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JOAN WICK PELLETIER
ROBERT D. ROSEBRUGH

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THE CATEGORY OF BANACH SPACES IN SHEAVES *)

by *Joan Wick PELLETIER and Robert D. ROSEBRUGH*

1. INTRODUCTION.

The notion of a Banach space in the topos *Sh(X)* of sheaves over a topological space *X* was first introduced by Mulvey [11]. Interest in the category of Banach spaces in *Sh(X)*, which we hereafter denote by *ban(X)*, has converged from many sides, due to Mulvey's realization that *ban(X)* is equivalent both to the category of Banach fibre spaces over *X*, studied by Hofmann [5,6], and to the category of Banach (Q- or approximation) sheaves on *X*, studied by Auspitz [1] and Banaschewski [2]. Thus, the results obtained in *ban(X)* by using the techniques of intuitionistic mathematics available in *Sh(X)* can be transferred freely to these other categories. Important among these results is the Hahn-Banach Theorem for *R-valued functionals in *Sh(X)* due to Burden [3], where *R denotes the order completion of the Dedekind reals.

Our object in this paper is to begin an investigation of the category *ban(X)*. We are interested in the existence of internal and external limits in *ban(X)*. We study the construction of dual spaces and applications of the Hahn-Banach Theorem. We calculate dual spaces of quotients and subspaces. It evolves that duality behaves best when restricted to a subcategory of *ban(X)* consisting of «*normed» Banach spaces. We examine *closure in *ban(X)*, describing it in terms of annihilators. Finally, in bringing together many of the above notions, we show that *normed, *complete Banach spaces can be characterized in terms of the behaviour of their duals. We see that this latter subcategory arises naturally and is reflective in *ban(X)*.

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We recall for the convenience of our readers that the Dedekind real numbers object in $Sh(X)$, denoted R, is the sheaf of continuous real-valued functions on X. A normed space B in $Sh(X)$ is a sheaf of R-modules equipped with a morphism (the «norm») $N: \mathrm{R}^+ \to \Omega^B$ satisfying (in the internal language of $Sh(X)$, see [7] or [10]):

(1N) $\forall b \in B, \ \exists r \in \mathrm{R}^+ \quad b \in N(r),$

(2N) $\forall b \in B, \ \forall r \in \mathrm{R}^+ \ (b \in N(r) \longleftrightarrow \exists r' < r, \quad b \in N(r')),$

(3N) $\forall b, b' \in B, \quad \forall r, r' \in \mathrm{R}^+$
$$(b \in N(r) \wedge b' \in N(r') \to b + b' \in N(r + r')),$$

(4N) $\forall b \in B, \quad \forall r, s \in \mathrm{R}^+, \forall \alpha \in \mathrm{R}$
$$(b \in N(r) \wedge |\alpha| < s \to \alpha b \in N(rs)),$$

(5N) $\forall b \in B \quad (b = 0 \Longleftrightarrow \forall r \in \mathrm{R}^+, \ b \in N(r)).$

A Cauchy approximation, on a normed space B, is a morphism $C: \mathrm{N} \to \Omega^B$, where N denotes the natural numbers objects in $Sh(X)$ (the locally constant natural numbers-valued functions on X) satisfying:

(1A) $\forall n \in \mathrm{N}, \quad \exists b \in B \quad b \in C(n),$

(2A) $\forall n \in \mathrm{N}, \ \exists m \in \mathrm{N}, \ \forall k, k' \geq m$
$$(b \in C(k) \wedge b' \in C(k') \to b - b' \in N(1/n)).$$

The Cauchy approximation C is said to converge if

$$\exists b_0 \in B, \ \forall n \in \mathrm{N}, \exists m \in \mathrm{N} \ \forall k \geq m \ (b \in C(k) \to b - b_0 \in N(1/n)).$$

A Banach space in $Sh(X)$ is a normed space B for which (in the internal language of $Sh(X)$) every Cauchy approximation converges.

If B' and B'' are Banach spaces, a linear map $f: B' \to B''$ is said to be continuous if:

$$\forall r \in \mathrm{R}^+, \quad \exists s \in \mathrm{R}^+ \ (b \in N_{B'}(s) \to f(b) \in N_{B''}(r)) \ ;$$

f is said to be bounded if:

$$\exists x \in \mathrm{R}^+, \ \forall b \in B', \ \forall r \in \mathrm{R}^+ \ (b \in N_{B'}(r) \to f(b) \in N_{B''}(xr)).$$

It is easy to see that f is continuous iff it is bounded. The category $ban(X)$ will have as its morphisms all linear maps bounded by 1 (i.e., the norm decreasing linear maps).

Both the Dedekind reals R and the Dedekind-Mac Neille reals *R will be used in this paper. The former object is quite standard by now, while the latter has only more recently become an object of study. *R is the (internal) Dedekind-Mac Neille completion of the rational numbers object Q ; it is also known as the extended reals (Burden [3]). Formally, it is defined to be all pairs (L, U) of subobjects of Q , the sheaf of locally constant rational-valued functions on X, satisfying:

(1R) $\exists q \in Q\ (q \in L) \wedge \exists q' \in Q\ (q' \in U)$,

(2R) $L = \{ q \in Q \mid \exists q' > q\ \ \forall q'' \in U\ (q'' > q') \}$,

$U = \{ q \in Q \mid \exists q' < q\ \ \forall q'' \in L\ (q'' < q') \}$.

Mulvey has identified the sections of *R as pairs of functions $(\check{f}, \hat{f})$ where $\check{f}$ is lower semicontinuous, $\hat{f}$ is upper semicontinuous and $\hat{f}$ is the least upper semicontinuous function greater than $\check{f}$, and dually.

2. LIMITS IN $ban(X)$.

In this section we consider internal and external limits in $ban(X)$.

To study internal limits it will be necessary to endow $ban(X)$ with the structure of a $Sh(X)$-indexed category. Recall that this requires that, for each object I in $Sh(X)$, there be given a category $ban(X)^I$ of «I-indexed families» of objects of $ban(X)$ and for each morphism $a: J \to I$ in $Sh(X)$, there be a functor $a^*: ban(X)^I \to ban(X)^J$, called «substitution along a of I-indexed families in J-indexed families». Substitution is required to be compatible with composition and identities in $Sh(X)$ up to canonical isomorphism. (For details, see Paré and Schumacher [12].) For example, $Sh(X)$ is itself a $Sh(X)$-indexed category with

$$Sh(X)^I = Sh(X)/I$$

and the substitution functors defined by pulling back. We note further that the unique morphism $I \to 1$ is denoted I and substitution along it by I^*.

For I an object of $Sh(X)$, we define $ban(X)^I$ to be the category of Banach spaces in the topos $Sh(X)/I \approx Sh(X)^I$ so that an I-indexed family of Banach spaces is just a «Banach space in I-indexed families» (of objects). Next suppose that $a: J \to I$ is a morphism in $Sh(X)$ and B is an object of $ban(X)^I$ with norm N_B. We wish to describe the object a^*B in $ban(X)^J$. Since a^* is a logical functor from $Sh(X)^I$ to $Sh(X)^J$ it preserves natural and rational numbers objects as well as the sentences in the language of $Sh(X)^I$ defining norm and completeness of a norm. Thus, denoting the Dedekind reals in $Sh(X)^I$ by R_I and the underlying object of B by B, we have $a^*\mathrm{R}_I = \mathrm{R}_J$ and so $a^* B$ is a R_J-module in $Sh(X)^J$. $a^* B$ comes equipped with, and is complete with respect to, the norm $N_{a^*B} = a^* N_B$. Clearly, morphisms of $ban(X)^I$ are preserved by a^* as well, and the substitution functors so defined inherit compatibility with composition and identities from the $Sh(X)$-indexed category $Sh(X)$. Thus, $ban(X)$ is a $Sh(X)$-indexed category.

To see that $ban(X)$ is internally complete we need only show that it and $ban(X)^I$ have stable finite limits (finite limits preserved by a^*) and internal products (see [12]).

2.1. LEMMA. *$ban(X)^I$ has stable finite limits.*

PROOF. Finite limits will be constructed in $ban(X)$ and then those in $ban(X)^I$ exist by localization. Their stability is clear from the constructions given. First, let B' and B'' be in $ban(X)$ with norms $N_{B'}$ and $N_{B''}$. The underlying R-module of $B' \times B''$ is the R-module $B' \times B''$. The norm $N_{B' \times B''}$ is defined by

$$(b', b'') \in N_{B' \times B''}(r) \iff b' \in N_{B'}(r) \wedge b'' \in N_{B''}(r).$$

To see that $N_{B' \times B''}$ is a norm on $B' \times B''$ requires some verification. For example, to show that $N_{B' \times B''}$ satisfies (2N), we use the fact (about R^+) that

$$\exists r', r'' \in \mathrm{R}^+ \; (r' < r) \wedge (r'' < r) \to \exists r''' \in \mathrm{R}^+, \; r', r'' \leq r''' < r.$$

Then

$$\forall (b', b'') \in B' \times B'', \; \forall r \in \mathrm{R}^+,$$
$$((b', b'') \in N_{B' \times B''}(r) \iff b' \in N_{B'}(r) \wedge b'' \in N_{B''}(r)$$
$$\iff \exists r' < r, \; b' \in N_{B'}(r') \wedge \exists r'' < r, \; b'' \in N_{B''}(r'')$$

$$\Longleftrightarrow \exists r''' < r,\ b' \in N_{B'}(r''') \wedge b'' \in N_{B''}(r''')$$
$$\Longleftrightarrow \exists r''' < r,\ (b', b'') \in N_{B' \times B''}(r''')).$$

Verification of the other properties of $N_{B' \times B''}$ is similar and is left to the reader. Moreover, showing that $B' \times B''$ is complete with respect to $N_{B' \times B''}$ also follows the standard argument and is omitted.

To construct the equalizer of a pair of maps

$$B' \overset{f}{\underset{g}{\rightrightarrows}} B''$$

in $ban(X)$, we simply form the equalizer $e: B_0 \rightarrowtail B'$ in R-modules, and endow it with norm N_{B_0} which is the pullback of $N_{B'}$ along

$$B_0 \times \mathrm{R}^+ \rightarrowtail B' \times \mathrm{R}^+ .$$

This makes e a morphism of Banach spaces. Verification is omitted since e is equally well defined as the kernel of $f - g$ (Burden [4]). Now, since $ban(X)$ has finite products and equalizers, the lemma is proved. ■

Before constructing arbitrary internal products in $ban(X)$, we recall the construction of products in ban, the category of Banach spaces and norm decreasing linear maps in set. If $(B_i)_{i \in I}$ is a family of Banach spaces, their product is the set

$$\{ (b_i) \in \prod_{i \in I} B_i \mid \sup_{i \in I} \| b_i \| < \infty \}$$

with coordinatewise operations and $\| (b_i) \| = \sup_{i \in I} \| b_i \|$.

Recall that what is required for the construction of internal products in $ban(X)$ is a right adjoint to

$$I^*: ban(X) \to ban(X)^I$$

for each I in $Sh(X)$. Now suppose B is in $ban(X)^I$ with norm N_B. Let η be the front adjunction for $I^* \dashv \Pi_I: Sh(X)^I \to Sh(X)$ and

$$\eta_I : \mathrm{R}^+ \to \Pi_I I^* \mathrm{R}^+ = \Pi_I \mathrm{R}_I^+ .$$

($Sh(X)$ is itself internally complete, being a topos.)

By using exponential adjointness and the fact that Ω classifies subobjects, we may write N_B as a subobject of $B \times R_I^+$ (as will be convenient for the remainder of this section). Let the top square in the following diagram be a pullback in $Sh(X)$:

$$\begin{array}{ccc}
\Pi_I N_B & \xrightarrow{\ v\ } & \Pi_I(B \times R_I^+) \approx \Pi_I B \times \Pi_I R_I^+ \\
\uparrow & \text{P.B.} & \uparrow \Pi_I B \times \eta_I \\
N_{\bar{B}} & \xrightarrow[\ v'\ ]{} & \Pi_I B \times R^+ \\
\downarrow & & \downarrow p_I \\
\bar{B} & \rightarrowtail & \Pi_I B
\end{array}$$

where v, which is Π_I applied to the inclusion $N_B \rightarrowtail B \times R^+$, is a monomorphism since Π_I is a right adjoint. $\bar{B}$ is the image in $\Pi_I B$ of v' followed by the projection p_I on $\Pi_I B$. We then have the following lemma:

2.2. LEMMA. *$\bar{B}$ is in $ban(X)$ with norm $N_{\bar{B}}$ and is the internal product of B.*

PROOF. We first note that $\Pi_I B$ in $Sh(X)$ is an R-module since B is an R-module and the forgetful functor from R-modules to $Sh(X)$ creates internal products [12]. Consequently, the back adjunction $\epsilon : I^*\Pi_I B \to B$ is an R-module morphism which will be used below. Before showing that $\bar{B}$ is a submodule of $\Pi_I B$, we will need some properties of $N_{\bar{B}}$. By the definition of $N_{\bar{B}}$ as a pullback in $Sh(X)$, for $(b, r) \in N_{\bar{B}}(U)$ (U open in X) we have:

$$\begin{array}{ccc}
U & \xrightarrow{(b,r)} & N_{\bar{B}} \\
\hline
U & \xrightarrow{(b,r)} & \Pi_I B \times R^+ \\
\downarrow & & \downarrow \\
\Pi_I N_B & \rightarrowtail & \Pi_I B \times \Pi_I R_I^+ \\
\hline
I^*U & \xrightarrow{(I^*b,\, I^*r)} & I^*\Pi_I B \times I^*R^+ \\
(c,s)\downarrow & & \downarrow \epsilon \times I^*R^+ \\
N_B & \rightarrowtail & B \times I^*R^+
\end{array}$$

where the second bijection uses $I^* \dashv \Pi_I$, and so elements $(b,r) \in N_{\bar{B}}(U)$ correspond bijectively with elements $(c,s) \in N_B(I^*U)$ such that

$$c = \epsilon(I^*b) \quad \text{and} \quad s = I^*r.$$

With this we can show that properties (4N) and (3N) of a norm on $\Pi_I B$ hold for $N_{\bar{B}}$. For example, using the transpose of (4N) for the norm N_B , if $(b,r) \in N_{\bar{B}}$ and $|a| < s$, then

$$(\epsilon I^*b, I^*r) \in N_B \wedge |I^*a| < I^*s \rightarrow (I^*a\epsilon(I^*b), I^*r I^*s) \in N_B$$
$$\rightarrow (\epsilon(I^*a I^*b), I^*r I^*s) \in N_B \rightarrow (\epsilon(I^*a b), I^*rs) \in N_B$$

using that ϵ is an R-module homomorphism, so $(ab, rs) \in N_{\bar{B}}$. Similarly,

$$(a,r) \in N_{\bar{B}} \wedge (b,s) \in N_{\bar{B}} \rightarrow (a+b, r+s) \in N_{\bar{B}} .$$

Using these, we can show that $\bar{B}$ is a submodule of $\Pi_I B$. Notice first that (1N) for a norm on $\bar{B}$ holds for $N_{\bar{B}}$ by the definition of $\bar{B}$. Now, if $b \in \bar{B}$ and $s \in \mathrm{R}$ we have

$$\exists r \in \mathrm{R}^+, (b,r) \in N_{\bar{B}} \wedge |s| < |s| + 1$$
$$\rightarrow \exists r \in \mathrm{R}^+ (sb, r(|s|+1)) \in N_{\bar{B}}$$
$$\rightarrow \exists r' \in \mathrm{R}^+ (sb, r') \in N_{\bar{B}} \rightarrow sb \in \bar{B}.$$

Thus $\bar{B}$ is closed under scalar multiplication and an even simpler argument shows it closed under addition. The arguments above show that $N_{\bar{B}}$ satisfies (1N), (3N) and (4N) in the definition of norm with respect to $\bar{B}$. The remaining two conditions may be shown similarly and are left to the reader. We conclude that $\bar{B}$ is a normed space.

Suppose that $C: \mathrm{N} \rightarrow \Omega^{\bar{B}}$ is a global Cauchy approximation in $\bar{B}$. Then I^*C is a Cauchy approximation in $I^*\bar{B}$ and the R_I-module morphism

$$f: I^*\bar{B} \rightarrowtail I^*\Pi_I B \xrightarrow{\epsilon} B$$

is clearly norm-decreasing. Thus $f \circ I^*C$ is a Cauchy approximation in B and so there exists $b \in B$ to which it converges. Now b is unique so it exists globally and hence corresponds to $\bar{b} \in \Pi_I B$, to which C obviously converges. That $\bar{b}$ lies in $\bar{B}$ follows since $I^*C(n)$ is bounded by a constant implying that b and $\bar{b}$ are also. This argument can be localized to any open set in X and so (in the internal sense) every Cauchy approximation in $\bar{B}$ converges.

Finally, it remains to show that $\bar{B}$ acts like a product for B. Let $\phi : B' \to \bar{B}$ be a morphism in $ban(X)$. From it, using the f in the preceding paragraph, we obtain a morphism

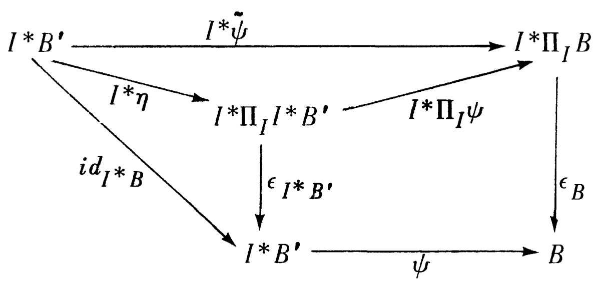
$$f \circ I^*\phi = \bar{\phi} : I^*B' \to B \quad \text{in} \quad ban(X)^I .$$

On the other hand, if $\psi : I^*B' \to B$ is a morphism in $ban(X)^I$, we obtain a morphism of R-modules $\bar{\psi} : B' \to \Pi_I B$, and we wish to show that $\bar{\psi}$ factors through $\bar{B}$. Now

$$b' \in B' \;\to\; \exists\, r \in \mathrm{R}^+ \;\; (b', r) \in N_{B'}$$

and $(b', r) \in N_{B'}$ implies $(I^*b', I^*r) \in N_{I^*B'}$. Hence, $(\psi I^*b', I^*r) \in N_B$ since ψ is norm-decreasing. But

$$\begin{aligned} \psi I^*b' &= \psi(\epsilon I^*. I^*\eta) I^*b' && \text{using } I^* \dashv \Pi_I \\ &= \epsilon I^*(\Pi_I\psi \circ \eta\, b') && \text{by naturality (see diagram below)} \\ &= \epsilon I^*(\bar{\psi}\, b') && \text{by definition of } \bar{\psi} . \end{aligned}$$



Thus by the characterization at the beginning of this proof, $(\bar{\psi}\, b', r) \in N_{\bar{B}}$. So we have

$$\begin{aligned} b' \in B' &\to \exists\, r \in \mathrm{R}^+ \;\; (b', r) \in N_{B'} \\ &\to \exists\, r \in \mathrm{R}^+ \;\; (\bar{\psi}\, b', r) \in N_{\bar{B}} \;\to\; \bar{\psi}\, b' \in \bar{B}. \end{aligned}$$

Moreover, $\bar{\psi}$ is also clearly norm-decreasing by the preceding remarks. That the correspondences $\phi \mapsto \bar{\phi}$ and $\psi \mapsto \bar{\psi}$ are mutually inverse is left to the reader. The stability of the construction of $\bar{B}$ follows from the construction of $N_{\bar{B}}$ and $\bar{B}$ itself, since pullbacks and images are indexed. ■

Combining Lemmas 2.1 and 2.2 we obtain the following proposition.

2.3. PROPOSITION. *$ban(X)$ is internally complete.* ■

We wish to show also that $ban(X)$ is externally complete. Suppose I is a set and $(B_i)_{i\in I}$ is a family of objects of $ban(X)$. The (external) product in $Sh(X)$ of $(B_i)_{i\in I}$ is an R-module and we denote it by $\Pi_{i\in I}B_i$. To define the product of the B_i in $ban(X)$ consider the diagram

$$\begin{array}{ccc}
\Pi_{i\in I}N_{B_i} & \rightarrowtail & \Pi_{i\in I}B_i \times \Pi_{i\in I}\mathrm{R}^+ \\
\uparrow & \text{P.B.} & \uparrow{\scriptstyle id\times\Delta} \\
N_{\bar{B}} & \rightarrowtail & \Pi_{i\in I}B_i \times \mathrm{R}^+ \\
\downarrow & & \downarrow{\scriptstyle p_I} \\
\bar{B} & \rightarrowtail & \Pi_{i\in I}B_i
\end{array}$$

in which the top square is a pullback and $\bar{B}$ is the image of $N_{\bar{B}}$ in $\Pi_{i\in I}B_i$. Notice that $\bar{B}$ is the subobject of $\Pi_{i\in I}B_i$ defined by the formula giving the usual product in ban. $\bar{B}$ is the product of the B_i in $ban(X)$ with norm $N_{\bar{B}}$. The proof proceeds in much the same way as that of Lemma 2.2. It is perhaps worthwhile to note that the elements of $N_{\bar{B}}(U)$ are pairs $((b_i), r)$, where

$$(b_i, r) \in N_{B_i}(U) \quad \text{for all } i \text{ in } I,$$

which is similar to the characterization of such elements obtained early in Lemma 2.2. Again combining with 2.1, we obtain the desired result.

2.4. PROPOSITION. *$ban(X)$ is externally complete.* ■

In closing this section we remark that the definition of a Banach space makes sense in any topos having a natural numbers object, although to date spatial topoi have provided the customary setting. Moreover, the constructions used in 2.1-2.3 do not depend upon the nature of the topos considered, nor do the results (except to require the existence of the natural numbers object). It is also worth noting that the constructions given here show that the category of normed spaces in a topos (of sheaves) is internally (and externally) complete.

3. THE DUAL SPACE.

In analogy to ordinary Banach space theory, we could consider the dual space of the Banach space B to be the sheaf of R- or *R-valued continuous linear functionals on B. We choose the latter in view of the fact that Burden [3] has proved a Hahn-Banach Theorem for *R-valued functionals. The results which follow from the adoption of this definition are stated below. They have been proved by Burden [4] and independently by the present authors.

Given $B_1, B_2 \in ban(X)$, it is easily seen that $ban(X)(B_1, B_2)$ is the unit ball of a normed space $HOM(B_1, B_2)$ which consists of all bounded linear transformations from B_1 to B_2. The norm N_{B_1,B_2} on $HOM(B_1, B_2)$ is given by

$$f \in N_{B_1,B_2}(x) \iff \exists\, y < x \quad \forall b \in B_1 \quad \forall r \in \mathrm{R}^+$$
$$(\, b \in N_{B_1}(r) \rightarrow f(b) \in N_{B_2}(yr)\,).$$

One can readily prove that the completeness of $HOM(B_1, B_2)$ depends only on the completeness of B_2, so that $HOM(B_1, B_2) \in ban(X)$. If B_3 is another Banach space, we also have

$$(1) \quad HOM(B_1, HOM(B_2, B_3)) \approx HOM(B_2, HOM(B_1, B_3)),$$

where we understand by $\approx$ an isometric isomorphism, i.e. a R-module isomorphism $\phi : A_1 \rightarrow A_2$ such that

$$a \in N_{A_1}(r) \iff \phi(a) \in N_{A_2}(r).$$

We denote by B^* the space $HOM(B, {}^*\mathrm{R})$. Clearly, $(\ )^*$ is a contravariant endofunctor on $ban(X)$, with $f^* : B_2^* \rightarrow B_1^*$ defined by

$$f^*(g) = g \circ f \quad \text{for} \quad f \in ban(X)(B_1, B_2).$$

We observe that $f \in N_{B_1,B_2}(x)$ implies $f^* \in N_{B_2^*,B_1^*}(x)$.

By relation (1) above, we see that $(\ )^*$ is adjoint to itself on the right. We also have

$$HOM(B^*, B^*) \approx HOM(B, B^{**}),$$

and we denote by $i_B : B \to B^{**}$ the morphism which corresponds to the identity on B^*. Usual Banach space theory relies strongly on the fact that i_B is an isometric inclusion, which fact itself uses the Hahn-Banach Theorem for its proof. In our setting the Hahn-Banach Theorem is insufficient to obtain this result and there are non-trivial Banach spaces having trivial dual spaces. The following example illustrates our predicament.

Let X denote the Sierpinski space (X is a two-point space with one point open). Then $Sh(X) \approx set^2$ and a Banach space $B = (B_1, B_2, f)$ in $Sh(X)$ is a pair of (ordinary) Banach spaces together with a norm-decreasing linear transformation $f: B_1 \to B_2$. Since X is extremally disconnected, *R and R agree on set^2 (Johnstone [8]). We can easily calculate that this object is $1_R : R \to R$ where R denotes the ordinary real numbers. Then the dual $B^* = (A_1, A_2, g)$ of the Banach space B consists of $HOM((B_1, B_2, f), (R, R, 1_R))$. Thus we see that

$$A_1 = \{ (\phi_1, \phi_2) \mid \phi_1 \in B_1^*,\ \phi_2 \in B_2^*,\ \phi_2 \circ f = \phi_1 \} \approx B_2^*,$$

$$A_2 = B_2^* \text{ and } g = 1_{B_2^*}, \text{ i.e. } B^* = (B_2^*, B_2^*, 1_{B_2^*}).$$

Hence, in particular, the dual of (B_1, B_2, f) is 0 whenever B_2 is.

We remark that the above discussion characterizes dual spaces in set^2 as ordinary dual spaces equipped with the identity map.

The condition which salvages the situation and ensures that i_B is an isometric inclusion is the existence of a *norm on B, so named by Burden [4], who has explored the relation between normed, *normed and co-normed spaces.

A norm N on the space B is said to be a *norm if it satisfies:

$$b \in N(r) \iff \exists s < r \ \neg\neg\ b \in N(s).$$

Burden [4] justifies his terminology in showing that the existence of a *norm N on B is equivalent to the existence of a morphism $\| \ \| : B \to {}^*\mathrm{R}$ with the usual properties of a norm (we define

$$\| b \| = inf \{ r \mid b \in N(r) \}).$$

The following result summarizes the importance of the *norm.

3.1. PROPOSITION. *(1) The dual norm on B^* is a *norm for any B in $ban(X)$.*

*(2) B is *normed iff $i_B : B \to B^{**}$ is an isometric inclusion.*

We remark that R and *R are examples of *normed spaces in $Sh(X)$ for any X, but that their quotient, which is defined and discussed later on, need not be *normed, the latter fact being an observation of Mulvey.

We further remark that the full subcategory of $ban(X)$ consisting of *normed spaces is reflective. We refer to 3.12 for an analogous result.

We now investigate some examples of Banach spaces and their duals. First, recall that in set^2 a Banach space is given by a pair of Banach spaces B_1, B_2 and a norm-decreasing linear map $f: B_1 \to B_2$. The Banach space (B_1, B_2, f) will be *normed iff there is a map

$$\| \ \| : (B_1, B_2, f) \to (R, R, 1)$$

satisfying the properties of a norm. Such a map clearly corresponds to the existence of norms

$$\| \ \|_i : B_i \to \mathrm{R} \quad \text{such that} \quad \| \ \|_2 \circ f = \| \ \|_1 .$$

Thus, (B_1, B_2, f) is *normed iff f is an isometric inclusion.

Let X be the three-point space with two discrete points. Then $Sh(X) \approx set^A$, where

$$A = \quad \cdot \begin{array}{l} \nearrow \cdot \\ \searrow \cdot \end{array}$$

In this topos, R is

$$R \begin{array}{l} \xrightarrow{1_R} R \\ \xrightarrow[1_R]{} R \end{array}$$

but R* is

$$R \times R \begin{array}{l} \xrightarrow{\pi_1} R \\ \xrightarrow[\pi_2]{} R \end{array} .$$

A Banach space consists of a triple of (ordinary) Banach spaces with two norm-decreasing linear maps

$$A \begin{array}{l} \xrightarrow{\ f\ } B \\ \xrightarrow[\ g\ ]{} C\ . \end{array}$$

The dual space of (A, B, C, f, g) will be (A', B', C', f', g') where

$$A' = \{ (\phi_1, \phi_2, \phi_3) \mid \phi_2 \in B^*,\ \phi_3 \in C^*,\ \phi_1 : A \to R \times R \text{ satisfying}$$

$$\phi_2 \circ f = \pi_1 \circ \phi_1,\ \phi_3 \circ g = \pi_2 \circ \phi_1 \} \approx B^* \times C^*,$$

$$B' = B^*,\quad C' = C^*,\quad f' = \pi_1,\quad g' = \pi_2\ .$$

Hence, dual spaces are product diagrams of ordinary dual spaces. Since *normed spaces must be isometrically embedded in their second duals, we see that (A, B, C, f, g) is *normed iff

$$A \subset B \times C,\quad f = \pi_1|_A,\quad g = \pi_2|_A\ .$$

Virtually all results on Banach spaces and their duals require the Hahn-Banach Theorem. The Hahn-Banach Theorem in $Sh(X)$ as proved by Burden [3] states that every *R-valued functional defined on a subspace of a normed space B can be extended to all of B without increasing the norm. In ordinary analysis one immediately would apply this theorem to show that the dual space separates the points of the original space. However, we already observed an example in which B^* is trivial and B is non-trivial, so dual spaces certainly do not in general separate points. On the other hand, we can obtain a useful corollary of the Hahn-Banach Theorem which is akin to the ordinary separation of points result.

To do this it will be necessary to introduce a new notion of closure for a subobject of a Banach space which was introduced by Burden [3]. A subobject A of a Banach space B is said to be *closed if

$$a \in A \iff \neg\ \exists r \in \mathrm{R}^+\ \forall b \in B\ (b \in A \to \neg (b - a) \in N_B(r)).$$

Burden has shown that the *closure of A, denoted $^*Cl(A)$, consists of those elements of B to which some Cauchy *approximation converges. A Cauchy *approximation in B is a morphism $C : \mathrm{N} \to \Omega^B$ satisfying condition

(2A) above of a Cauchy approximation and of

$(1^*A)\ \forall n \in \mathrm{N}\ \neg\ (C(n) = \emptyset),$

$(3^*A)\ \exists r \in \mathrm{R}^+\ \exists m \in \mathrm{N}\ \forall k \geq m\ (b \in C(k) \rightarrow b \in N_B(r)).$

3.2. THEOREM *(Corollary of Hahn-Banach Theorem). Let* A, B *be in* $ban(X)$ *and* $j : A \rightarrow B$ *be an isometric inclusion. Then*

$$\forall b \in B\ \neg\ (b \in {}^*Cl(jA)) \rightarrow \exists f \in B^*\ \neg\ (f(b) = 0) \wedge f \circ j = 0.$$

The proof requires a preliminary lemma. For a Banach space B, a subspace $j : A \rightarrow B$, and $b \in B$, we define

$$K = \{\ r \in \mathrm{R}^+ \mid \neg\ \forall a \in A\ \neg\ (b - ja) \in N(r)\}.$$

3.3. LEMMA. $\exists r \in \mathrm{R}^+\ \forall a \in A\ \neg\ (b - ja) \in N(r) \rightarrow \neg\ (\inf K = 0).$

PROOF. Rewriting K as

$$\{\ r \in \mathrm{R}^+ \mid \neg\neg\ \exists a \in A\ (b - ja) \in N(r)\},$$

it is easy to see that K is closed under larger elements. Thus,

$$\forall a \in A\ \neg\ (b - ja) \in N(r) \wedge s \in K \wedge s \leq r$$

entails both $r \in K$ and $\neg\ (r \in K)$. Hence we have

$$(\forall a \in A\ \neg\ (b - ja) \in N(r) \wedge s \in K) \rightarrow \neg\ (s \leq r) \rightarrow r/2 \leq s,$$

so

$$\exists r \in \mathrm{R}^+\ \forall a \in A\ \neg (b - ja) \in N(r) \rightarrow (s \in K \rightarrow r/2 \leq s)$$
$$\rightarrow (\forall s \in K\ r/2 \leq s) \rightarrow r/2 \leq \inf K \rightarrow \neg\ (\inf K = 0).$$

PROOF of 3.2. Let A_b be the Banach subspace of B defined by

$$A_b = \{\ \lambda b + ja \mid \lambda \in \mathrm{R},\ a \in A\ \}.$$

Define $\hat{f} : A_b \rightarrow {}^*\mathrm{R}$ by

$$\hat{f}(\lambda b + ja) = \lambda\delta, \quad \text{where} \quad \delta = \inf K.$$

We observe by Lemma 3.3 that $\neg\ (\inf K) = 0$ since

$$\neg\ (b \in {}^*Cl(jA)) \Longleftrightarrow \neg\neg\ \exists r \in \mathrm{R}^+\ \forall a \in A\ \neg (b - ja) \in N(r) \rightarrow$$
$$\rightarrow \neg\neg\neg\ (\inf K = 0) \Longleftrightarrow \neg\ (\inf K = 0).$$

Clearly, $\hat{f}$ is linear and bounded by 1. By the Hahn-Banach Theorem $\hat{f}$ may be extended to $f \in B^*$ without increasing its norm, and we have

$$f \circ j = 0 \text{ and } f(b) = \delta \rightarrow \neg(f(b) = 0). \quad \blacksquare$$

Separation of points of a Banach space by functionals does not follow from 3.2 because of the strength of the statement

$$\neg(b = 0) \quad \text{(equivalent to } \neg(b \in {}^*Cl(0))\text{)},$$

which requires that b remain non-zero when restricted to any open set. For example, in set^2 any Banach space of the form $(B', 0, f)$ has no elements b satisfying $\neg(b = 0)$ since all non-zero global elements are zero when restricted to the other open.

We remark that Burden ([4] 4.7) gives a corollary to the Hahn-Banach Theorem very similar to our own in which a slightly stronger hypothesis is used and a slightly stronger result is deduced.

For a linear subspace $A \subset B$ we define the annihilator $A^{\perp}$, of A, as follows:

$$A^{\perp} = \{ f \in B^* \mid \forall a \in A \ f(a) = 0 \}.$$

For a linear subspace M of B^* we define the annihilator ${}^{\perp}M$ of M by

$${}^{\perp}M = \{ b \in B \mid \forall f \in M \ f(b) = 0 \}.$$

Clearly, $A^{\perp}$ and ${}^{\perp}M$ are closed linear subspaces of B^* and B, respectively.

As an application of 3.2, we shall show that the *closure of a subspace of B is its double annihilator. The following lemma due to Burden ([4] 3.14) will be used.

3.4. LEMMA. *If $f: B_1 \rightarrow B_2$ is a bounded linear map of Banach spaces, where B_2 is *normed, then*

$$ker\, f = \{ b \in B_1 \mid f(b) = 0 \}$$

*is *closed.*

3.5. PROPOSITION. *Let A and M be linear subspaces of B and B^*, respectively. Then $A^{\perp}$ and ${}^{\perp}M$ are *closed linear subspaces of B^* and B, respectively.*

PROOF. Lemma 3.4 gives us our result immediately if we note that

$$^{\perp}M = \bigcap_{f \in M} ker\, f \quad \text{and} \quad A^{\perp} = \bigcap_{a \in A} ker\, i_B(a),$$

where $i_B : B \to B^{**}$ is the map described above and the (internal) intersections exist by 2.3. ■

3.6. THEOREM. *For any linear subspace A of a Banach space B,*

$$*Cl(A) = {}^{\perp}(A^{\perp}).$$

PROOF. It is clear that $A \subset {}^{\perp}(A^{\perp})$. Thus, since ${}^{\perp}(A^{\perp})$ is *closed by 3.5 we have $*Cl(A) \subset {}^{\perp}(A^{\perp})$.

To show the opposite inclusion, we note

$$b \in {}^{\perp}(A^{\perp}) \wedge \neg(b \in *Cl(A))$$

implies by 3.2 that

$$\exists f \in B^* \; (f|_{*Cl(A)} = 0 \wedge \neg(f(b) = 0)).$$

However, by definition of ${}^{\perp}(A^{\perp})$,

$$b \in {}^{\perp}(A^{\perp}) \wedge \neg(b \in *Cl(A)) \to \forall f \in B^* (f|_A = 0 \to f(b) = 0).$$

Thus, we may conclude that $\neg\neg(b \in *Cl(A)$. Since $*Cl(A)$ is *closed, we have

$$\begin{aligned}
&\neg\neg(b \in *Cl(A)) \\
&\Longleftrightarrow \neg\neg\neg\, \exists r \in \mathrm{R}^+ \; \forall a \in B \; (a \in A \to \neg(a-b) \in N_B(r)) \\
&\Longleftrightarrow \neg\, \exists r \in \mathrm{R}^+ \; \forall a \in B \; (a \in A \to \neg(a-b) \in N_B(r)) \\
&\Longleftrightarrow b \in *Cl(A),
\end{aligned}$$

i.e., $*Cl(A)$ is double-negation closed, which completes the proof. ■

3.7. COROLLARY. ${}^{\perp}(\mathrm{R}^{\perp}) = *\mathrm{R}$. ■

Given a closed linear subspace A of B, we can define the quotient space B/A and the quotient map $\pi : B \to B/A$ in the obvious way. We norm B/A to make it into a Banach space as follows:

$$x \in N_{B/A}(r) \Longleftrightarrow \exists s < r \; \exists b \in B \; (\pi(b) = x \wedge b \in N_B(s)).$$

As we remarked above, the property of having a *norm is not necessarily inherited by the quotient space. In particular, the quotient of *R by R, both *normed spaces, is not *normed. One can see this because there are many non-zero elements x in *R/R for which the statement

$$\forall r \neg\neg x \in N_{*R/R}(r)$$

holds when the base space X, say, is $[0, 1]$.

The annihilator helps to describe the duals of the subspace A and the quotient space B/A.

3.8. PROPOSITION. *If A is a closed linear subspace of a Banach space B, then*

$$(1)\ A^* \approx B^*/A^{\perp} \qquad \textit{and} \qquad (2)\ (B/A)^* \approx A^{\perp}.$$

PROOF. (1) This has been proved by Burden ([4] 4.9).

(2) We define $\psi : (B/A)^* \to A^{\perp}$ by $\psi(g) = g \circ \pi$, where π is the quotient map. Clearly, $\psi(g) \in A^{\perp}$, and ψ is linear. Given $f \in A^{\perp}$, since A is contained in $ker f$ we have a well-defined linear map $g : B/A \to {}^*R$ given by

$$g(x) = f(b), \quad \text{where} \quad \pi(b) = x.$$

If $x \in N_{B/A}(r)$, then

$$\exists s < r\ \exists b \in B\ (\pi(b) = x \wedge b \in N_B(s)).$$

Hence, if f is bounded by y, then $f(b) = g(x) \in N_{*R}(ys)$ so g is also bounded by y and $g \in (B/A)^*$. Thus, ψ is onto.

To show that ψ is an isometry, if $f = \psi(g) \in N_{B^*}(y)$, then

$$\exists y' < y\ (b \in N_B(r) \to f(b) \in N_{*R}(y'r)).$$

If $x \in N_{B/A}(r)$, then

$$\exists s < r\ \exists b \in B\ (\pi(b) = x \wedge b \in N_B(s)).$$

Hence,

$$f(b) = g(x) \in N_{*R}(y's), \quad \text{so} \quad g \in N_{(B/A)^*}(y).$$

Since ψ is clearly norm-decreasing, we have that ψ is an isomorphism. ∎

We conclude by investigating *complete Banach spaces, i.e. normed linear spaces in which each *approximation converges. As we have seen above, the *closure of a subspace of a Banach space is its double annihilator. One can easily see that if A is a *closed subspace of a *complete space, then A is *complete. Moreover, it is not difficult to show that B^* is *complete for all $B \in ban(X)$.

We remark that *completeness does not imply *normed nor the converse. In set^2, $1/2: \mathrm{R} \to \mathrm{R}$ is *complete but not *normed (recall that *normed spaces are pairs of Banach spaces with isometric inclusions). In any spatial topos in which $\mathrm{R} \neq {}^*\mathrm{R}$, R is an example of a *normed space which is not *complete since its *closure is *R.

We will denote the full subcategory of $ban(X)$ consisting of all *complete, *normed spaces by $^*ban(X)$. The following theorem, based on the «Pullback Lemma» of Linton [9], leads us to a characterization of the elements of $^*ban(X)$.

3.9. THEOREM. *Let $j: A \to B$ be an isometric inclusion of Banach spaces, where A is *complete. Then the following diagram is a pullback in $ban(X)$*

$$\begin{array}{ccc} A & \xrightarrow{\quad j \quad} & B \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_B} \\ A^{**} & \xrightarrow{\quad j^{**} \quad} & B^{**} \end{array} \quad .$$

PROOF. Let $\alpha \in A^{**}$, $b \in B$ be such that $j^{**}(\alpha) = i_B(b)$. Suppose that $\neg(b \in j(A))$. By 3.2

$$\exists f \in B^* \, (f \circ j = 0 \wedge \neg(f(b) = 0))$$

since A is *closed. Then

$$\alpha(f \circ j) = 0 = j^{**}(\alpha)(f) = i_B(b)(f) = f(b),$$

which contradicts $\neg(f(b) = 0)$. Hence, $\neg\neg(b \in j(A))$. Since A is *complete, it is double-negation closed, as we showed in the proof of 3.6, so $b \in j(A)$. Finally letting $b = j(a)$ (a is unique, since j is an isometry), we have

$$j^{**} i_A(a) = i_B(b) = j^{**}(\alpha),$$

and the monotonicity of j^{**} implies that $i_A(a) = \alpha$, which proves the result. ■

3.10. COROLLARY. *$A \in {}^*ban(X)$ iff the following diagram is a pullback* *in ban(X):*

$$\begin{array}{ccc} A & \xrightarrow{\quad i_A \quad} & A^{**} \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_{A^{**}}} \\ A^{**} & \xrightarrow[\quad (i_A)^{**} \quad]{} & A^{****} \end{array}$$

PROOF. One direction follows immediately from 3.1 and 3.9.

In the other direction, if the above diagram is a pullback, then

$$A = ker\,(i_{A^{**}} - (i_A)^{**}).$$

Hence, i_A is an isometry and by 3.1, A is *normed. Moreover, A is *closed in A^{**} by 3.6 and, since A^{**} is *complete, so is A. ■

Finally, we point out that the above corollary actually shows that the functor $Q: A \mapsto {}^*Cl(i_A(A))$ is the idempotent triple associated to the double dualization triple on *ban(X)* obtained by localization. (For similar examples of localizations, see [13].) Then

$${}^*ban(X) = \{\, A \in ban(X) \mid QA \approx A \,\}$$

and, by general results on localizations, *ban(X)* is a reflective subcategory of *ban(X)*.

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Joan Wick Pelletier :
Department of Mathematics, York University
and University of Massachussetts

Robert D. Rosebrugh :
Department of Mathematics,
Mc Gill University
MONTREAL, CANADA.