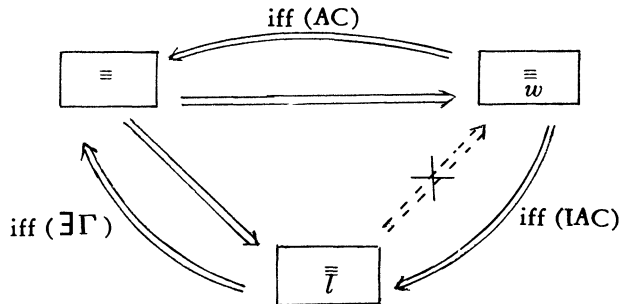
2. STACKS FOR THE REGULAR EPIMORPHISM TOPOLOGY.

In [4] and [5], J. Giraud considers the notion of stack (called *champ* in French) over a site A : a fibred category $p: \underline{F} \to A$ is a stack over A iff it satisfies an additional condition expressing that objects and morphisms satisfying descent conditions (cf. [6]) descend. Specifically $p: \underline{F} \to A$ is a stack iff, for each covering crible $\underline{R} \subset A/X$, the induced functor

$$Cart(A/X, \underline{F}) \to Cart(\underline{R}, \underline{F})$$

is an equivalence of categories. If for an object X of A, $\underline{F}_X$ denotes the fibre above X, this canonical functor can be described, roughly, as follows. A cartesian functor $A/X \to \underline{F}$ determines an object a of $\underline{F}_X$; for each $f: Y \to X$ in $\underline{R}$, f^*a is an object of $\underline{F}_Y$, and the objects obtained in this way are compatible up to isomorphism. To say that $p: \underline{F} \to A$ is a stack is to say that, given any covering $\{ f_\alpha : Y_\alpha \to X \}_{\alpha \epsilon I}$ and any family $\{ a_\alpha \}_{\alpha \epsilon I}$, of objects a_α of $\underline{F}_{Y_\alpha}$, compatible up to coherent isomorphisms, there exists an object a of $\underline{F}_X$ with

$$(f_\alpha)^*a \approx a_\alpha \quad \text{for each } \alpha \epsilon I ,$$

and this object is unique up to unique compatible isomorphisms. A similar condition must hold for maps.

Any regular category $\underline{S}$ may be regarded as a site if one takes for coverings exactly the regular epimorphisms (this is the regular epimorphism topology of [1]). We shall say explicitly what it means for an $\underline{S}$-indexed category $\underline{A}$ (which is essentially the same as a fibred category over $\underline{S}$) to be a stack over $\underline{S}$ (or just « *a stack* » from here on, as we shall consider no others). First, we need to recall some definitions of [6]. Let $\alpha: J \twoheadrightarrow I$ be a regular epimorphism in $\underline{S}$, and let

$$J \underset{I}{\times} J \underset{I}{\times} J \quad \begin{matrix} \xrightarrow{\pi_{01}} \\ \xrightarrow{\pi_{02}} \\ \xrightarrow{\pi_{12}} \end{matrix} \quad J \underset{I}{\times} J \quad \begin{matrix} \xrightarrow{\pi_0} \\ \xrightarrow{\pi_1} \end{matrix} \quad J$$

be the canonical projections arising from taking pullbacks. One has the equations:

$$\pi_0 \, \pi_{01} = \pi_0 \, \pi_{02} \, , \quad \pi_1 \pi_{02} = \pi_1 \, \pi_{12} \, , \quad \pi_0 \, \pi_{12} = \pi_1 \, \pi_{01} \, .$$

An object a of $\underline{A}^J$ has *descent data relative to* $\alpha: J \twoheadrightarrow I$ if there is given an isomorphism $\theta: \pi_0^* a \overset{\approx}{\to} \pi_1^* a$ such that the diagram

$$
\begin{array}{ccc}
\pi_{01}^* \pi_0^* \, a & \xrightarrow{\;\;\pi_{01}^* \, \theta\;\;} & \pi_{01}^* \pi_1^* a \\[2pt]
{\scriptstyle \approx} \downarrow & & \downarrow {\scriptstyle \approx} \\[2pt]
\pi_{02}^* \pi_0^* \, a & (*) & \pi_{12}^* \pi_0^* \, a \\[2pt]
{\scriptstyle \pi_{02}^* \, \theta} \downarrow & & \downarrow {\scriptstyle \pi_{12}^* \, \theta} \\[2pt]
\pi_{02}^* \pi_1^* \, a & \xrightarrow{\;\;\approx\;\;} & \pi_{12}^* \pi_1^* \, a
\end{array}
$$

commutes in $\underline{A}^{J''}$ (where the labeled isomorphisms are the canonical isomorphisms of the indexed category $\underline{A}$, and J'' denotes $J \underset{I}{\times} J \underset{I}{\times} J$). If we are given objects a, b of $\underline{A}^J$, both with descent data as above, a morphism $f: a \to b$ is said to be *compatible with the descent data* provided the diagram

$$
\begin{array}{ccc}
\pi_0^* \, a & \xrightarrow{\;\;\theta\;\;} & \pi_1^* \, a \\[2pt]
{\scriptstyle \pi_0^* f} \downarrow & & \downarrow {\scriptstyle \pi_1^* f} \\[2pt]
\pi_0^* \, b & \xrightarrow{\;\;\theta'\;\;} & \pi_1^* \, b
\end{array}
$$

commutes in $\underline{A}^{J'}$ (J' denotes $J \underset{I}{\times} J$).

For any $\hat{a} \in |\underline{A}^I|$, $\alpha^* \hat{a}$ comes equipped with descent data

$$\pi_0^* \, \alpha^* \hat{a} \approx (\alpha \pi_0)^* \hat{a} = (\alpha \pi_1)^* \hat{a} \approx \pi_1^* \, \alpha^* \hat{a}$$

(the compatibility condition above follows from the coherence conditions which the canonical isomorphisms $\psi^* \phi^* \approx (\phi \psi)^*$ satisfy). Also for any morphism $\hat{f}: \hat{a} \to \hat{b}$ in $\underline{A}^I$, $\alpha^* \hat{f}: \alpha^* \hat{a} \to \alpha^* \hat{b}$ is compatible with the descent data. Thus α^* gives a functor $\alpha^*: \underline{A}^I \to \underline{D}_\alpha$ into the category of objects equipped with descent data.

(2.1) DEFINITION. An $\underline{S}$-indexed category $\underline{A}$ is a *stack over* $\underline{S}$ (or simply a *stack*) if for every regular epimorphism $\alpha: J \twoheadrightarrow I$ of $\underline{S}$, the functor α^*, $\alpha^*: \underline{A}^I \to \underline{D}_\alpha$ is an (ordinary) equivalence functor.

(2.2) PROPOSITION. *For any S-indexed category $\underline{A}$, the following are equivalent:*

   (i) $\underline{A}$ is a stack.

   (ii) For every regular epi $\alpha: J \twoheadrightarrow I$, the canonical functor $F_\alpha: J_\alpha \to I$ (see 1.12) induces an equivalence $\underline{A}^{F_\alpha}: \underline{A}^{I_{dis}} \to \underline{A}^{J_\alpha}$ of S-indexed categories.

PROOF. The category J_α was described earlier; its defining complex is none other than the diagram

$$
J \underset{I}{\times} J \underset{I}{\times} J \quad
\begin{array}{c} \xrightarrow{\ \pi_{01}\ } \\ \xrightarrow{\ \pi_{02}\ } \\ \xrightarrow{\ \pi_{12}\ } \end{array}
\quad J \underset{I}{\times} J \quad
\begin{array}{c} \xrightarrow{\ \pi_0\ } \\ \xleftarrow{\ \delta\ } \\ \xrightarrow{\ \pi_1\ } \end{array}
\quad J
$$

involved in stating descent. An object of $\underline{A}^{J_\alpha}$ (see [7], page 27) is given by a pair (a, θ), where

$$a \in |\underline{A}^J| \quad \text{and} \quad \theta: \pi_0^* a \to \pi_1^* a \quad \text{in} \quad \underline{A}^{J'} \quad (J' = J \underset{I}{\times} J),$$

satisfying the conditions

   (i) $\delta^*(\theta) = 1_a$,

   (ii) $\pi_{12}^*(\theta)\pi_{01}^*(\theta) = \pi_{02}^*(\theta)$,

where it is understood that canonical isomorphisms are to be inserted wherever necessary for these equations to make sense. Condition (ii) is exactly (*) above, and we shall show that in the presence of (ii), (i) is equivalent to θ being an isomorphism.

   Assume that (i) and (ii) are satisfied and let $\phi: J \underset{I}{\times} J \to J \underset{I}{\times} J \underset{I}{\times} J$ be the morphism (π_0, π_1, π_0). Then

$$\pi_{01}\phi = 1 \quad \text{and} \quad \pi_{02}\phi = \delta\pi_0,$$

so applying ϕ^* to (ii) we get

$$\phi^*\pi_{12}^*(\theta)\,\phi^*\pi_{01}^*(\theta) = \phi^*\pi_{02}^*(\theta),$$

i.e.,

$$(\phi^*\pi_{12}^*(\theta))\ \theta = \pi_0^*\delta^*(\theta) = 1$$

(we leave to the reader the task of checking that the canonical isomorphisms give no problem). Repeating the same argument with $\psi = (\pi_1, \pi_0, \pi_1)$

we see that $\theta \ (\psi^* \pi^*_{0\ 1}(\theta)) = 1$, and so θ is an isomorphism.

Now, assume that θ is an isomorphism and (ii) holds. Let $d: J \to J \underset{I}{\times} J \underset{I}{\times} J$ be the diagonal and note that $\pi_{ij} d = \delta$ for all i, j. Applying d^* to (ii) we get

$$d^* \pi^*_{12}(\theta) \ d^* \pi^*_{01}(\theta) = d^* \pi^*_{02}(\theta) , \quad \text{i.e.,} \quad \delta^*(\theta) \ \delta^*(\theta) = \delta^*(\theta)$$

and since $\delta^*(\theta)$ is an isomorphism, $\delta^*(\theta) = 1_a$.

Thus an object of $\underline{A}^{J_\alpha}$ is the same as an object of $\underline{A}^J$ equipped with descent data. A morphism in $\underline{A}^{J_\alpha}$ is exactly the same as a morphism compatible with the descent data, so $\underline{A}^{J_\alpha} \equiv \underline{D}_\alpha$ (above). $\underline{A}^{I_{dis}} = \underline{A}^I$ and $\underline{A}^{F_\alpha}$ is the functor α^* of (2.1). This proves the proposition. $\square$

(2.3) PROPOSITION *(Bénabou and Roubaud [10]). Let $\underline{A}$ be an $\underline{S}$-indexed category for which Σ (or Π) exists and satisfies the Beck condition. Then $\underline{A}$ is a stack iff for every regular epi $\alpha: J \twoheadrightarrow I$, the functor $\alpha^*: \underline{A}^I \to \underline{A}^J$ is tripleable (resp. cotripleable).*

PROOF (sketched). Let T be the triple on $\underline{A}^J$ which is induced by the adjoint pair $\Sigma_\alpha \dashv \alpha^*$. There is a comparison functor Φ in

We compare $(\underline{A}^J)^T$ with the $\underline{A}^{J_\alpha}$ of (2.2) and find that they are equivalent categories; moreover, modulo this equivalence, Φ corresponds to $\underline{A}^{F_\alpha}$. The result then follows from (2.2). The case with Π follows by duality (being a stack is self-dual).

A T-algebra is a pair (a, ξ) with

$$a \in |\underline{A}^J| \quad \text{and} \quad \xi : \alpha^* \Sigma_\alpha a \to a,$$

satisfying the unit and associative laws. The Beck condition, applied to the pullback

$$\begin{array}{ccc} J' & \xrightarrow{\quad \pi_0 \quad} & J \\ {\scriptstyle \pi_1} \downarrow & & \downarrow {\scriptstyle \alpha} \\ J & \xrightarrow{\quad \alpha \quad} & 1 \end{array} \;,$$

says that the canonical morphism

$$\begin{array}{ccc} \underline{A}^{J'} & \xleftarrow{\quad \pi_0^* \quad} & \underline{A}^J \\ {\scriptstyle \Sigma_{\pi_1}} \downarrow & \longrightarrow & \downarrow {\scriptstyle \Sigma_\alpha} \\ \underline{A}^J & \xleftarrow{\quad \alpha^* \quad} & \underline{A}^I \end{array}$$

is an isomorphism. Thus ξ corresponds to

$$\Sigma_{\pi_1} \pi_0^* a \xrightarrow{\;\approx\;} \alpha^* \Sigma_\alpha a \xrightarrow{\;\xi\;} a$$

which, by the adjointness $\Sigma_{\pi_1} \dashv \pi_1^*$, corresponds to $\theta : \pi_0^* a \to \pi_1^* a$. It is now a computation to show that the unit and associative laws for ξ correspond to conditions (i) and (ii) of the proof of (2.2), thus making (a, θ) into a functor from J_α to $\underline{A}$. $\square$

(2.4) COROLLARY. *Any finitely complete exact category* $\underline{S}$, *indexed by itself in the usual way, is a stack.*

PROOF. An exact category is a regular category in which equivalence relations are kernel pairs (see [1]). For $a : J \to I$, $\underline{S}/I$ and $\underline{S}/J$ are exact categories [1, page 19] and $a^* : \underline{S}/I \to \underline{S}/J$ is exact. Since a^* has a left adjoint Σ_a and the Σ satisfy the Beck condition, we are in a position to apply (2.3). If α is a regular epi, a^* is faithful and so reflects isomorphisms. Duskin's Theorem [3, page 91] shows immediately that a^* is tripleable. $\square$

If $\underline{S}$ and $\underline{S}'$ are regular categories and $\Delta : \underline{S} \to \underline{S}'$ an exact functor, then $\underline{S}'$ can be indexed by $\underline{S}$ by defining $\underline{S}'^I = \underline{S}'/\Delta I$.

(2.5) COROLLARY. *With the above notation, if* $\underline{S}'$ *is a finitely complete exact category, then it is a stack over* $\underline{S}$.

PROOF. For a regular epi $a: J \twoheadrightarrow I$ in $\underline{S}$,

$$a^*: \underline{S}^{\prime I} \to \underline{S}^{\prime J} \quad \text{is} \quad (\Delta a)^*: \underline{S}'/\Delta I \to \underline{S}'/\Delta J.$$

Since Δa is also a regular epi, and $\underline{S}'$ has Σ, $(\Delta a)^*$ is tripleable. $\square$

(2.6) COROLLARY. *If $\underline{S}$ is a topos and $\underline{E}$ an $\underline{S}$-topos, then $\underline{E}$ is a stack over $\underline{S}$.* $\square$

We pointed out in the proof of (1.12) that, if $a: J \twoheadrightarrow I$ is a regular epi in $\underline{S}$, the internal functor $F_a: J_a \to I$ is a weak equivalence functor. Then in (2.2) we saw that to be a stack, $\underline{A}$ must have the property that $\underline{A}^{F_a}$ be an equivalence for all regular epis a. We may ask whether arbitrary weak equivalence functors do not play the same role, and indeed they do, as the following shows.

The notation $\underline{A}^{\underline{B}}$, where both $\underline{A}$ and $\underline{B}$ are indexed categories, indicates (see [7, page 61]) the internal category of indexed functors from $\underline{B}$ to $\underline{A}$. Given indexed categories $\underline{A}$, $\underline{B}$, $\underline{C}$ and an indexed functor $F: \underline{B} \to \underline{C}$, there is an indexed functor $\underline{A}^F: \underline{A}^{\underline{C}} \to \underline{A}^{\underline{B}}$. In fact, all the usual theorems about manipulating functor categories are true in the context of indexed categories. We may now prove the following characterization of stacks, by means of a statement actually proposed by A. Joyal (in public lectures, unpublished) as a definition of «stack». Remark that its contents suggest that the stacks should be viewed as the «2-sheaves» for the «2-topology» (on account of (1.2)) of weak equivalence indexed functors on $\underline{S}$-indexed Cat.

(2.7) PROPOSITION. *An indexed category $\underline{A}$ is a stack iff for every weak equivalence functor $F: \underline{B} \to \underline{C}$, the functor $\underline{A}^F: \underline{A}^{\underline{C}} \to \underline{A}^{\underline{B}}$ is an equivalence of indexed categories.*

PROOF. By the above remarks and (2.2), the condition is sufficient.

Let $\underline{A}$ be a stack and $F: \underline{B} \to \underline{C}$ a weak equivalence functor. We shall show that $\underline{A}^F$ is an equivalence at I, and the result will follow by localization since all concepts are stable under localization (see [7, page 16]). We shall construct a functor $G: \underline{A}^{\underline{B}} \to \underline{A}^{\underline{C}}$ which is «in-

verse» to $\underline{A}^F$.

Let $\Phi: \underline{B} \to \underline{A}$ be an indexed functor. We wish to construct an indexed functor

$$\Psi: \underline{C} \to \underline{A} \quad \text{such that} \quad \Psi F \approx \Phi .$$

Let $c \in |\underline{C}^I|$, then there exist a regular epi $\alpha: J \twoheadrightarrow I$, an object $b \in |\underline{B}^J|$ and an isomorphism $\phi: F^J b \stackrel{\approx}{\to} \alpha^* c$. Since $\alpha^* c$ has descent data for α and since $F^J b \approx \alpha^* c$, so has $F^J b$. Since $F^{J'}$ and $F^{J''}$ are fully faithful, b also has descent data for α . Thus $\Phi^J(b)$ has descent data and since $\underline{A}$ is a stack there exists

$$a \in \underline{A}^I \quad \text{such that} \quad \alpha^* a \approx \Phi^J(b).$$

This a is unique up to a unique isomorphism compatible with the descent data. If we choose some other α and some other b as above, by taking a common refinement of the two covers (e.g., the pullback) and using the uniqueness just mentioned, we see that there is a unique isomorphism between the new a and the old one, compatible with the respective descent data. Choose one a and define $\Psi^I(c) = a$. If $h: c \to c'$ in $\underline{C}^I$, then by choosing a common refinement, we can assume that the same $\alpha: J \twoheadrightarrow I$ works for c and c' . Thus there exist b and b' in $\underline{B}^J$ such that

$$F^J b \approx \alpha^* c \quad \text{and} \quad F^J b' \approx \alpha^* c'.$$

Since F^J is fully faithful there exists a unique $g: b \to b'$ such that

$$
\begin{array}{ccc}
F^J b & \xrightarrow{\approx} & \alpha^* c \\
F^J g \downarrow & & \downarrow \alpha^* h \\
F^J b' & \xrightarrow{\approx} & \alpha^* c'
\end{array}
$$

commutes. Furthermore, g is compatible with the descent data on b and b' . Thus $\Phi^J(g): \Phi^J(b) \to \Phi^J(b')$ is compatible with the corresponding descent data. Thus there exists a unique $f: a \to a'$ (a and a' are the objects chosen for c and c' before) such that the following diagram commutes. Define $\Psi^I(g) = f$. It is routine to check that Ψ is an indexed functor.

$$\begin{array}{ccc}
\Phi^J(b) & \xrightarrow{\approx} & \alpha^*a \\
{\scriptstyle\Phi^J(g)}\downarrow & & \downarrow{\scriptstyle\alpha^*f} \\
\Phi^J(b') & \xrightarrow{\approx} & \alpha^*a'
\end{array}$$

We define $G(\Phi) = \Psi$. In a similar manner we define $G(t)$ for in-dexed natural transformations $t: \Phi \to \Phi'$. Then

$$\underline{A}^F G(\Phi) = \underline{A}^F \Psi = \Psi F.$$

For any $b \in |\underline{B}^I|$, to calculate $\Psi^I F^I(b)$ as above, we may choose

$$\alpha = 1_I : I \to I \quad \text{since} \quad F^I b \approx I_I^* F^I b.$$

Then $\Psi^I F^I(b) = a$ for some $a \in |\underline{A}^I|$ such that $I_I^* a \approx \Phi^I(b)$, thus $\Psi^I F^I(b) \approx \Phi^I(b)$. Similarly, for morphisms $b \to b'$ and so

$$\Psi F \approx \Phi, \quad \text{i.e.,} \quad \underline{A}^F G \approx 1_{\underline{A}^B}.$$

For any indexed functor $\Psi': \underline{C} \to \underline{A}$, let

$$G(\underline{A}^F \Psi') = G(\Psi' F) = \Psi.$$

To calculate $\Psi^I(c)$ we choose, as above,

$$\alpha: J \twoheadrightarrow I \quad \text{and} \quad b \in |B^J| \quad \text{such that} \quad F^J b \approx \alpha^* c,$$

and then find the «unique» $a \in |\underline{A}^J|$ such that $\Psi'^J F^J(b) \approx \alpha^* a$. But

$$\Psi'^J F^J(b) \approx \Psi'^J(\alpha^* c) \approx \alpha^* \Psi'^I(c),$$

so $\Psi^I(c) \approx \Psi'^I(c)$. This isomorphism is natural, and so $\Psi \approx \Psi'$, which shows that $G\underline{A}^F \approx 1_{\underline{A}^C}$, once we have checked the naturality. Hence, $\underline{A}^F$ is an equivalence. □

(2.8) COROLLARY. *For any weak equivalence functor* $F: \underline{B} \to \underline{C}$ *between* $\underline{S}$-*indexed categories, where* $\underline{S}$ *is exact, the induced functor* $\underline{S}^F: \underline{S}^{\underline{C}} \to \underline{S}^{\underline{B}}$ *is an equivalence. Also, if* $\underline{S}$ *satisfies the axiom of choice, every* $\underline{S}$-*in-dexed category* $\underline{A}$ *is an* $\underline{S}$-*stack.* □

(2.9) COROLLARY. *If* $\underline{A}$ *is a stack and* $\underline{D}$ *is any indexed category, then* $\underline{A}^{\underline{D}}$ *is also a stack.*

PROOF. If $F: \underline{B} \to \underline{C}$ is a weak equivalence functor, then so is $D \times F:$ $\underline{D} \times \underline{B} \to \underline{D} \times \underline{C}$ and so $\underline{A}^{D \times F}: \underline{A}^{D \times C} \to \underline{A}^{D \times B}$ is an equivalence. But $\underline{A}^{D \times F} \approx (\underline{A}^D)^F$.   □

(2.10) DEFINITION. Let $\underline{A}$ be an $\underline{S}$-indexed category. The *associated stack*, or *stack completion*, *of* $\underline{A}$ (if it exists) is given by an indexed functor $F: \underline{A} \to \tilde{\underline{A}}$ where $\tilde{\underline{A}}$ is a stack and such that for any other stack $\underline{X}$, composition with F gives an equivalence of the category of indexed functors $\tilde{\underline{A}} \to \underline{X}$ with the category of indexed functors $\underline{A} \to \underline{X}$.

In more concrete terms, this definition means that for every indexed functor $G: \underline{A} \to \underline{X}$ ($\underline{X}$ a stack), there exists an indexed functor $H: \tilde{\underline{A}} \to \underline{X}$, unique up to a unique isomorphism, such that

$$
\begin{array}{ccc}
\underline{A} & \xrightarrow{\quad F \quad} & \tilde{\underline{A}} \\
 & {\scriptstyle G} \searrow & \big\downarrow {\scriptstyle H} \\
 & & \underline{X}
\end{array}
$$

commutes to within isomorphism. It follows that any two stack completions of $\underline{A}$ must be strongly equivalent categories.

(2.11) COROLLARY. *Let* $F: \underline{A} \to \underline{B}$ *be a weak equivalence functor between* $\underline{S}$-*indexed categories where* $\underline{B}$ *is a stack. Then* $F: \underline{A} \to \underline{B}$ *is «the» associated stack of* $\underline{A}$.

PROOF. If $\underline{X}$ is a stack, (2.7) says that $\underline{X}^F$ is an equivalence of categories and therefore an equivalence at 1. This is exactly Definition (2.10).   □

(2.12) COROLLARY. *Let* $\underline{A}$ *and* $\underline{B}$ *be indexed categories and let*

$$F: \underline{A} \to \tilde{\underline{A}} \quad and \quad G: \underline{B} \to \tilde{\underline{B}}$$

be weak equivalence functors with $\tilde{\underline{A}}$ *and* $\tilde{\underline{B}}$ *stacks. If* $\underline{A} \underset{w}{\equiv} \underline{B}$, *then* $\tilde{\underline{A}} \equiv \tilde{\underline{B}}$, *and conversely.*

PROOF. If $\underline{A} \underset{w}{\equiv} \underline{B}$, because of the weak equivalence functors

$$\underline{A} \xleftarrow{\quad H \quad} \underline{E} \xrightarrow{\quad K \quad} \underline{B},$$

then by (2.11), $FH: \underline{E} \to \tilde{\underline{A}}$ and $GK: \underline{E} \to \tilde{\underline{B}}$ are both stack completions of $\underline{E}$, so $\tilde{\underline{A}} \equiv \tilde{\underline{B}}$. □

In [2], the associated stack of any locally internal $\underline{S}$-indexed category $\underline{A}$ is constructed. From the nature of the construction it will be seen that $F: \underline{A} \to \tilde{\underline{A}}$ is always a weak equivalence. The construction also gives a large class of examples.

3. EXAMPLE.

We end up by illustrating the notions just introduced in the case where $\underline{S}$ is the topos Set^2. An internal category A in Set^2 consists of two categories and a functor $F: \underline{A}_0 \to \underline{A}_1$. An internal functor $\Phi : A \to B$ is a commutative square of functors

$$
\begin{array}{ccc}
\underline{A}_0 & \xrightarrow{\quad \Phi_0 \quad} & \underline{B}_0 \\
{\scriptstyle F} \downarrow & & \downarrow {\scriptstyle G} \\
\underline{A}_1 & \xrightarrow{\quad \Phi_1 \quad} & \underline{B}_1
\end{array}
$$

An internal natural transformation $t: \Phi \to \Phi'$ consists of a pair of natural transformations

$$t_0 : \Phi_0 \to \Phi_0' \quad \text{and} \quad t_1 : \Phi_1 \to \Phi_1' \quad \text{such that} \quad G t_0 = t_1 F.$$

All the constructions involved in Proposition (1.5) are finite limit constructions, and so are pointwise in Set^2 (i.e., are performed independently on the domain and codomain). Since epis are also pointwise, (1.5) tells us that $\Phi : A \to B$ is a weak equivalence functor iff Φ_0 and Φ_1 are ordinary equivalence functors. This does not imply that for every object I of Set^2, the externalization $[\Phi]^I$ is an equivalence functor, as the following example shows. Let

$\underline{A}_0 = \underline{B}_0 =$ discrete category with two objects,

$\underline{A}_1 =$ indiscrete category with two objects, $\underline{B}_1 = 1$,

$F: \underline{A}_0 \to \underline{A}_1$ the inclusion, $\Phi_0: \underline{A}_0 \to \underline{B}_0$ the identity,

$G: \underline{B}_0 \to \underline{B}_1$ and $\Phi_1: \underline{A}_1 \to \underline{B}_1$ the unique functor into 1 .

$$\Phi : A \to B$$

If $I = (\, 2 \to 1\,)$, then $([A]^I)_0$ is the discrete category with two objects whereas $([B]^I)_0$ is the discrete category with four objects.

If $\Phi : A \to B$ is a weak equivalence, then Φ_0 and Φ_1 have «inverses» Ψ_0 and Ψ_1 respectively, but

only commutes up to isomorphism, and so is not an internal functor. If there exist Ψ_0 and Ψ_1 making this square commute (up to equality) then Φ is a strong equivalence functor. This is equivalent to saying that for every I in Set^2, $[\Phi]^I$ is an equivalence functor.

Since supports split in Set^2, local equivalence is the same as strong equivalence.

(3.1) PROPOSITION. *If $\Phi : A \to B$ is a weak equivalence functor and C any internal category, then $C^\Phi : C^B \to C^A$ is full and faithful.*

PROOF. Let $\Psi, \Psi': B \to C$ be two internal functors and $t: \Psi\Phi \to \Psi'\Phi$ an internal natural transformation. Thus we have

$$t_i: \Psi_i\Phi_i \to \Psi'_i\Phi_i, \quad i = 0, 1, \quad \text{such that} \quad H\, t_0 = t_1\, F.$$

Since Φ_i is an equivalence functor, there exist unique natural transformations

$$u_i: \Psi_i \to \Psi'_i \quad \text{such that} \quad u_i\Phi_i = t_i .$$
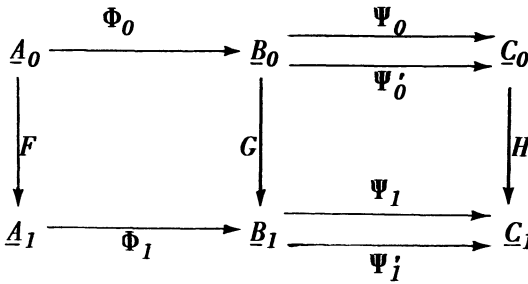
Then,

$$H u_0 \Phi_0 = H t_0 = t_1 F = u_1 \Phi_1 F = u_1 G \Phi_0 ,$$

so by uniqueness $H u_0 = u_1 G$. Thus there is a unique internal natural transformation $u : \Psi \to \Psi'$ such that $u \Phi = t$. This shows that C^{Φ} is full and faithful. □

Note that this proof used the fact that the proposition is true in *Set*. This is easily seen since weak equivalences are strong equivalences and in this case C^{Φ} is also an equivalence.

(3.2) DEFINITION. Say that a functor $H : \underline{C}_0 \to \underline{C}_1$ has the *isomorphism lifting property* (ILP) if for every $c_0 \in |\underline{C}_0|$ and every isomorphism $\sigma : c_1 \overset{\approx}{\to} H c_0$ in $\underline{C}_1$, there exist $c_0' \in |\underline{C}_0|$ and an isomorphism

$$\theta : c_0' \overset{\approx}{\to} c_0 \quad \text{such that} \quad H \theta = \sigma.$$

(3.3) THEOREM. *If $H : \underline{C}_0 \to \underline{C}_1$ has the ILP, then $C = (H : \underline{C}_0 \to \underline{C}_1)$ is a stack.*

PROOF. Let $\Phi : A \to B$ be a weak equivalence functor. We shall prove that C^{Φ} is an (ordinary) equivalence functor. By Proposition (3.1), C^{Φ} is full and faithful. Let $\Gamma : A \to C$ be an internal functor. Since Φ_i is an equivalence of categories, there exists
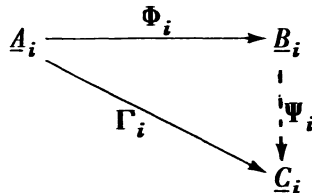
$$\Psi_i : \underline{B}_i \to \underline{C}_i \quad \text{such that} \quad \alpha_i : \Psi_i \Phi_i \stackrel{\approx}{\to} \Gamma_i .$$

Ψ_i is unique up to isomorphism. Now

$$\Psi_1 G \Phi_0 = \Psi_1 \Phi_1 F \xrightarrow[\alpha_1 F]{\approx} \Gamma_1 F = H \Gamma_0 \xrightarrow[H \alpha_0^{-1}]{\approx} H \Psi_0 \Phi_0$$

and so by uniqueness there is an isomorphism

$$\sigma : \Psi_1 G \stackrel{\approx}{\to} H \Psi_0 \quad \text{such that} \quad \sigma \Phi_0 = H \alpha_0^{-1} . \alpha_1 F .$$

$$
\begin{array}{ccc}
\underline{B}_0 & \xrightarrow{\quad \Psi_0 \quad} & \underline{C}_0 \\[2pt]
\scriptstyle G \Big\downarrow & \substack{\approx \\ \overset{\stackrel{\approx}{\to}}{\sigma}} & \Big\downarrow \scriptstyle H \\[2pt]
\underline{B}_1 & \xrightarrow{\quad \Psi_1 \quad} & \underline{C}_1
\end{array}
$$

For every $b_0 \epsilon \,|\,\underline{B}_0\,|$, $\sigma b_0 : \Psi_1 G b_0 \stackrel{\approx}{\to} H \Psi_0 b_0$, so by the ILP, there exist $\theta b_0 : \Psi_0' b_0 \stackrel{\approx}{\to} \Psi_0 b_0$ such that

$$H \Psi_0' b_0 = \Psi_1 G b_0 \quad \text{and} \quad H \theta b_0 = \sigma b_0 .$$

Now, Ψ_0' extends uniquely to a functor such that θ is natural. Then $H \Psi_0' = \Psi_1 G$ as functors, and we get an internal functor $\Psi' : B \to C$. We also have an isomorphism

$$\alpha_0' : \Psi_0' \Phi_0 \xrightarrow{\quad \theta \Phi_0 \quad} \Psi_0 \Phi_0 \xrightarrow{\quad \alpha_0 \quad} \Gamma_0 ,$$

and

$$H \alpha_0' = H (\alpha_0 . \theta \Phi_0) = H \alpha_0 . H \theta \Phi_0 = H \alpha_0 . \sigma \Phi_0 =$$
$$= H \alpha_0 . H \alpha_0^{-1} . \alpha_1 F = \alpha_1 F ,$$

so (α_0', α_1) gives an isomorphism $\alpha' : \Psi' \Phi \stackrel{\approx}{\to} \Gamma$. Thus C^{Φ} is an equivalence functor and so C is a stack. $\square$

If $H : \underline{C}_0 \to \underline{C}_1$ is any functor, construct a category $\tilde{\underline{C}}_0$ whose objects are triples (c_0, c_1, σ) where

$$c_0 \epsilon \,|\,\underline{C}_0\,|, \quad c_1 \epsilon \,|\,\underline{C}_1\,| \quad \text{and} \quad \sigma : c_1 \to H c_0 \quad \text{is an isomorphism.}$$

A morphism $(c_0, c_1, \sigma) \to (c_0', c_1', \sigma')$ is a pair (f_0, f_1) where

$$f_0 : c_0 \to c_0' \quad \text{and} \quad f_1 : c_1 \to c_1'$$

and

$$c_1 \xrightarrow{\ \sigma\ } H c_0$$

$$f_1 \downarrow \qquad \equiv \qquad \downarrow H f_0$$

$$c_1' \xrightarrow{\ \sigma_1\ } H c_0'$$

There is a functor $\tilde{H} : \tilde{\underline{C}}_0 \to \underline{C}_1$ defined by $\tilde{H}(c_0, c_1, \sigma) = c_1$. There is also a functor $\Phi_0 : \underline{C}_0 \to \tilde{\underline{C}}_0$ given by $\Phi_0(c_0) = (c_0, H c_0, 1_{H c_0})$, and

$$\underline{C}_0 \xrightarrow{\ \Phi_0\ } \tilde{\underline{C}}_0$$

$$H \downarrow \qquad \qquad \downarrow \tilde{H}$$

$$\underline{C}_1 \xrightarrow{\ 1_{\underline{C}_1}\ } \underline{C}_1$$

commutes. Thus for every internal category C we have constructed $\Phi : C \to \tilde{C}$. It is easily seen that Φ is a weak equivalence functor and that $\tilde{H}$ has the ILP and so $\tilde{C}$ is a stack. Thus by Corollary (2.11), $\tilde{C}$ is the stack completion of C. Then C is a stack iff $\Phi : C \to \tilde{C}$ is a strong equivalence. For example, it is easily seen that the category A introduced at the beginning of this section is not a stack.
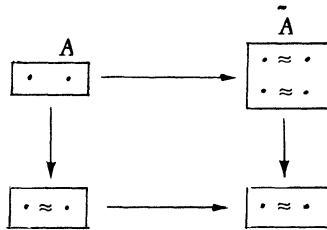
REFERENCES.

1. M. BARR, Exact categories, in Exact categories and categories of sheaves, *Lecture Notes in Math.* 236, Springer (1970), 1- 120.

2. M. BUNGE, Stack completions and Morita equivalence for categories in a topos, *Cahiers Topo. et Géo. Diff.*, same issue.

3. J. DUSKIN, Variations on Beck's tripleability criterion, in Reports of the Midwest Cat. Seminar III, *Lecture Notes in Math.* 106, Springer (1969), 74.

4. J. GIRAUD, *Cohomologie non abélienne*, Springer, Berlin, 1971.

5. J. GIRAUD, Classifying topos, in Toposes, Algebraic Geometry and Logic, *Lecture Notes in Math.* 274, Springer (1972), 43-56.

6. A. GROTHENDIECK, Technique et descente et théorèmes d'existence en Géométrie Algébrique, *Séminaire Bourbaki* n⁰ 190 (1959/ 60).

7. R. PARE & D. SCHUMACHER, Abstract families and the adjoint functor Theorems, in Indexed categories and their applications, *Lecture Notes in Math.* 661, Springer (1978), 1- 125.

8. G. WRAITH, Lecture on elementary topoi, in Model theory and Topoi, *Lecture Notes in Math.* 445, Springer (1975), 114- 206.

9. G. WRAITH, *Locally equivalent toposes*, Manuscript, Univ. Sussex, 1977.

10. J. BENABOU & J. ROUBAUD, Monades et descente, *C. R. A. S. Paris* 270 (1970), A, 96-98.

M. BUNGE: Department of Mathematics
Mc Gill University
MONTREAL, Québec, CANADA.

R. PARE: Department of Mathematics
Dalhousie University
HALIFAX, N. S., CANADA.