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RUDOLF-E. HOFFMANN

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NOTE ON UNIVERSAL TOPOLOGICAL COMPLETION

by Rudolf-E. HOFFMANN

In [8] (1.3) H. Herrlich has introduced a universal topological completion $E^3: (\underline{A}, U) \to (\underline{A}^3, U^3)$ of a faithful and amnestic 1) functor $U: \underline{A} \to \underline{X}$. His examples [8] (3.1 a,b) are in a sense «negative». The purpose of this paper is to obtain - essentially by the aid of our former investigations [9, 12] - some «positive» examples, i.e. satisfactory interpretations of the «universal topological completion category» $\underline{A}^3$ for several familiar functors $U: \underline{A} \to \underline{X}$. As an essential application of these interpretations we shall see that even for the forgetful functor

U: *Comp*-T_2 (compact T_2-spaces and continuous maps) $\to$ *Ens*

the MacNeille-Antoine-completion is strictly smaller than the universal topological completion.

We give a necessary and sufficient condition for a functor $U: \underline{A} \to \underline{X}$ admitting a factorization $(\underline{J}, \underline{M})$ of *relative cones* (as indicated in [12], page 288) in order that its universal topological completion coincides with its $\underline{J}$-canonical extension [12] (1.4). So we only verify the condition under which this happens for the examples obtained in [12] Section 3 (and similarly for some examples of [9]) in order to ensure all of them being examples of the present situation. The general condition obtained implies that the functor U in question has to be *topologically-algebraic* in the sense of S.S. and Y.H. Hong [15, 16].

We note that there is a formal analogy between this paper and [14] concerning the fact that the results of this paper depend on a crucial idea, that of a *basis of a W-co-sieve* (1.2), parallel to (but very much different from) the concept of a subbasis in [14]. (The present results are, of course, not comparable with those in [14].)

1) The hypothesis of amnesticity may be discarded as is pointed out in [13]. Then E^3 as constructed in [8], 1.3 will be non-injective on objects, i.e. E^3 is only full and faithful. (Replacing $\underline{A}^3$ by an equivalent copy one still obtains an embedding $(U, \underline{A}) \to (U, \underline{A}^3)$.)

We are working in a fixed universe $\underline{U}$. Other than in [8] the constructions, if necessary, will be carried out in the next universe $\underline{U}^+$ (with $\underline{U} \in \underline{U}^+$). Recall that a category $\underline{C}$ is *$\underline{U}$-legitimate* iff

$$Ob\,\underline{C},\ Mor\,\underline{C} \subset \underline{U} \quad \text{and} \quad Hom(A, B) \in \underline{U}$$

for every pair A, B of objects of $\underline{C}$; a set M is *$\underline{U}$-small* (resp. a *$\underline{U}$-class*) iff $M \in \underline{U}$ (resp. $M \subset \underline{U}$).

1.1. Suppose $\mathbb{W}: \underline{A} \to \underline{X}$ is a faithful functor. The objects of $\underline{A}^u$ (Herrlich's $\underline{A}^3$, [8] page 103) are certain $\mathbb{W}$-co-sieves [13, 22] ; i.e. pairs (X, ξ) with $X \in Ob\,\underline{X}$ and ξ a collection of pairs (u, A) with

$$A \in Ob\,\underline{A}, \quad u \in Hom_{\underline{X}}(X, \mathbb{W}A)$$

subject to the following requirements:

(i) If $(u, A) \in \xi$ and $g: A \to B$ is an $\underline{A}$-morphism, then $(\mathbb{W}(g)u, B) \in \xi$, i.e. (X, ξ) is a $\mathbb{W}$-co-sieve.

(ii) If $(A, \{f_i: A \to A_i\}_{i \in I})$ is a $\mathbb{W}$-co-identifying cone (= $\mathbb{W}$-initial source in [6, 8]) indexed by a $\underline{U}^+$-small set I and $u: X \to \mathbb{W}A$ is an $\underline{X}$-morphism such that $(\mathbb{W}(f_i)u, A_i) \in \xi$ for every $i \in I$, then $(u, A) \in \xi$.

The morphisms of $\underline{A}^u$ from (X, ξ) into (X', ξ') are those $\underline{X}$-morphisms $v: X \to X'$ such that

$$(uv, A) \in \xi \quad \text{whenever} \quad (u: X' \to \mathbb{W}A,\ A) \in \xi'.$$

(Hom-sets have to be made disjoint in the canonical way.) Composition in $\underline{A}^u$ is «the same» as in $\underline{X}$.

A faithful functor $\mathbb{W}^u: \underline{A}^u \to \underline{X}$ is obtained by the assignment

$$(X, \xi) \mapsto X, \quad u \mapsto u.$$

A full and faithful functor $F: \underline{A} \to \underline{A}^u$ is described by

$$A \mapsto (\mathbb{W}A, \{(\mathbb{W}g, B) \mid B \in Ob\,\underline{A},\ g \in Hom(A, B)\}).$$

(F is an embedding iff $\mathbb{W}$ is amnestic.)

Note that $\underline{A}^u$ need not be $\underline{U}$-legitimate [8] (3.1 b). We observe that $\mathbb{W}$ is a topological functor in the universe $\underline{U}^+$ ($\mathbb{W}$ trivially satisfies the $\underline{U}^+$-smallness condition [10], 2.1 (a) 1). In view of [8] (1.3.3),

$F:(\underline{A}, W) \to (\underline{A}^u, W^u)$ may be called the *universal topological completion of the faithful functor* $W: \underline{A} \to \underline{X}$.

1.2. For application in the proof of our criterion (1.7) we need the notion of a *W-basis of a W-co-sieve* consisting of an $\underline{X}$-object X and an arbitrary collection β of pairs (u, A) with

$$A \in Ob\underline{A}, \quad u \in Hom_{\underline{X}}(X, WA).$$

The W-co-sieve (X, β') *generated by* (X, β) is

$$(X, \{ (v: X \to WB,\ B) \mid \text{there is some } (u, A) \in \beta \text{ and some } g \in Hom_{\underline{A}}(A, B) \text{ with } v = W(g)u \}).$$

1.3. (a) Suppose $W: \underline{A} \to \underline{X}$ is a functor. A *W-relative cone* consists of a set I, an I-indexed family $\{A_i\}_{i \in I}$ of $\underline{A}$-objects and a cone

$$(X, \{u_i : X \to WA_i\}_{i \in I})$$

in $\underline{X}$. *A factorization of W-relative cones* consists of a class $\underline{J}$ of W-epimorphisms [2)] and a class $\underline{M}$ of cones in $\underline{A}$ indexed by sets (up to a suitable size - see below) subject to the following requirements:

(0) For every $\underline{A}$-isomorphism $k: A \to B$ holds $(Wk, B) \in \underline{J}$. (As a consequence, W has to be *faithful*.) Furthermore, if

$$(u, A) \in \underline{J}, \quad (B, \{m_i : B \to B_i\}_{i \in I}) \in \underline{M},$$

and $k: A \to B$ is an $\underline{A}$-isomorphism, then

$$(W(k)u,\ B) \in \underline{J} \quad \text{and} \quad (A, \{m_i k : A \to B_i\}_{i \in I}) \in \underline{M}.$$

(1) For every W-relative cone $(X; \{u_i : X \to WA_i,\ A_i\}_{i \in I})$ whose index set does not exceed a certain size (to be specified below) we assume the existence of members $(u: X \to WA, A)$ and $(A, \{m_i : A \to A_i\}_{i \in I})$ of $\underline{J}$ and, resp., $\underline{M}$ such that $W(m_i)u = u_i$ for every $i \in I$.

(2) $(\underline{J}, \underline{M})$ satisfies a *diagonal condition*: Whenever

2) A W-epimorphism is a pair (u, A) with $A \in Ob\underline{A}$, $u \in Hom_{\underline{X}}(X, WA)$, such that, whenever $W(f)u = W(g)u$ for some $f, g \in Hom_{\underline{A}}(A, B)$ with $B \in Ob\underline{A}$, then $f = g$.

$$\begin{array}{ccc} X & \xrightarrow{\quad u \quad} & \mathbb{W}A \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle \mathbb{W}f_i} \\ \mathbb{W}B & \xrightarrow{\quad \mathbb{W}m_i \quad} & \mathbb{W}B_i \end{array}$$

commutes for every $i \in I$ with

$$(u, A) \in \underline{J} \quad \text{and} \quad (B, \{m_i : B \to B_i\}_{i \in I}) \in \underline{M},$$

$$v \in Hom_{\underline{X}}(X, \mathbb{W}B), \quad f_i \in Hom_{\underline{A}}(A, B_i) \quad (i \in I),$$

then there exists a (necessarily) unique $\underline{A}$-morphism

$$h: A \to B \quad \text{with} \quad \mathbb{W}(h)u = v,$$

(hence) $m_i h = f_i$ for every $i \in I$ (since $\mathbb{W}$ is faithful and (u, A) is a $\mathbb{W}$-epimorphism).

(3) The upper cardinal bound for the index sets I may be chosen as the cardinal of the universe $\underline{U}$ itself.

For the following it will be relevant to observe that a $\mathbb{W}$-relative cone $(X, \{u_i : X \to \mathbb{W}A_i, A_i\}_{i \in I})$ whose index set I is too large can be suitably re-indexed; Take the identity on the set $\{(u_i, A_i) \mid i \in I\}$ of «$\mathbb{W}$-morphisms». After factorizing this $\mathbb{W}$-relative cone according to (1), one may re-index the $\underline{M}$-cone thus obtained by the set I. Thus the meaning of $\underline{M}$-cone is extended to the case of arbitrary index sets: (0), (1), (2) remain valid (cf. [10] 2.0[3]).

Generalizing H. Herrlich's concept in [7], this definition was proposed in [12] page 288, but not explicitly given. A (variant of a) special case was studied earlier by Y.H. and S.S. Hong [15, 16]. The concept also appears in a recent series of preprints of M.B. Wischnewsky and W. Tholen; some details given below were worked out independently by these authors and by myself (unpublished).

(b) Under the hypothesis that $\mathbb{W}$ admits a factorization $(\underline{J}, \underline{M})$ of relative cones, one readily observes that:

[3] Note that in [10] the terms «cone» and «co-cone» are interchanged.

(1) W has a left adjoint L, given by factoring the relative cones with domain $X \epsilon Ob\underline{X}$ and «co-domains» varying over all $A \epsilon Ob\underline{A}$.

(2) With

$$\underline{E} =: \{ f \epsilon Mor_{\underline{A}}(A,B) \mid A, B \epsilon Ob\underline{A},\ (Wf, B) \epsilon \underline{J} \}$$

we obtain a factorization of cones in $\underline{A}$ subject to the requirement that $\epsilon_A \epsilon \underline{E}$ for the co-unit $\epsilon: LWA \to A$ of the adjunction $L \dashv W$. (*)

(3) Conversely, given a (faithful) right adjoint functor $W: \underline{A} \to \underline{X}$ with unit $\eta: id_{\underline{X}} \to WL$ and a factorization $(\underline{E}, \underline{M})$ of cones in $\underline{A}$ subject to condition (*), then W admits a relative factorization $(\underline{J}, \underline{M})$ of cones with

$$\underline{J} = \{ (u: X \to WA,\ A) \mid A \epsilon Ob\underline{A},\ u \epsilon Mor\underline{X},\ \hat{u}: LX \to A \text{ in } \underline{E} \text{ with } \hat{u} \text{ determined by } W(\hat{u})\eta_X = u \}.$$

From (1), (2), (3) and the theorem on factorizations in [11] 1.1, we deduce by the intersection property ([11], 1.2 (c)) that, if W admits a factorization of relative cones, then W also admits a smallest factorization $(\underline{J}, \underline{M})$ of relative cones in the sense that $\underline{J}$ is the smallest possible class of W-epimorphisms inducing a factorization of W-relative cones.

Furthermore, by (1), (2), (3) many of the standard results on factorizations of cones carry over to factorizations of relative cones (cf. [11] Section 0), e.g. that $\underline{J}$ determines $\underline{M}$ and vice versa.

1.4. (a) With the preceding definition one has the following Lawvere type comma construction (cf. [12] 1.4) which we shall call the *$\underline{J}$-canonical extension of W*.

Suppose $W: \underline{A} \to \underline{X}$ is a (faithful) functor admitting a factorization $(\underline{J}, \underline{M})$ of W-relative cones. The objects of $\underline{A}^{\underline{J}}$ are precisely the members of $\underline{J}$. The morphisms of $\underline{A}^{\underline{J}}$ from $(u: X \to WA,\ A)$ to $(v: Y \to WB,\ B)$ are pairs $(s, g) \epsilon Hom_{\underline{X}}(X, Y) \times Hom_{\underline{A}}(A, B)$ such that

$$\begin{array}{ccc} X & \xrightarrow{\quad u \quad} & WA \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle Wg} \\ Y & \xrightarrow{\quad v \quad} & WB \end{array}$$

commutes. (Hom-sets are made pairwise disjoint in the standard way.) Composition is defined co-ordinatewise.

There is a full embedding $J: \underline{A} \hookrightarrow \underline{A}^{\underline{J}}$ with

$$A \in Ob\underline{A} \mapsto (id_{WA}, A) \in Ob\underline{A}^{\underline{J}}$$

and a faithful functor $W^{\underline{J}}: \underline{A}^{\underline{J}} \to \underline{X}$ mapping (u, A), (s, g) into their first coordinate.

(b) Moreover $W^{\underline{J}}: \underline{A}^{\underline{J}} \to \underline{X}$ is a topological functor (in the wider sense of [4; 6; 10, 2.1 (b)]): Suppose $\{(u_i: X_i \to WA_i, A_i)\}_{i \in I}$ is a family of objects of $\underline{A}^{\underline{J}}$ and $(X, \{s_i: X \to X_i\}_{i \in I})$ is a cone in $\underline{X}$, then factor the W-relative cone

$$(X, \{u_i s_i: X \to WA_i, A_i\}_{i \in I}),$$

in order to obtain a member $(u: X \to WA, A)$ of $\underline{J}$ and a member

$$(A, \{m_i: A \to A_i\}_{i \in I})$$

of $\underline{M}$. In an obvious way these data are interpreted as a cone in $\underline{A}^{\underline{J}}$ which turns out to be the $W^{\underline{J}}$-co-identifying lift of the given (lifting) datum.

1.5. LEMMA. *Suppose $W: \underline{A} \to \underline{X}$ admits a factorization $(\underline{J}, \underline{M})$ of relative cones. If $(B, \{m_i: B \to B_i\}_{i \in I}) \in \underline{M}$, then $(B, \{m_i\}_{i \in I})$ is W-co-identifying.*

PROOF. For a cone $(A, \{f_i: A \to B_i\}_{i \in I})$ in $\underline{A}$ and a morphism

$$v: WA \to WB \text{ in } \underline{X} \text{ with } W(f_i) = W(m_i)v \text{ for every } i \in I,$$

consider

$$\begin{array}{ccc} WA & = & WA \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle Wf_i} \\ WB & \xrightarrow{Wm_i} & WB_i \end{array} .$$

By 1.3 (2) there exists a morphism $h: A \to B$ with $Wh = v$. Since W is faithful, h is uniquely determined.

1.6. THEOREM. *Suppose $W: \underline{A} \to \underline{X}$ admits a factorization $(\underline{J}, \underline{M})$ of relative cones, then there is a full and faithful functor $K: \underline{A}^u \to \underline{A}^{\underline{J}}$ with:*

$$W^J \circ K = W^u \quad \textit{and} \quad K \circ F \approx J.$$

PROOF. Let $(X,\xi) \in Ob\underline{A}^u$. (X,ξ) may be considered as a W-relative cone and may be factored by 1.3 (1) into a member $(p: X \to WB,\ B)$ of $\underline{J}$ and an $\underline{M}$-cone

$$(B, \{m_{(u,A)}: B \to A\}_{(u,A)\in\xi}).$$

If $s: (X,\xi) \to (X',\xi')$ in $Mor\underline{A}^u$, then there is a mapping $\phi : \xi' \to \xi$ with

$$\phi(u', A') := (u's, A') \in \xi,$$

thus the following square commutes

$$\begin{array}{ccc}
X & \xrightarrow{\quad p \quad} & WB \\
{\scriptstyle s}\downarrow & & \\
X' & & \downarrow{\scriptstyle W(m_{\phi(k)})} \\
{\scriptstyle p'}\downarrow & & \\
WB' & \xrightarrow{\quad W(m'_k) \quad} & WB_k
\end{array}$$

with

$$k := (u', A') \in \xi', \quad B_k := A'.$$

By the diagonal condition 1.3 (2) there exists a morphism

$$h: B \to B' \quad \text{with} \quad W(h)p = p's.$$

Thus a functor $K: \underline{A}^u \to \underline{A}^J$ with $W^J \circ K = W^u$ is defined. One readily checks that $K \circ F \approx J$. Now suppose that we are given the commutative square

$$\begin{array}{ccc}
X & \xrightarrow{\quad p \quad} & WB \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle Wh} \\
X' & \xrightarrow{\quad p' \quad} & WB'
\end{array}$$

with $h \in Mor_{\underline{A}}(B, B')$ and with p, p' arising from (X,ξ) and, resp., (X',ξ') by factoring (but no further hypothesis on s). If $(u', A') \in \xi'$, then $u' = W(m'_{(u',A')})p'$, hence

$$u's = W(m'_{(u',A')})p's = W(m'_{(u',A')}h)p.$$

Since $(p, B) \in \xi$, $(u's, A') \in \xi$, hence $s: (X,\xi) \to (X',\xi')$ is an $\underline{A}^u$-morphism. In consequence, $K: \underline{A}^u \to \underline{A}^{\underline{J}}$ is full.

1.7. THEOREM. *Suppose $W: \underline{A} \to \underline{X}$ is a faithful functor. Then the following assertions are equivalent:*

(a) W admits a factorization $(\underline{J}, \underline{M})$ of relative cones such that there is an equivalence $K: \underline{A}^u \to \underline{A}^{\underline{J}}$ with $W^{\underline{J}} \circ K = W^u$ and $K \circ F \approx J$.

(b) There is a factorization

$$(\underline{J}^*, \{ W\text{-co-identifying cones indexed by } \underline{U}\text{-classes} \})$$

of W-relative cones.

If (a) or (b) is satisfied, then

$$\underline{J} = \underline{J}^*, \quad \underline{M} = \{ W\text{-co-identifying cones indexed by } \underline{U}\text{-classes} \}.$$

PROOF. (a) $\Rightarrow$ (b): Since F preserves co-identifying cones [8] (1.3.1), so does J. Suppose

$$(B, \{ g_i: B \to A_i \}_{i \in I})$$

is W-co-identifying, thus $(u: WB \to WA, A)$ as constructed in 1.3 (b) from

$$\{ id_{WA_i}: WA_i \to WA_i, A_i \}_{i \in I} \text{ and } (WB, \{ Wg_i: WB \to WA_i \}_{i \in I})$$

(replacing the corresponding data in 1.3 (a)) must be of the form

$$(W(k), A) \text{ for an isomorphism } k: B \to A.$$

In consequence,

$$(B, \{ g_i: B \to A_i \}_{i \in I}) \in \underline{M}.$$

Now, conversely, suppose that $(B, \{ g_i: B \to A_i \}_{i \in I}) \in \underline{M}$, then $(B, \{ g_i \}_I)$ is W-co-identifying by 1.5.

(b) $\Rightarrow$ (a): We use the same functor K as constructed in the proof of 1.6 when applied to

$$(\underline{J}^*, \{ W\text{-co-identifying cones indexed by } \underline{U}\text{-classes} \}).$$

Suppose that $(v: X \to WB, B) \in \underline{J}^*$ and consider the W-co-sieve (X, ξ) generated by the W-basis $\{(v, B)\}$. We have to verify 1.1 (ii) for ξ: If

$$(W(f_i)u, A_i) = (u_i: X \to WA_i, A_i) \in \xi$$

for every $i \in I$, then - by hypothesis - $u_i = W(g_i)v$ for some $g_i: B \to A_i$ in $\underline{A}$, hence we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad v \quad} & WB \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle Wg_i} \\ WA & \xrightarrow{\quad Wf_i \quad} & WA_i \end{array} .$$

Since $(v, B) \in J$ and $(A, \{f_i\}_{i \in I})$ is W-co-identifying, there is - by hypothesis - a «diagonal» $h: B \to A$ with $u = W(h)v$, hence $(u, A) \in \xi$. In consequence, $(X, \xi) \in \underline{A}^u$. The «composite» of (v, B) and the W-co-identifying (!) cone of all $\underline{A}$-morphisms with domain B (including id_B!) is (X, ξ); hence $K(X, \xi)$ is equivalent to (v, B).

1.8. REMARKS. (i) If (a) and (b) of 1.7 are satisfied, then

$$(\underline{E}, \{W\text{-co-identifying cones in } \underline{A}\})$$

is the smallest factorization of relative cones which W admits.

(ii) Under the hypothesis of 1.6, $\underline{A}^u$ is $\underline{U}$-legitimate (cf. [14]).

We observe that the $\underline{J}$-canonical extension of a faithful functor W admitting a factorization of relative cones itself has a very natural universal property which is completely analogous to the result in [9] 1.1 (since the $\underline{J}$-canonical extension is analogous and, moreover, in some sense a generalization of the construction in [6] 9.1):

1.9. THEOREM. *Suppose $W: \underline{A} \to \underline{X}$ admits a factorization $(\underline{J}, \underline{M})$ of relative cones; suppose $T: \underline{Y} \to \underline{X}$ is a topological functor which lifts isomorphisms uniquely. Suppose $G: \underline{A} \to \underline{Y}$ with $T \circ G = W$ takes every $\underline{M}$-cone in $\underline{A}$ into a T-co-identifying cone in $\underline{Y}$. Then there is a unique functor $G^{\underline{J}}: \underline{A}^{\underline{J}} \to \underline{Y}$ with*

$$G^{\underline{J}} \circ J = G \quad \textit{and} \quad T \circ G^{\underline{J}} = W^{\underline{J}}$$

which takes all $W^{\underline{J}}$-co-identifying cones into T-co-identifying cones.

PROOF. $G^{\underline{J}}$ maps $(u: X \to WA, A) \in \underline{J}$ into the domain B of the unique (!) T-co-identifying morphism $f: B \to A$ with $T(f) = u$. The remaining considerations are completely given in the proof of the analogous result [9] 1.1.

1.10. REMARKS. (a) If $W: \underline{A} \to \underline{X}$ is a topological functor, then it obviously satisfies the conditions of 1.7 (b).

(b) If $T: \underline{A} \to \underline{X}$ is an $(\underline{I}, \underline{N})$-topological functor in the sense of [6] - where $(\underline{I}, \underline{N})$ denotes a factorization of cones in $\underline{X}$ - then $(\underline{I}_T, \underline{N}_T)$ is a factorization of cones in $\underline{A}$ satisfying the condition in 1.3 (b) (2), with

$$\underline{I}_T := \{ f \in Mor\underline{A} \mid T(f) \in \underline{I} \}$$

and

$\underline{N}_T :=$ { all T-co-identifying cones indexed by $\underline{U}$-classes which are taken by T into $\underline{N}$-cones}.

SECTION 2.

In this section we investigate examples. In order to get an interpretation of the universal topological completion for $W: \underline{A} \to \underline{X}$ we shall in some cases verify the conditions of 1.7 and, moreover, that $\underline{J}^*$ of 1.7 (b) is the class of all W-epimorphisms. Then we obtain the examples of [12] Section 3. In some other cases we will use that $\underline{A}^u$ coincides with the construction in [6] 9.1 [4]. Then we shall obtain examples from [6] and [9] 1.8, 2.8.

We begin with a list of examples whose universal topological completions will be investigated. The underlying sets of the structures mentioned are always supposed to be $\underline{U}$-small.

2.1. EXAMPLES.

(E 1) The forgetful functor $W: Gr \to Ens$ with

$$Gr = \text{groups and homomorphisms.}$$

(E 2) The forgetful functor $W: AbGr \to Ens$ with

$$AbGr = \text{abelian groups and homomorphisms.}$$

[4] We observe that this construction yields a topological functor only if the functor to which it is applied is a relatively topological functor in the sense of [6].

(E 3) The forgetful functor $W: Sob \to Ens$ with

Sob = sober spaces and continuous maps.

(E 4) The forgetful functor $W: C\text{-}Unif \to Ens$ with

$C\text{-}Unif$ = Cauchy-complete separated uniform spaces and uniformly continuous maps.

(E 5) The forgetful functor $W: Comp\text{-}T_2 \to Ens$ with

$Comp\text{-}T_2$ = compact T_2-spaces and continuous maps.

(E 6) The forgetful functor $W: C\text{-}{}^qMet \to Ens$ with

$C\text{-}{}^qMet$ = Cauchy-complete separated ^{q}metric spaces and non-expansive maps.

$d: M \times M \to [0, \infty]$ is called a *q-metric on* M iff

(1) $d(x,y) = d(y,x)$,

(2) $d(x,x) = 0$,

(3) $d(x,y) \leq d(x,z) + d(z,y)$,

for any elements $x, y, z \in M$ (Separated means

$$d(x,y) = 0 \Rightarrow x = y\ ).$$

(E 7) The forgetful functor $W: {}^qBan_K \to Vec_K$ with

Vec_K = K-vector spaces and K-linear maps (K = R or C),

${}^qBan_K$ = category of ^{q}Banach K-spaces and non-expansive maps.

The prefix «^{q}» indicates that $\| x \| = \infty$ has to be admitted (in order to make this functor W right adjoint).

(E 8) The forgetful functor $W_i: \underline{T}_i \to Ens$ with

$\underline{T}_i$ = T_i-spaces and continuous maps

($i = 0, 1, 2, 3$; T_3 = regular and T_0).

(E 9) The forgetful functor $W: Sep\text{-}Unif \to Ens$ with

$Sep\text{-}Unif$ = separated uniform spaces and uniformly continuous maps.

(Similarly for proximity spaces.)

(E 10) The forgetful functor $W: Poset \to Ens$ with

$Poset$ = *P*artially *o*rdered *sets* and isotone maps.

(E 11) The forgetful functor $\mathbb{W}: Sep\text{-}{}^qMet \to Ens$ with

$Sep\text{-}{}^qMet$ = separated ^{q}metric spaces and non-expansive maps.

(E 12) The forgetful functor $\mathbb{W}: Sep\text{-}{}^qn\text{-}Vec_K \to Vec_K$ with

Vec_K = K-vector spaces and K-linear maps,

$Sep\text{-}{}^qn\text{-}Vec_K$ = separated quasi-normed K-vector spaces and K-linear non-expansive maps

(quasi-normed means that

$$\| x \| = \infty \quad \text{and} \quad \| y \| = 0 \text{ for } y \neq 0$$

is admitted; separated means that $\| y \| = 0$ implies $y = 0$).

(E 13) The forgetful functors

$\mathbb{W}$: *T_0-k-spaces, sequential T_0-spaces*, resp. *locally connected T_0-spaces* (and continuous maps) $\to$ *Ens*.

2.2. In case of the category *Gr* of groups it is due to J.C. Taylor [21] (example 4) that the $\mathbb{W}$-co-identifying cones are precisely the joint-injective cones (in case that the index set is $\emptyset$, we add that it is precisely the 0-group). It is well known that

$$(\{ Gr\text{-epimorphisms}\}, \{ \text{joint-injective cones in } Gr\})$$

is a factorization of cones in *Gr* which satisfies the requirement of 1.3 (b) (2) above (note that

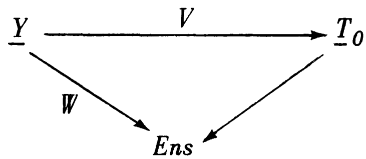
$$Gr\text{-epimorphism} = \text{surjective homomorphism}$$

The elements of the associated class $\underline{J}$ are now easily recognized as being groups G together with a map $\phi: M \to G$ such that the image $\phi[M]$ «generates» G, i.e., groups G equipped with a distinguished family of generators.

The situation with *AbGr* is analogous (and simpler).

Some topological examples will be handled with by the aid of the following lemma, a special case of which I learned from [17] ($\underline{Y} = \underline{T}_1$):

Suppose $\mathbb{W}: \underline{Y} \to Ens$ is a faithful functor and let



commute for some functor V and the usual forgetful functor $\underline{T}_0 \to Ens$. Furthermore, we assume the following conditions to be satisfied:

(i) For every $X, Y \in Ob\underline{Y}$ with $WX \neq \emptyset$ and every constant map $\phi : WX \to WY$ there is a morphism $f: X \to Y$ in $\underline{Y}$ with $W(f) = \phi$.

(ii) There is some $Y_0 \in Ob\underline{Y}$ such that VY_0 is a non-discrete space.

2.3. LEMMA. *Under the above hypotheses one has:*

If $(A, \{g_i : A \to B_i\}_{i \in I})$ *is a W-co-identifying cone in* $\underline{Y}$, *then*

$$(WA, \{W(g_i): WA \to WB_i\}_{i \in I})$$

is joint-injective (if $I = \emptyset$, *then* $card\, WA \leq 1$ *).*

PROOF. Suppose $(Wg_i)(x) = (Wg_i)(y)$ for some $x, y \in WA$, $x \neq y$ and every $i \in I$. Since VY_0 is non-discrete, $card\, WY_0 \geq 2$. Since VA is T_0, there exists an open subset U of VA containing x, but not containing y (or vice versa). Let S be any subset of WY_0 and let

$$\chi_S(s) = x \text{ for } s \in S \quad \text{and} \quad \chi_S(t) = y \text{ for } t \in WY_0 - S,$$

then $W(g_i)\chi_S : WY_0 \to WB_i$ is a constant map, hence

$$W(g_i)\chi_S = W(f_i) \text{ for some } f_i : Y_0 \to B_i \text{ in } \underline{Y}$$

by condition (i). In consequence, there is a morphism

$$h_S : Y_0 \to A \text{ in } \underline{Y} \text{ with } W(h_S) = \chi_S,$$

hence $\chi_S : VY_0 \to VA$ is a continuous map, so $S = \chi_S^{-1}(U)$ is open in VY_0 for every subset S of WY_0, hence VY_0 is discrete – contradicting our hypothesis (ii).

2.4. To (E3)-(E6) one may apply 2.3. All these categories have a factorization

$$(\{\text{epimorphisms}\}, \{W\text{-co-identifying mono-cones}\})$$

as is readily deduced from the characterization of epimorphisms in these categories (cf. [12] Section 3), hence by [12] 3.1-3.4 we obtain the following interpretations of $\underline{A}^u$ (the embedding $J: \underline{A} \hookrightarrow \underline{A}^u$ and the forgetful functor $W^u: \underline{A}^u \to Ens$ are to be understood as the obvious functors):

(E 3) $\underline{A}^u = Top$;

(E 4) $\underline{A}^u = Unif$;

(E 5) $\underline{A}^u = Prox$ (proximity spaces and uniformly continuous maps);

(E 6) $\underline{A}^u = {}^qMet$.

2.5. To (E 8)-(E 11) one can also apply 2.3. All these categories have a factorization

({morphisms f with Wf surjective}, {W-co-identifying mono-cones}).

The co-units of the adjunction are (pointwise) bijective, hence 1.3 (b) (2) is satisfied. The appropriate canonical $\underline{J}$-extension coincides in these cases with the construction in [6], 9.1, hence we have a description of $\underline{A}^u$ from [9] 2.8:

(E 8) $\underline{A}^u_0 = Top$, $\underline{A}^u_1 = \underline{R}_0$, $\underline{A}^u_2 = \underline{R}_1$, $\underline{A}^u_3 = \underline{R}_2$ with

$\underline{R}_i$ = R_i-spaces and continuous maps.

A space is R_i iff its T_0-quotient is T_{i+1} ($i = 0, 1, 2$) (A.S. Davis [5]). R_0-spaces are also known as *symmetric spaces*

$$(x \in cl\{y\} \Leftrightarrow y \in cl\{x\})$$

or as *weakly regular* spaces of A.N. Shanin [20]. A space X is R_1 iff whenever a filter $\underline{F}$ on X converges to both $x, y \in X$ then $cl\{x\} = cl\{y\}$. R_2-spaces = regular spaces (without T_0). (For similar definitions see H.J. Kowalsky [18].)

(E 9) $\underline{A}^u = Unif$.

(E 10) $\underline{A}^u = Preord$ (preordered sets and isotone maps) [5].

(E 11) $\underline{A}^u = {}^qMet$.

[5] A pre-ordered set X becomes a space X with a basis consisting of all sets:

$$U_x = \{ y \in X \mid x \leq y \} \quad (x \in X),$$

thus 2.3 applies to *Poset*.

2.6. By considerations similar to those for $\underline{T}_0$ (E 8) it follows for (E 13) that $\underline{A}^u$ is obtained by the construction of [6], 9.1. Then it follows by [9] 3.14 (on bi-co-reflective subcategories of *Top* containing a non-discrete space) that

$\underline{A}^u$ = *k-spaces, sequential spaces*, resp. *locally connected spaces* (and continuous maps).

The examples (E 7) and (E 12) are handled with by the following analogue of 2.3.

2.7. LEMMA. *Let* W: ${}^q Ban_K$, *resp.* $Sep\text{-}{}^q n\text{-}Vec_K \to Vec_K$ *denote the obvious forgetful functor; then, for every W-co-identifying cone*

$$(A, \{ g_i : A \to B_i \}_{i \in I}),$$

$(WA, \{ W(g_i) \}_{i \in I})$ *is joint-injective.*

PROOF. Suppose

$$C = \bigcap_I ker(g_i) \neq \{ 0 \},$$

then consider the K-vector space C endowed with the quasi-norm

$$\| \cdot \|' = \frac{1}{2} \| \cdot \|_A .$$

The inclusion $j : (C, \| \cdot \|') \to (A, \| \cdot \|_A)$ is expansive ($(A, \| \cdot \|_A)$ is separated, hence

$$\| x \| \neq 0 \quad \text{for every} \quad x \in C - \{ O \}),$$

but $g_i j = O$.

In view of 2.7 one may apply to ${}^q Ban_K$ and $Sep\text{-}{}^q n\text{-}Vec_K$ the constructions 1.4 (a) (with $\underline{J}$ = { W-epimorphisms}) and, resp., [6] 9.1 in order to obtain:

(E 7) $\underline{A}^u = {}^q n\text{-}Vec_K$,

(E 12) $\underline{A}^u = {}^q n\text{-}Vec_K$.

N. Bourbaki ([3] page 103) has mistakenly attributed a wrong universal property to the MacNeille completion of a poset. This error has been observed by P. Ringleb [19]. Ph. Antoine, who has introduced the analogue of the MacNeille completion [6] for *Ens*-valued faithful functors, has made (independently) an analogous false claim for a universal property of his construction [1, 2]. A counterexample showing that this is not true was given by H. Herrlich [8] (3.1 a). Maybe it is surprising to observe that even in a non-artificial application, namely for the forgetful functor $W: Comp\text{-}T_2 \to Ens$, the MacNeille completion is different from the universal topological completion.

2.8. LEMMA. *Let $W: A = Comp\text{-}T_2 \to Ens$, then the MacNeille-Antoine completion $MN(Comp\text{-}T_2)$ of W is properly embedded in $\underline{A}^u = Prox$ (compatibly with the embeddings and forgetful functors).*

PROOF. According to a result of [13], $MN(Comp\text{-}T_2)$ may be interpreted as the full subcategory of *Top* consisting of those completely regular spaces which are complete regularization of their associated k-spaces, i.e. $X = ckX$ (c, k designating completely regular reflection and, resp., k-space co-reflection; completely regular does not include T_0; k-space means having the final topology with regard to continuous maps from compact T_2-spaces). An embedding $MN(Comp\text{-}T_2) \hookrightarrow Prox$ is obtained by assigning to X its Stone-Čech-compactification. Since, of course, not every compactification is equivalent to a Stone-Čech-compactification, the result is established.

It will be shown elsewhere («MacNeille completion of the category Ban_1 of Banach spaces») that in cases (E 7), (E 12) the MacNeille completion coincides with the universal topological completion, in other words that it has the natural universal property.

After having finished this manuscript, I received a preprint of H. Herrlich and G.E. Strecker («Semi-universal maps and universal initial completions») in part overlapping, in part complementing, with this paper.

6) This analogy was observed by H. Herrlich [8] and by myself [13] independently.

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Universität Bremen
Studienbereich Mathematik
Achterstrasse 33
D-2800 BREMEN, R.F.A.