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THE STRUCTURE OF INITIAL COMPLETIONS

by J. ADAMEK, H. HERRLICH & G. E. STRECKER

Dedicated to Professor Charles Ehresmann

ABSTRACT.

The functor-structured categories over *Set* of Hedrlin, Pultr and Trnkova are generalized so as to admit any base category. The reflective modifications of such categories are shown to be (up to isomorphism) precisely all of the fibre-small initially complete concrete categories. Characterizations are also obtained for the full concrete (reflective) (resp. E-reflective) subcategories of functor-structured categories.

INTRODUCTION.

It has been noted long ago by Hedrlin, Pultr and Trnkova (see [HPT] and [P]) that most usual categories of sets with structure are «controlled» by set functors. More precisely each is a full concrete subcategory of a so-called functor-structured category $S(F)$, where F is a functor $F: Set \to Set$.

In this paper, we introduce functor-structured categories over arbitrary base categories, generalize results of Kučera and Pultr [KP], and simplify some of their proofs as well. In doing so we answer the following questions in full generality:

(1) What is the precise relationship between functor-structured categories and fibre-small initially complete categories?

(2) Which categories can be fully, concretely (and reflectively) (resp. E-reflectively) embedded into some functor-structured category? It turns out that these questions are closely related. In particular, a concrete category

(1) is fibre-small and initially complete iff it is concretely isomorphic to a reflective modification of some functor-structured category (2.5),

(2) can be fully, concretely (and reflectively) (resp. E-reflectively) embedded into some functor-structured category iff it can be fullly, concretely (and reflectively) (resp. E-reflectively) embedded into some fibre-small initially complete category (2.8, 3.2, 4.4).

Categories with the latter properties are characterized using previous results of the authors [AHS], Hoffmann [Ho_1, Ho_2], Wischnewsky and Tholen [Th] by very satisfactory internal conditions (2.8, 2.13, 3.2, 3.3, 4.4).

PRELIMINARIES.

1.1. For the most part we will use the terminology and notation of [AHS]. In particular we will work within a set-theoretic framework consisting of sets, classes and conglomerates, where every set is a class and every class is a conglomerate. If a conglomerate is in one-to-one correspondence with a class, it will be called *legitimate* ; and if it is in one-to-one correspondence with a set, it will be called *small*.

1.2. Throughout we let $\mathfrak{X}$ be a fixed «base» category. A *concrete category* (over $\mathfrak{X}$) is a category $\mathfrak{K}$ together with a faithful, amnestic functor: $|\ | : \mathfrak{K} \to \mathfrak{X}$, assigning to a morphism $f: V \to W$ in $\mathfrak{K}$, the morphism

$$f: |V| \to |W| \quad \text{in} \quad \mathfrak{X}$$

(denoted by the same symbol due to faithfulness, and called a *map* for distinction). For each object X in $\mathfrak{X}$ let

$$\mathfrak{K}[X] = \{ V \in \mathfrak{K} \mid |V| = X \}.$$

Amnesticity is equivalent to the antisymmetry of the following relation on $\mathfrak{K}[X]$ (for any X in $\mathfrak{X}$):

$$V_1 \leq V_2 \quad \text{iff} \quad 1_X: V_1 \to V_2 \text{ is a morphism in } \mathfrak{K}.$$

Thus $(\mathfrak{K}[X], \leq)$ is a partially-ordered class (called the *fibre of* $\mathfrak{K}$ *on* X). If each such fibre is a set, then $\mathfrak{K}$ is said to be *fibre-small.*

A functor $\phi : \mathfrak{K} \to \mathfrak{L}$ between concrete categories is called *concrete* provided that for objects $|\phi(V)| = |V|$ and for morphisms $\phi(f) = f$.

1.3. A full concrete subcategory $\bar{\mathfrak{L}}$ of a concrete category $\mathfrak{L}$ is called a *reflective modification of* $\mathfrak{L}$ iff $\bar{\mathfrak{L}}$ is a reflective subcategory of $\mathfrak{L}$ with reflections carried by identity maps; i.e., iff there is a concrete reflector $R: \mathfrak{L} \to \bar{\mathfrak{L}}$. (Example: the category of indiscrete spaces forms a reflective modification of the category of topological spaces.)

Dual notion: coreflective modification. (Example: discrete spaces in *Top*.)

1.4. Given a concrete category $\mathfrak{K}$, a *structured map with domain X in* $\mathfrak{K}$ is a pair (f, V) with V an object of $\mathfrak{K}$ and $f: X \to |V|$ a map. It is denoted by $X \xrightarrow{f} |V|$. Two structured maps

$$X \xrightarrow{f} |V| \quad \text{and} \quad X \xrightarrow{g} |W|$$

are called *structurally equivalent* provided that, given $|U| \xrightarrow{h} X$, then

$$f.h: U \to V \text{ is a morphism iff } g.h: U \to W \text{ is.}$$

$\mathfrak{K}$ is called *strongly fibre-small* iff for each object X in $\mathfrak{X}$ there exists a small system of representatives relative to structural equivalence of structured maps with domain X.

Dual notions: *structured map with codomain* X; $|V| \xrightarrow{f} X$; structurally equivalent; strongly fibre-small.

1.5. A class-indexed family of structured maps with common domain X is called a *source with domain* X, denoted by

$$S = (X \xrightarrow{f_i} |V_i|)_{i \in I} \quad \text{or just} \quad (X \xrightarrow{f_i} |V_i|).$$

A concrete category $\mathfrak{K}$ is called *initially complete* iff, for each source $S = (X \xrightarrow{f_i} |V_i|)$ there exists an object W on X (called the *initial lift of* S) such that:

a) every $f_i: W \to V_i$ is a morphism, and

b) if for a given $|U| \xrightarrow{h} X$, every $f_i.h: U \to V_i$ is a morphism, then $h: U \to W$ must also be a morphism.

An *initial completion* of a concrete category $\mathfrak{K}$ is an initially complete category $\mathfrak{L}$ in which $\mathfrak{K}$ is an *initially dense* full concrete subcategory (i.e., each object of $\mathfrak{L}$ is an initial lift of some source with codomains in $\mathfrak{K}$).

Dual notions: *sink with codomain* X ; $(|V_i| \xrightarrow{f_i} X)$; *finally complete; final lift; final completion; finally dense.*

1.6. A concrete category need not have any initial (or final) completion; see [He_1]. If it has any, it has a least one, called the *Mac Neille completion*, which is, up to isomorphism, the only initial completion that is a final completion as well. It can be described via so-called *closed sinks* : For each sink $S = (|V_i| \xrightarrow{f_i} X)$ denote by S^{op} the (opposite) source of all the structured maps $X \xrightarrow{g} |W|$ with the property that, for each i, $g \cdot f_i : V_i \to W$ is a morphism; analogously each source S gives rise to the opposite sink S^{op}. A sink (or source) is said to be *closed* iff

$$(S^{op})^{op} = S.$$

If all closed sinks form a legitimate conglomerate, then we can consider the category $\mathfrak{L}$ of closed sinks, where morphisms from

$$S = (|V_i| \xrightarrow{f_i} X) \quad \text{to} \quad T = (|W_j| \xrightarrow{g_j} Y)$$

are maps $p : X \to Y$ such that for each i there exists a j with

$$|V_i| \xrightarrow{p \cdot f_i} Y \quad \text{equal to} \quad |W_j| \xrightarrow{g_j} Y .$$

$\mathfrak{L}$ is a concrete category in the obvious sense, and the assignment of each object V in $\mathfrak{K}$ to the (closed!) sink S_V of all structured maps $|U| \xrightarrow{g} V$ which are morphisms $g : U \to V$ in $\mathfrak{K}$ gives a full concrete embedding of $\mathfrak{K}$ into $\mathfrak{L}$. Then $\mathfrak{L}$ is the Mac Neille completion of $\mathfrak{K}$. (Note that if S^V is defined dually, then

$$(S_V)^{op} = S^V \quad \text{and} \quad (S^V)^{op} = S_V \text{ .)}$$

1.7. A category $\mathfrak{X}$ is called an *(E, M)-category* provided E is a class of $\mathfrak{X}$-morphisms and M a conglomerate of $\mathfrak{X}$-sources, each closed under composition with isomorphisms, such that $\mathfrak{X}$ has the (E, M)-factorization property and the (unique) (E, M)-diagonalization property (for details see e.g. [HSV]). Note that even though singleton sources in M need not be monomorphisms, we will nevertheless call such singleton sources $m: Y \to X$ in M, *M-subobjects of* X. In this context $\mathfrak{X}$ will be said to be *M-well-powered* iff each $\mathfrak{X}$-object has at most a set of pairwise non-equivalent M-subobjects. The dual condition is called *E-co-well-powered.*

2. INITIAL COMPLETIONS AND STRONG FIBRE-SMALLNESS.

2.1. DEFINITION. Let $F: \mathfrak{X} \to Set$ be a functor. The concrete category whose objects are all pairs

$$(X, A) \quad \text{with} \quad X \text{ in } \mathfrak{X} \quad \text{and} \quad A \subset F(X)$$

and whose morphisms $f: (X, A) \to (Y, B)$ are all $\mathfrak{X}$-morphisms

$$f: X \to Y \quad \text{with} \quad Ff[A] \subset B$$

will be called the *functor-structured category determined by* F and will be denoted by $S(F)$.

2.2. PROPOSITION. *Every functor-structured category is fibre-small and initially complete.*

PROOF. Fibre-smallness is obvious. The initial lift of $(X \xrightarrow{f_i} |(Y_i, A_i)|)$ is (X, B) where $B = \cap F f_i^{-1}[A_i]$, and the final lift of $(|(Y_i, A_i)| \xrightarrow{f_i} X)$ is (X, B) where $B = \cup F f_i[A_i]$.

2.3. DEFINITION. For each fibre-small initially complete category $\mathfrak{K}$ (over $\mathfrak{X}$) we define a functor $F_{\mathfrak{K}}: \mathfrak{X} \to Set$, called the *fibre-functor of* $\mathfrak{K}$, as follows:

Each object X in $\mathfrak{X}$ is assigned to the fibre $F_{\mathfrak{K}}(X) = \mathfrak{K}[X]$, and each morphism $f: X \to Y$ in $\mathfrak{X}$ is assigned to the mapping

$$F_{\mathfrak{K}}(f): \mathfrak{K}[X] \to \mathfrak{K}[Y]$$

defined by: For $V \in \mathfrak{K}[X]$ put

$$F_{\mathfrak{K}}(f)(V) = fin(f, V) = \text{the final lift of the singleton sink } |V| \xrightarrow{f} Y.$$

(The preservation of composition follows from the fact that final lifts are unique (by amnesticity) and the composition of final morphisms is final. The preservation of identities is clear.)

2.4. LEMMA. *Any reflective modification of an initially complete category is initially complete.*

2.5. THEOREM. *For a concrete category* $\mathcal{K}$, *the following are equivalent:*

(i) $\mathcal{K}$ *is fibre-small and initially complete.*

(ii) $\mathcal{K}$ *is concretely isomorphic to a reflective modification of the functor-structured category* $S(F_{\mathcal{K}})$ *determined by the fibre-functor of* $\mathcal{K}$.

(iii) $\mathcal{K}$ *is concretely isomorphic to a reflective modification of some functor-structured category.*

PROOF. Clearly (ii) implies (iii) and by Proposition 2.2 and Lemma 2.4, (iii) implies (i). To show that (i) implies (ii) define $\phi : \mathcal{K} \to S(F_{\mathcal{K}})$ by: $\phi(V) = (X, A[V])$, where

$$X = |V| \quad \text{and} \quad A[V] = \{ W \in \mathcal{K}[X] \mid W \leq V \},$$

and $\phi(f) = f$. If $f: V \to U$ is a $\mathcal{K}$-morphism and $W \in A[V]$, then

$$W \xrightarrow{1_X} V \xrightarrow{f} U$$

is a morphism; hence $fin(f, W) \leq U$. Hence, $F_{\mathcal{K}} f[A[V]] \subset A[U]$; so that $\phi(f)$ is an $S(F_{\mathcal{K}})$-morphism. Since ϕ clearly preserves identities and compositions and is one-to-one, it is an embedding. If

$$f: (X, A[V]) \to (Y, A[U])$$

is an $S(F_{\mathcal{K}})$-morphism, then since $V \in A[V]$, $fin(f, V) \in A[U]$. Thus

$$V \xrightarrow{f} fin(f, V) \xrightarrow{1_Y} U$$

is a $\mathcal{K}$-morphism. Thus ϕ is full. We need only show that $\phi[\mathcal{K}]$ is a reflective modification of $S(F_{\mathcal{K}})$. For each $A \subset \mathcal{K}[X]$ let $sup\, A$ be the final lift of the sink $(|V| \xrightarrow{1_X} X)_{V \in A}$. Then

$$(X, A) \xrightarrow{1_X} (X, A[sup\, A])$$

is a morphism since $A \subset A[sup\, A]$. We shall prove that this is a reflection of the object (X, A) in $\phi[\mathcal{K}]$. Now if

$$(X, A) \xrightarrow{f} \phi(V) = (Y, A[V])$$

is an $S(F_{\mathcal{K}})$-morphism, then for each $W \in A$,

$$W \xrightarrow{f} fin(W, f) \xrightarrow{1_Y} V$$

is a $\mathcal{K}$-morphism and

$$W \xrightarrow{f} V = W \xrightarrow{1_X} sup\ A \xrightarrow{f} V$$

implies that $f: sup\, A \to V$ is a $\mathcal{K}$-morphism, so that for each $Z \leq sup\ A$, $f: Z \to V$ is a $\mathcal{K}$-morphism. Thus

$$F_{\mathcal{K}}(f)[A[sup\, A]] \subset A[V],$$

so that f is an $S(F_{\mathcal{K}})$-morphism. Hence $\phi[\mathcal{K}]$ is a reflective modification of $S(F_{\mathcal{K}})$.

2.6. EXAMPLE. Let $\mathcal{X}$ be the trivial one-morphism category. Then a category that is concrete over $\mathcal{X}$ equals its (unique) fibre. Hence, for this situation, «fibre-small» means «small» and «initially complete» means «(possible large) complete lattice». A functor-structured category has the form $(exp M, \subset)$ for some set M. Thus the above theorem (2.5) asserts that every complete lattice is isomorphic to a *inf*-complete sub-semilattice of some $(exp M, \subset)$.

2.7. REMARK. A structural theory of modifications of functor-structured categories is exhibited by Menu and Pultr [MP_1]. These authors also characterize concrete categories that are isomorphic to entire functor-structured categories.

2.8. THEOREM. *For a concrete category* $\mathcal{K}$ *over* $\mathcal{X}$, *the following are equivalent:*

(i) $\mathcal{K}$ *can be fully and concretely embedded in some functor-structured category.*

(ii) $\mathcal{K}$ *can be fully and concretely embedded in some fibre-small initially complete category.*

(iii) $\mathcal{K}$ *has a fibre-small Mac Neille completion.*

(iv) $\mathcal{K}$ *is strongly fibre-small (1.4).*

PROOF. The equivalence of (ii), (iii) and (iv) has previously been established, see [AHS]. That (i) implies (ii) is clear from Proposition 2.2

and that (ii) implies (i) is immediate from Theorem 2.5.

2.9. We now address ourselves to the question of which categories are strongly fibre-small (and thus are full concrete subcategories of functor-structured ones). It turns out that under extremely mild side conditions these are precisely the fibre-small concrete categories. Before showing this we wish to mention a pathological example of a subcategory of *Set* (concrete in the obvious sense) that fails to be strongly fibre-small. The objects are all sets, and the morphisms are all bijections and all constant maps between sets of equal cardinality (see [KP]).

2.10. Let $\mathfrak{X}$ be an (E,M)-category (1.7). A concrete category $\mathfrak{K}$ is said to:

(i) *have weak factorizations* if each morphism $f: V \to W$ factors (not necessarily uniquely) as

$$V \xrightarrow{e} U \xrightarrow{m} W \quad \text{where (as maps) } e \in E \text{ and } m \in M.$$

(ii) be *transportable* if for each isomorphism $f: X \to Y$ in $\mathfrak{X}$ and each $V \in \mathfrak{K}[X]$ there exists some $W \in \mathfrak{K}[Y]$ such that $f: V \to W$ is an isomorphism in $\mathfrak{K}$.

Notice that many concrete categories over *Set* have both properties (i) and (ii) above (when E = surjections and M = one-to-one maps, or more precisely point-separating sources).

The following theorem is a generalization of a result of Kučera and Pultr [KP].

2.11. THEOREM *(Strong fibre-smallness Criterion). Let $\mathfrak{X}$ be an (E,M)-category which is both M-well-powered and E-co-well-powered (1.7). Then for every transportable concrete category over $\mathfrak{X}$ with weak factorizations, strong fibre-smallness is equivalent to fibre-smallness.*

PROOF. Clearly strong fibre-smallness implies fibre-smallness. Suppose that $\mathfrak{K}$ is a fibre-small concrete category with the above properties. We shall prove that $\mathfrak{K}$ is strongly fibre-small.

A) For each structured map $X \xrightarrow{f} |V|$ denote by $L(f,V)$ the conglomerate of all triples (m, e, V') with $m: X' \to X$ a map in M, V' an object of $\mathcal{K}$ and $e: X' \to |V'|$ a map in E, such that there exists a morphism $p: V' \to V$ in $\mathcal{K}$ for which the square

$$\begin{array}{ccc} X' & \xrightarrow{\quad e \quad} & |V'| \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\quad f \quad} & |V| \end{array}$$

commutes. For two structured maps

$$X \xrightarrow{f} |V| \quad \text{and} \quad X \xrightarrow{g} |W|$$

we shall show that $L(f, V) = L(g, W)$ implies that (f, V) and (g, W) are structurally equivalent (1.4). Consider a structured map $|U| \xrightarrow{h} X$. Assuming that $f.h: U \to V$ is a morphism we shall prove that $g.h: U \to W$ is one also (so that by symmetry we obtain a structural equivalence). By hypothesis $f.h$ can be factored as

$$U \xrightarrow{e} V' \xrightarrow{m} V$$

where, as maps, $e \in E$ and $m \in M$. Furthermore we have a factorization (in $\mathcal{X}$) of the map h as

$$|U| \xrightarrow{\hat{e}} X' \xrightarrow{\hat{m}} X$$

with $\hat{e} \in E$ and $\hat{m} \in M$. Thus by the (E, M)-diagonalization property there is a map $f': X' \to |V'|$ such that the diagram

$$\begin{array}{ccccc} |U| & \xrightarrow{\quad \hat{e} \quad} & X' & \xrightarrow{\quad \hat{m} \quad} & X \\ {\scriptstyle e}\downarrow & \swarrow{\scriptstyle f'} & & & \downarrow{\scriptstyle f} \\ |V'| & & \xrightarrow{\qquad m \qquad} & & |V| \end{array}$$

commutes. Now f' must be in E. Thus

$$(\hat{m}, f', V') \in L(f, V) = L(g, W),$$

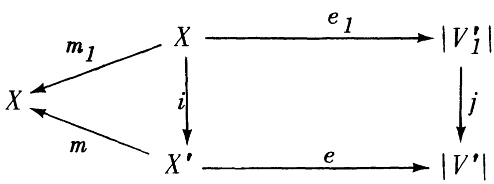
so that there exists a morphism

$$p: V' \to W \quad \text{with} \quad p.f' = g.\hat{m}.$$

Hence $g.h = p.e$ is a morphism.

B) $\mathcal{K}$ *is strongly fibre-small.* For each $X \xrightarrow{f} |V|$ the above conglomerate $L(f, V)$ is a subconglomerate of $\Omega(X)$, the conglomerate of all triples (m, e, V') with $m: X' \to X$ in M and $e: X' \to |V'|$ in E. Consider the equivalence relation $\equiv$ on $\Omega(X)$ defined by:

$(m, e, V') \equiv (m_1, e_1, V_1')$ iff there exist isomorphisms i in $\mathcal{X}$ and j in $\mathcal{K}$ such that the following diagram commutes:



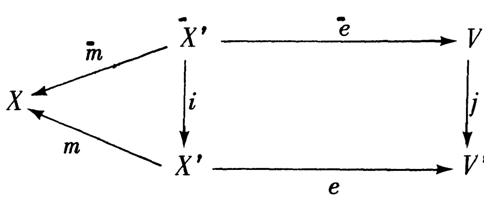
Clearly $L(f, V)$ is stable under this equivalence, i.e., if (m, e, V') is in $L(f, V)$, then

$$(m_1, e_1, V_1') \equiv (m, e, V')$$

implies that $(m_1, e_1, V_1') \in L(f, V)$. (Given a morphism $p: V' \to V$, consider the morphism $p.j: V_1' \to V$.) Thus by Part A it suffices to show that $\Omega(X)$ has a small system of representatives with respect to $\equiv$. Since $\mathcal{X}$ is M-well-powered, we have a small system of representatives D of M-subobjects $m: X' \to X$. Since $\mathcal{X}$ is E-co-well-powered, for each $m \in D$ we have a small system of representatives $D(m)$ of E-quotients $e: X' \to Y$. Define $D^* \subset \Omega(X)$ by:

$$D^* = \{ (m, e, V') \mid m \in D,\ e \in D(m) \text{ and } |V'| \text{ is the codomain of } e \}.$$

Since $\mathcal{K}$ is fibre-small, D^* is a set. For each $(\bar{m}, \bar{e}, \bar{V}')$ in $\Omega(X)$ we have $m \in D$ isomorphic to $\bar{m}$ (via an isomorphism i), and $e \in \bar{D}(m)$ isomorphic to $\bar{e}.i^{-1}$. Since $\mathcal{K}$ is transportable, there exists a $\mathcal{K}$-object V' such that $j: \bar{V}' \to V'$ is an isomorphism in $\mathcal{K}$ and



commutes. Thus $(\bar{m}, \bar{e}, \bar{V}') \equiv (m, e, V') \in D^*$. Hence D^* is a system of representatives for $\equiv$.

2.12. REMARK. The hypothesis that $\mathcal{K}$ be transportable cannot be omitted. For example, a large discrete category made concrete over *Set* by an arbitrary one-to-one functor to singleton sets is fibre-small and has suitable weak factorizations, yet fails to be strongly fibre-small. Indeed, given a singleton set X, two structured maps

$$X \xrightarrow{f} |V| \quad \text{and} \quad X \xrightarrow{g} |W|$$

are non-equivalent whenever $V \neq W$. Consider $|V| \xrightarrow{f^{-1}} X$, then

$$f.f^{-1}: V \to V$$

is a morphism but $g.f^{-1}: V \to W$ is not.

2.13. COROLLARY. *Let $\mathcal{K}$ be a transportable concrete category with weak factorizations over an (E, M)-category that is M-well-powered and E-co-well-powered. Then the following are equivalent:*

(i) $\mathcal{K}$ can be fully and concretely embedded in some functor-structured category.

(ii) $\mathcal{K}$ can be fully and concretely embedded in some fibre-small initially complete category.

(iii) $\mathcal{K}$ has a fibre-small Mac Neille completion.

(iv) $\mathcal{K}$ is fibre-small.

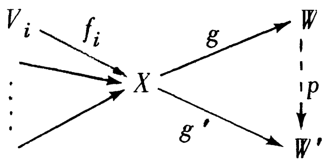
3. REFLECTIVE INITIAL COMPLETIONS.

An important generalization of initial (or final) completeness has been introduced by Trnková [Tr] (weak inductive generation), Hoffmann [Ho_1] (semi-identifying functors), Wischnewsky [W] and Tholen [Th] (semi-topological functors). See also [E_1] (functors with quasi-quotients). Here we call such concrete categories *finally semi-complete*.

3.1. A concrete category $|\ | : \mathcal{K} \to \mathcal{X}$ is called *finally semi-complete* provided that each sink $(|V_i| \xrightarrow{f_i} X)$ has a *semi-final lift*, i.e., a structured map $X \xrightarrow{g} |W|$ with the properties:

(i) each $g.f_i : V_i \to W$ is a morphism ;

(ii) whenever $X \xrightarrow{g'} |W'|$ is a structured map such that each $g'.f_i$ is a morphism, there exists a unique morphism $p: W \to W'$ with $g' = p.g$.



3.2. THEOREM. *For any concrete category* $|\ |: \mathcal{K} \to \mathcal{X}$, *the following are equivalent:*

(i) $\mathcal{K}$ *has a reflective, fibre-small initial completion.*

(ii) $\mathcal{K}$ *has a full reflective concrete embedding in a functor-structured category.*

(iii) $\mathcal{K}$ *is finally semi-complete and strongly fibre-small.*

Furthermore if $\mathcal{K}$ *is co-well-powered and* $\mathcal{X}$ *is cocomplete, the above are equivalent to:*

(iv) $\mathcal{K}$ *has free objects (i.e.,* $|\ |$ *has a left adjoint), is cocomplete and is strongly fibre-small.*

PROOF. (i)$\Rightarrow$(ii). Let $\mathcal{L}$ be a fibre-small initial completion of $\mathcal{K}$. By Theorem 2.5, $\mathcal{L}$ is concretely isomorphic to a reflective modification of some functor-structured category $S(F)$. If $\mathcal{K}$ is reflective in $\mathcal{L}$, then it will also be reflective in $S(F)$.

(ii)$\Rightarrow$(iii). Any concrete category that has a reflective initial completion is well-known to be finally semi-complete [Th, HS_2]. By Theorem 2.8 any such category that has a full concrete embedding in a functor-structured category must be strongly fibre-small.

(iii)$\Rightarrow$(i). Any concrete category that is finally semi-complete is well-known to have a reflective Mac Neille completion [Th, HS_2] and by Theorem 2.8 such a completion must be fibre-small.

(iii)$\Rightarrow$(iv). This is established in [Ho_1] and [Th].

3.3. COROLLARY. *Let* $\mathcal{K}$ *be a co-well-powered, transportable concrete category over Set in which each extremal epimorphism is surjective. Then*

$\mathcal{K}$ can be fully reflectively embedded in a functor-structured category iff $\mathcal{K}$ is fibre-small, cocomplete and has free objects.

PROOF. Clearly if $\mathcal{K}$ can be so embedded, it must be fibre-small, cocomplete and have free objects. To show the converse, apply Theorem 2.11 with $\mathcal{X} = Set$, E = surjective maps and M = point separating sources (= mono-sources). Indeed, a co-well-powered cocomplete category always has (extremal epi; mono)-factorizations of morphisms [HS_1]. Since $\mathcal{K}$ has free objects, monomorphisms are one-to-one, and by hypothesis extremal epimorphisms are surjective. Hence $\mathcal{K}$ has weak factorizations (2.10) and so it is strongly fibre-small. Thus Theorem 3.2 can be applied.

3.4. REMARKS. (a) A reflective full concrete embedding of $\mathcal{K}$ in a functor-structured category does not guarantee that $\mathcal{K}$ is co-well-powered. In fact Herrlich [He_3] has exhibited a reflective subcategory of the category of topological spaces that is not co-well-powered.

(b) There is a fibre-small concrete category that is not strongly fibre-small in spite of being finally semi-complete. See [He_4]. Thus, a concrete fibre-small category can have a reflective Mac Neille completion that fails to be fibre-small.

4. E-REFLECTIVE INITIAL COMPLETIONS.

4.1. If the base category $\mathcal{X}$ is an (E, M)-category (1.7), then a concrete category $\mathcal{K}$ over $\mathcal{X}$ is said to be:

(i) *initially M-complete* (= (E, M)-topological in [He_2] and [Ho]) iff every source $(X \xrightarrow{f_i} |V_i|)$ that is carried by an M-source $(f_i : X \to Y_i)$ has an initial lift;

(ii) *finally E-semi-complete* iff it is finally semi-complete and each semi-final lift $X \xrightarrow{g} |W|$ is carried by an E-map, i.e. $g \in E$;

(iii) *M-hereditary* iff each singleton source $X \xrightarrow{m} |V|$ in M has an initial lift;

(iv) *topologically algebraic* iff each source has a (generating[1]),

[1] A structured map (f, V) is said to be *generating* provided that for each $\mathcal{K}$-object W and each pair of morphisms $r, s : V \to W$, $r.f = s.f$ implies $r = s$.

initial-)factorization (see $[HS_2]$).

4.2. THEOREM. *If $\mathfrak{X}$ is an (E, M)-category then for each transportable concrete category $\mathfrak{K}$ over $\mathfrak{X}$, the following are equivalent:*

(i) $\mathfrak{K}$ is initially M-complete.

(ii) $\mathfrak{K}$ is topologically-algebraic and M-hereditary.

(iii) $\mathfrak{K}$ is finally E-semi-complete.

(iv) $\mathfrak{K}$ is E-reflective in its Mac Neille completion.

(v) $\mathfrak{K}$ is E-reflective in each of its final completions and has at least one final completion.

PROOF. (i) ⇒ (ii). Let $S = (X \xrightarrow{f_i} |V_i|)$ be an arbitrary source. The underlying map-source factors as an E-map $g: X \to Y$ followed by an M-source $(Y \xrightarrow{\hat{f}_i} |V_i|)$. The latter has an initial lift, say W, and we have a factorization of S as a generating map $X \xrightarrow{g} |W|$ followed by an initial morphism-source $(\hat{f}_i: W \to V_i)$. Thus $\mathfrak{K}$ is topologically-algebraic. That $\mathfrak{K}$ is M-hereditary is immediate.

(ii) ⇒ (iii). Given a sink $S = (|U_j| \xrightarrow{h_j} X)$, consider the opposite source $S^{op} = (X \xrightarrow{f_i} |V_i|)$. By (ii) S^{op} has a factorization as a generating structured map $X \xrightarrow{g} |W|$ followed by an initial source $(\hat{f}_i: W \to V_i)$. Let

$$X \xrightarrow{e} Y \xrightarrow{m} |W|$$

be the (E, M)-factorization of g. Then, by the M-hereditary property, there exists an object V such that $Y = |V|$ and $m: V \to W$ is initial. Since for each i, j we have a morphism

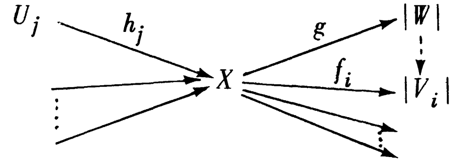
$$f_i . h_j : U_j \to V_i \ (= U_j \xrightarrow{e . h_j} |V| \xrightarrow{\hat{f}_i . m} V_i),$$

initiality implies that each $e . h_j : U_j \to V$ is a morphism. Clearly $X \xrightarrow{e} |V|$ is a semi-final lift of S.

(iii) ⇒ (v). Since $\mathfrak{K}$ is finally semi-complete it has a (reflective) final completion $\mathfrak{L}$. We wish to show that $\mathfrak{K}$ is E-reflective in $\mathfrak{L}$. Now each object P of $\mathfrak{L}$ is the final lift of some sink $(|V_i| \xrightarrow{f} X)$ with each V_i in $\mathfrak{K}$. By finality the semi-final lift $X \xrightarrow{g} |W|$ (with $g \in E$) induces an $\mathcal{L}$-morphism $g: P \to W$ that is obviously an E-reflection.

(v) ⇒ (iv). Clear.

(iv) ⇒ (i). Let $S = (X \xrightarrow{f_i} |V|)$ be an M-source. Its (closed) opposite sink $S^{op} = (|U_j| \xrightarrow{h_j} X)$ is an object of the Mac Neille completion. Hence it has an E-reflection $g: S^{op} \to S_W$ where W is an object of $\mathcal{K}$. For each i, j, $f_i . h_j : U_j \to V_i$ is a morphism, so that by reflectivity each f_i factors through g.



Since S is an M-source that factors through $g \in E$, it follows that g is an isomorphism. Clearly W is the initial lift of S.

4.3. REMARKS. The equivalence of (i) and (iv) in the above theorem was established in [He_2] and the equivalence of (i) and (ii) in [Ho_1]. *Universal initial completions* form an important type of initial completions (see [He_1]). In [HS_2] it is proved that a concrete category has a reflective universal initial completion iff it is topologically-algebraic. This is a strictly stronger condition than having a reflective Mac Neille completion (i.e., stronger than final semi-completeness); see [BT] and [HNST]. (For related results, see [E_2].) In contrast, given an (E, M)-category $\mathcal{X}$:

A concrete category $\mathcal{K}$ over $\mathcal{X}$ has an E-reflective Mac Neille completion iff it has an E-reflective universal initial completion.

PROOF. If $\mathcal{K}$ has an E-reflective Mac Neille completion then by Theorem 4.2 it is topologically-algebraic and M-hereditary. Thus the category of semi-closed sources (which is the universal initial completion) is legitimate (see [AHS, 3.2]). Factoring a semi-closed source S as $X \xrightarrow{g} |W|$ with $g \in E$, followed by an initial source, we see that $X \xrightarrow{g} |W|$ must be a member of S. Hence $g: S \to S^W$ is a reflection for S in $\mathcal{K}$; so that $\mathcal{K}$ has an E-reflective universal initial completion. The converse is clear.

4.4. THEOREM. *If the base category $\mathcal{X}$ is an (E,M)-category, then for* *each transportable concrete category* $\mathcal{K}$ *over* $\mathcal{X}$, *the following are equivalent:*

(i) $\mathcal{K}$ *has an E-reflective, fibre-small initial completion.*

(ii) $\mathcal{K}$ *has a full concrete E-reflective embedding in a functor-structured category.*

(iii) $\mathcal{K}$ *is initially M-complete and strongly fibre-small.*

Furthermore if $\mathcal{X}$ *is complete and E-co-well-powered, the above are equivalent to:*

(iv) $\mathcal{K}$ *is fibre-small, M-hereditary and has concrete products* [1].

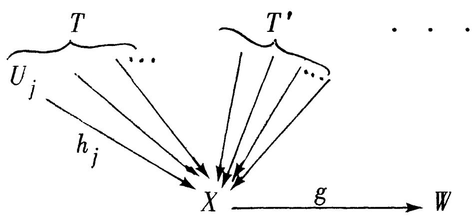
PROOF. (iii)$\Rightarrow$(ii). By Theorem 2.8 the Mac Neille completion $\mathfrak{L}$ (i.e., the category of closed sinks) is fibre-small. Thus by Theorem 2.5, $\mathfrak{L}$ is a reflective modification of $S(F_{\mathfrak{L}})$ when each object S of $\mathfrak{L}$ (i.e., each closed sink in $\mathcal{K}$) is identified with $(X, A[S])$ where

$$X = |S| \quad \text{and} \quad A[S] = \{\, T \in \mathfrak{L}[X] \mid T \subset S \,\}.$$

Recall from 1.6 that each object V of $\mathcal{K}$ is identified with the sink S_V in $\mathfrak{L}$. Thus, now we identify each V in $\mathcal{K}$ with

$$\hat{V} = (X, A[S_V]) \quad \text{in} \quad S(F_{\mathfrak{L}}), \quad \text{where} \quad X = |V|.$$

We shall verify that $\mathcal{K}$ (or, more precisely, the full subcategory determined by objects $\hat{V}$ with $V \in \mathcal{K}$) is an E-reflective subcategory of $S(F_{\mathfrak{L}})$. An object (X, A) in $S(F_{\mathfrak{L}})$ is a collection A of closed sinks with codomain X. The union of all these sinks is a (not necessarily closed) sink $T^* = \cup A$ which, by Theorem 4.2 (iii), has a semi-final lift $X \xrightarrow{g} |W|$ with $g \in E$.



We claim that $g: (X, A) \to \hat{W}$ is a reflection map for (X, A).

(a) We claim that $g: (X, A) \to (Y, A[S_W])$ is a morphism in $S(F_{\mathfrak{L}})$ (where $Y = |W|$). For each sink $T = (|U_j| \xrightarrow{h_j} X)_{j \in J}$ in A we must show

1) *concrete products* are products preserved by the forgetful functor.

that $fin(g, T) \subset S_W$. Now $fin(g, T)$ is the final lift (in $\mathcal{L}$) of $|T| \xrightarrow{g} Y$. This is the least closed sink with codomain Y that contains

$$|U_j| \xrightarrow{g.h_j} Y \quad \text{for each } j \in J.$$

Since $T \subset T^*$, it is clear that each $g.h_j: U_j \to W$ is a morphism in $\mathcal{K}$; in other words each $|U_j| \xrightarrow{g.h_j} Y$ is an element of S_W. Thus the sink S_W is closed, so that $fin(g, T) \subset S_W$.

(b) Suppose that $g': (X, A) \to (Z, A[S_V])$ is another morphism in $S(F\mathcal{L})$. Since $X \xrightarrow{g} |W|$ is a semi-final lift of T^*, to show that g' factors through g it suffices to show that, for each i, $g'.f_i: V_i \to V$ is a morphism. But since g' is an $S(F\mathcal{L})$-morphism, $T \in A$ implies

$$g'[T] \subset fin(g, T) \in A[S_V].$$

Thus $g'[T^*] \subset A[S_V]$, so that $g'.f_i$ is a morphism.

(ii) ⇒ (i). Immediate from Proposition 2.2.

(i) ⇒ (iii). Immediate from Theorems 2.8 and 4.2.

(iii) ⇒ (iv). We need only show the existence of concrete products. If $P = (X \xrightarrow{p_i} X_i)$ is a product in $\mathcal{X}$ with each $X_i = |V_i|$, then P is an M-source and its initial lift clearly gives the product of (V_i) in $\mathcal{K}$.

(iv) ⇒ (iii). By the M-hereditary property, each structured map from $X \in \mathcal{X}$ can be written as

$$X \xrightarrow{e} |U| \xrightarrow{m} |V|$$

with $e \in E$ and $m: U \to V$ an initial morphism. Given a structured map $|Q| \xrightarrow{h} X$, then $m.e.h: Q \to V$ is a morphism iff $e.h: Q \to U$ is. It follows that:

(a) two structured maps

$$X \xrightarrow{e_i} |U_i| \xrightarrow{m_i} |V_i| \quad (i = 1, 2)$$

are structurally equivalent iff

$$X \xrightarrow{e_i} |U_i| \quad (i = 1, 2)$$

are. Since $\mathcal{K}$ is transportable and fibre-small and since $\mathcal{X}$ is E-co-well-powered, it is clear that $\mathcal{K}$ is strongly fibre-small.

(b) For each source S

$$S = (X \xrightarrow{e_i} |U_i| \xrightarrow{m_i} |V_i|)$$

any initial lift is the same as the initial lift of $S' = (X \xrightarrow{e_i} |U_i|)$. Hence by E-co-well-poweredness it suffices to consider only sources indexed by sets. Now given a small M-source $S = (X \xrightarrow{f_i} |W_i|)_{i \in I}$, let

$$Y = \prod_{i \in I} |W_i| = |\prod_{i \in I} W_i| .$$

The induced map $f: X \to Y$ belongs to M, so that by the M-hereditary property we have an initial lift W of $X \xrightarrow{f} |\Pi W_i|$. Thus W is the initial lift of S.

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