CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

ANDRÉE EHRESMANN
CHARLES EHRESMANN

Multiple functors. IV. Monoidal closed structures on Cat_n

Cahiers de topologie et géométrie différentielle catégoriques, tome 20, nº 1 (1979), p. 59-104
<http://www.numdam.org/item?id=CTGDC_1979__20_1_59_0>

CAHIERS DE TOPOLOGIE *Vol. XX-1 (1979)*
ET GEOMETRIE DIFFERENTIELLE

MULTIPLE FUNCTORS
IV. MONOIDAL CLOSED STRUCTURES ON Cat_n

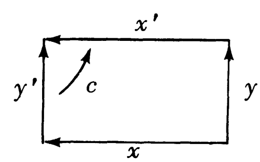
by Andrée and Charles EHRESMANN

INTRODUCTION.

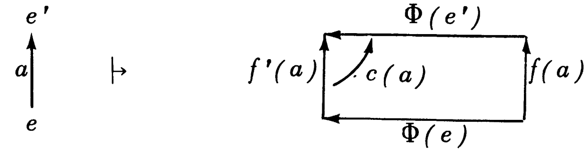
This paper is Part IV of our work on multiple categories whose Parts I, II and III are published in [3, 4, 5]. Here we «laxify» the constructions of Part III (replacing equalities by cells) in order to describe monoidal closed structures on the category Cat_n of n-fold categories, for which the internal Hom functors associate to (A, B) an n-fold category of «lax hypertransformations» between n-fold functors from A to B.

As an application, we prove that all double categories are (canonically embedded as) double sub-categories of the double category of squares of a 2-category; by iteration this gives a complete characterization of multiple categories in terms of 2-categories. Hence the study of multiple categories reduces «for most purposes» to that of 2-categories and of their squares, and generalized limits of multiple functors [4, 5] are just lax limits (in the sense of Gray [7], Bourn [2], Street [10],...), taking somewhat restricted values.

More precisely, if C is a category, the double category Q(C) of its (up-)squares



is a laxification of the double category of commutative squares of the category of 1-morphisms of C; a lax transformation Φ between two functors from a category A to $|C|^1$ «is» a double functor $\Phi : A \to Q(C)$:



(Φ «is» a natural transformation iff $c(a)$ is an identity for each a in A). Similarly, to an n-fold category A, we associate in Section A the $(n+1)$-fold category *Cub* B (of cubes of B), which is a laxification of the $(n+1)$-fold category *Sq* B (of squares of B) used in Part III to explicit the cartesian closure functor of Cat_n.

In Section B, the construction (given in Part III) of the left adjoint *Link* of the functor *Square* from Cat_n to Cat_{n+1} is laxified in order to get the left adjoint *LaxLink* of the functor $Cube: Cat_n \to Cat_{n+1}$. While *Link* A, for an $(n+1)$-fold category A, is generated by classes of strings of objects of the two last categories A^{n-1} and A^n, the n-fold category *LaxLink* A is generated by classes of strings of strings of objects of «alternately» A^{n-2} and A^{n-1} or A^n (so we introduce objects of A^{n-2} instead of equalities).

LaxLink is a left inverse (Section C) of the functor *Cylinder* from n-*Cat* to $(n+1)$-*Cat* associating to an n-category B the greatest $(n+1)$-category included in *Cub* B.

The functor $Cub_{n,m}$ from Cat_n to Cat_m is defined by iteration as well as its left adjoint. They give rise to a closure functor $LaxHom_n$ on Cat_n mapping the couple (A, B) of n-fold categories onto the n-fold category $Hom(A, (Cub_{n,2n}B)^{\gamma})$, where:

- *Hom* is the internal Hom of the monoidal closed category (considered in Part II) $(\coprod_n Cat_n, \blacksquare, Hom)$,

- $(Cub_{n,2n}B)^{\gamma}$ is the $2n$-fold category deduced from $Cub_{n,2n}B$ by the permutation of the compositions γ:

$$(0, \dots, 2n-1) \mapsto (0, 2, \dots, 2n-2, 1, 3, \dots, 2n-1).$$

The corresponding tensor product on Cat_n admits as a unit the n-fold category on 1.

«Less laxified» monoidal closed structures on Cat_n are defined by replacing at some steps the functor *Cube* by the functor *Square* ; the «most rigid» one is the cartesian closed structure (where only functors *Square* are considered [5]). For 2-categories, Gray's monoidal structure is also obtained.

Existence theorems for the «lax limits» corresponding to these closure functors are given in Section D. In fact, we prove that, if B is an n-fold category whose category $|B|^{n-1}$ of objects for the $(n-1)$-th first compositions admits (finite) usual limits, then the representability of B implies that of the $(n+1)$-fold categories $K = Sq\,B$, $Cub\,B$, $Cyl\,B$; therefore, according to the theorem of existence of generalized limits given in Part II, Proposition 11, all (finite) n-fold functors toward B admit K-wise limits. In particular, the existence theorem for lax limits of 2-functors given by Gray [7], Bourn [2], Street [10] is found anew, with a more structural (and shorter) proof (already sketched in Part I, Remark page 271, and exposed in our talk at the Amiens Colloquium in 1975) *.

The notations are those of Parts II and III. If B is an n-fold category, $\underline{B}$ is the set of its blocks and, for each sequence $(i_0, \dots, i_{p-1})$ of distinct integers lesser than n, the p-fold category whose j-th category is B^{i_j} is denoted by $B^{i_0,\dots,i_{p-1}}$.

* NOTE ADDED IN PROOFS. We have just received a mimeographed text of J.W. Gray, *The existence and construction of lax limits*, in which a very similar proof is given for this particular theorem. The only difference is that *Cat* is considered as the inductive closure of {1, 2, 3} (instead of {2}) and that the proof is not split in two parts:

1° existence of generalized limits (those limits are not used by Gray),

2° representability of $Q(C)$ and $Cyl\,C$ for a 2-category C (though this result is implicitely proved).

BIBLIOGRAPHY.

1. J. BENABOU, Introduction to bicategories, *Lecture Notes in Math.* 47, Springer (1967).

2. D. BOURN, Natural anadeses and catadeses, *Cahiers Topo. et Géo. Diff.* XIV-4 (1973), p. 371-380.

3. A. & C. EHRESMANN, Multiple functors I, *Cahiers Topo. et Géo. Diff.* XV-3 (1978), 215-292.

4. A. & C. EHRESMANN, Multiple functors II, *Id.* XIX-3 (1978), 295-333.

5. A. & C. EHRESMANN, Multiple functors III, *Id.* XIX-4 (1978), 387-443.

6. C. EHRESMANN, Structures quasi-quotients, *Math. Ann.* 171 (1967), 293-363.

7. J. W. GRAY, Formal category theory, *Lecture Notes in Math.* 391, Springer (1974).

8. J. PENON, Catégories à sommes commutables, *Cahiers Topo. et Géo. Diff.* XIV-3 (1973).

9. C.B. SPENCER, An abstract setting for homotopy pushouts and pullbacks, *Cahiers Topo. et Géo. Diff.* XVIII-4 (1977), 409-430.

10. R. STREET, Limits indexed by category-valued 2-functors, *J. Pure and App. Algebra* 8-2 (1976), 149-181.

U.E.R. de Mathématiques
33 rue Saint-Leu
80039 AMIENS

A. The cubes of a multiple category.

The aim is to give to Cat_n a monoidal closed structure whose tensor product «laxifies» the (cartesian) product (by introducing non-degenerate blocks in place of some identities). The method is the same as that used in Part III to construct the cartesian closed structure of Cat_n.

The first step is the description of a functor *Cube* from Cat_n to Cat_{n+1}, admitting a left adjoint which maps an $(n+1)$-fold category $\mathbb{A}$ onto an n-fold category $LaxLk\mathbb{A}$, obtained by «laxification» of the construction of $Lk\mathbb{A}$.

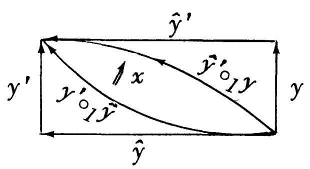
1° The «model» double category $\mathbb{M}$.

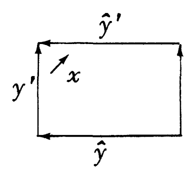
To define the *Square* functor, we used as a basic tool the double category of squares of a category $\mathcal{C}$, whose blocks «are» the functors from 2×2 to $\mathcal{C}$. The analogous tool will be here the triple category of cubes of a double category, obtained by replacing the category 2×2 by the «model» double category $\mathbb{M}$ described as follows:

Consider the 2-category Q with four vertices, six 1-morphisms

$$y,\ y',\ \hat{y},\ \hat{y}',\ \hat{y}'\circ_1 y,\ y'\circ_1 \hat{y},$$

and only one non-degenerate 2-cell $x: y'\circ_1 \hat{y} \to \hat{y}'\circ_1 y$ in Q^0:

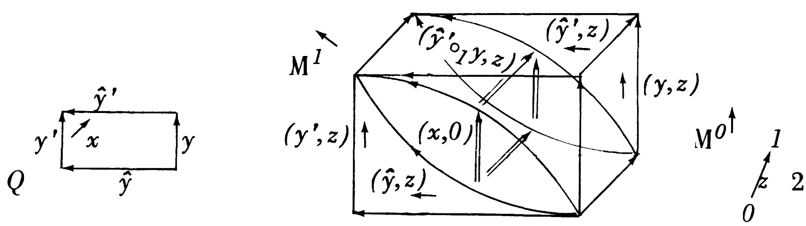


(also represented by 

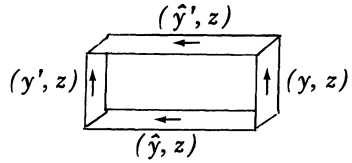
).

(Intuitively, Q consists of a square «only commutative up to a 2-cell».)

The model double category $\mathbb{M}$ is the double category $Q\times(2,\underline{2}^{dis})$, product of Q with the double category $(2,\underline{2}^{dis})$:

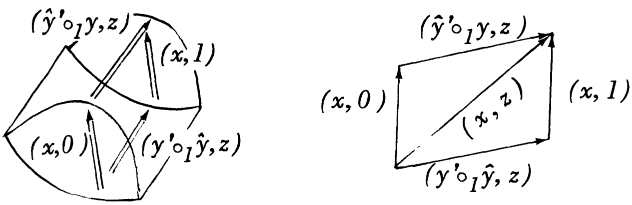


It is generated by the blocks forming the non-commutative square



of $\mathrm{M}^1 = Q^1 \times \underline{2}^{dis}$

and those forming the commutative square («cylinder») of $\mathrm{M}^0 = Q^0 \times 2$:

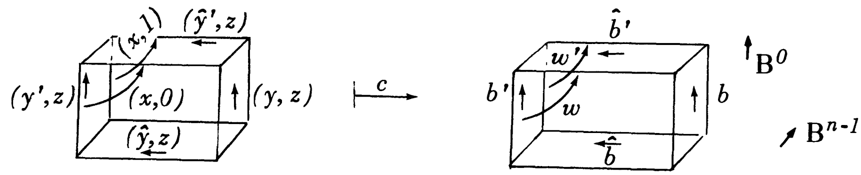


whose diagonal is (x, z).

2° The multiple category of cubes of an n-fold category.

Let n be an integer such that $n \geqslant 2$. We denote by B an n-fold category, by α^i and β^i the source and target maps of B^i, for $i < n$.

DEFINITION. A double functor $c : \mathrm{M} \to \mathrm{B}^{n-1,0}$ from the model double category M to the double category $\mathrm{B}^{n-1,0}$ (whose compositions are the $(n-1)$-th and 0-th compositions of B) is called *a cube of* B.



The cube c will be identified with the 6-uple $(b', \hat{b}', w', w, \hat{b}, b)$ where

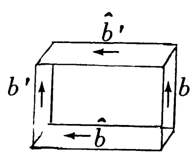
$$b = c(y, z), \quad b' = c(y', z), \quad \hat{b} = c(\hat{y}, z), \quad \hat{b}' = c(\hat{y}', z),$$
$$w = c(x, 0), \quad w' = c(x, 1)$$

(which determines the cube c uniquely).

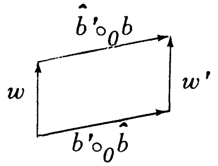
In other words, a cube c of B may also be defined as a 6-uple

$$c = (b', \hat{b}', w', w, \hat{b}, b)$$

of blocks of B such that



is a non-commutative square of B^0,



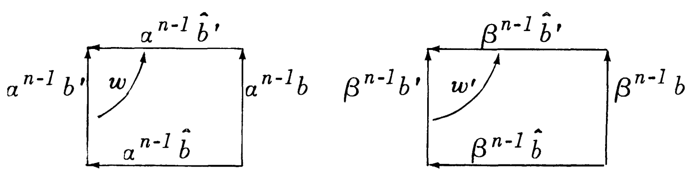
is a commutative square of B^{n-1}.

The diagonal of this last square:

$$(\hat{b}'\circ_0 b)\circ_{n-1} w = w'\circ_{n-1}(b'\circ_0\hat{b})$$

is called the *diagonal of the cube* c, and denoted by ∂c.

Remark that w and w' are 2-cells of the greatest 2-category contained in $\mathrm{B}^{n-1,0}$, and that in the cube c (represented by a «geometric» cube), the «front» and «back» faces are up-squares of this 2-category:



On the set $Cub\mathrm{B}$ of cubes of B, we have the $(n-2)$-fold category $Hom(\mathrm{M},\mathrm{B}^{n-1,0,1,\ldots,n-2})$, whose i-th composition is deduced pointwise from the $(i+1)$-th composition of B, for $i < n-2$. With the notations above (we add everywhere indices if necessary), the i-th composition is written:

$$c_1\circ_i c = (b'_1\circ_{i+1} b', \hat{b}'_1\circ_{i+1}\hat{b}', w'_1\circ_{i+1} w', w_1\circ_{i+1} w, \hat{b}_1\circ_{i+1}\hat{b}, b_1\circ_{i+1} b),$$

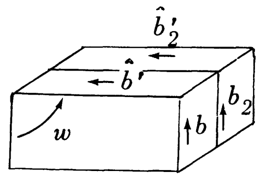
iff the six composites are defined.

Now, we define three other compositions on $Cub\mathrm{B}$ so that, by adding these new compositions, we obtain an $(n+1)$-fold category $Cub\mathrm{B}$:

- We denote by $(Cub\mathrm{B})^{n-2}$ the category whose composition is deduced «laterally pointwise» from that of B^{n-1}:

$$c_2\circ_{n-2} c = (b'_2\circ_{n-1} b', \hat{b}'_2\circ_{n-1}\hat{b}', w'_2, w, \hat{b}_2\circ_{n-1}\hat{b}, b_2\circ_{n-1} b)$$

iff $w_2 = w'$ and the four composites are defined.



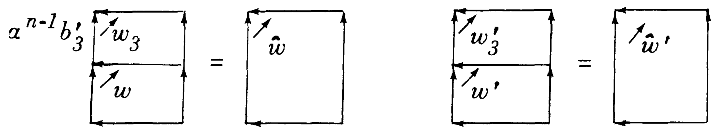
$\nearrow B^{n-1}$

The source and target of c are the degenerate cubes determined by the front and back faces of c.

- Let $(Cub\mathrm{B})^{n-1}$ be the category whose composition is the «vertical» composition of cubes (also denoted by $\boxminus$):

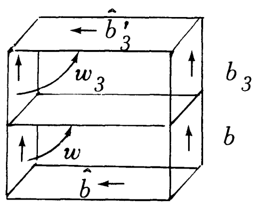
$$c_3 \circ_{n-1} c = (b'_3 \circ_0 b', \hat{b}'_3, \hat{w}', \hat{w}, \hat{b}, b_3 \circ_0 b) \text{ iff } \hat{b}' = b_3,$$

where $\hat{w}$ and $\hat{w}'$ are the 2-cells of the vertical composites of the front and back up-squares:



(hence:

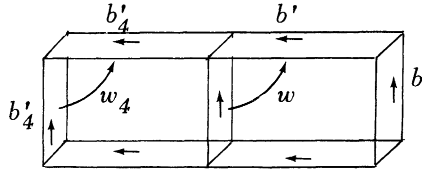
$$\hat{w} = (w_3 \circ_0 \alpha^{n-1} b) \circ_{n-1} (\alpha^{n-1} b'_3 \circ_0 w), \ \hat{w}' = (w'_3 \circ_0 \beta^{n-1} b') \circ_{n-1} (\beta^{n-1} b'_3 \circ_0 w') \ ).$$



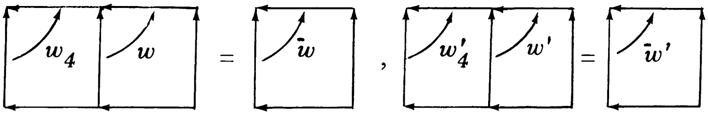
$\uparrow B^0$

- Finally, $(Cub\mathrm{B})^n$ is the category whose composition is the «horizontal» composition of cubes (also denoted by $\square\!\square$):

$$c_4 \circ_n c = (b'_4, \hat{b}'_4 \circ_0 \hat{b}', \bar{w}', \bar{w}, \hat{b}_4 \circ_0 \hat{b}, b) \text{ iff } b' = b_4,$$



where $\bar{w}$ and $\bar{w}'$ are the 2-cells of the horizontal composites of the front and back up-squares:



REMARK. $(Cub\mathrm{B})^{n-1,n}$ is the double category of up-squares of the 2-category of cylinders $(Cyl\mathrm{B})^{n,n-1}$, which is the greatest 2-category contained in the double category $(Sq(\mathrm{B}^{n-1,0}))^{2,0}$ (with the notations of Section C).

From the permutability axiom satisfied by B it follows that we have an $(n+1)$-fold category on the set of cubes of B, denoted by $Cub\,\mathrm{B}$, such that:

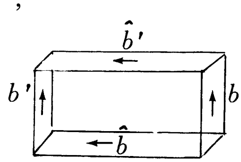
- $(Cub\mathrm{B})^{0,\ldots,n-2} = Hom(\mathrm{M}, \mathrm{B}^{n-1,0,1,\ldots,n-2})$,
- the $(n-2)$-th, $(n-1)$-th and n-th compositions are those defined above.

DEFINITION. This $(n+1)$-fold category $Cub\mathrm{B}$ is called the *$(n+1)$-fold category of cubes of* B.

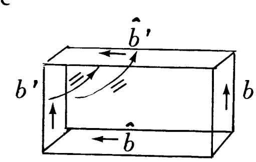
Summing up, the i-th category $(Cub\mathrm{B})^i$ is deduced pointwise from B^{i+1} for $i < n-2$ and «laterally pointwise» from B^{n-1} for $i = n-2$, while $(Cub\mathrm{B})^{n-1}$ and $(Cub\mathrm{B})^n$ are the «vertical» and «horizontal» categories of cubes.

EXAMPLE. If B is a double category, $Cub\mathrm{B}$ is a triple category whose 0-th composition is deduced laterally pointwise from B^1.

If a square s of B^0,

$s =$ 

is identified with the cube

$c =$ 

(with the same «lateral» faces) in which w and w' are the degenerate 2-cells $\alpha^{n-1}(b' \circ_0 \hat{b})$ and $\beta^{n-1}(\hat{b}' \circ_0 b)$, then the $(n+1)$-fold category $Sq\mathrm{B}$

of squares of B (see Part III [5]) becomes an $(n+1)$-fold subcategory of CubB, which has the same objects than CubB for the $(n-1)$-th and n-th categories. It follows that we may still consider the isomorphisms

$$-^{\boxminus} : B^{1,\ldots,n-1,0} \rightarrow |(Cub\mathrm{B})^{n-1}|^{0,\ldots,n-2,n} : b \mapsto b^{\boxminus},$$

$$-^{\boxplus} : B^{1,\ldots,n-1,0} \rightarrow |(Cub\mathrm{B})^{n}|^{0,\ldots,n-1} : b \mapsto b^{\boxplus}$$

from $B^{1,\ldots,n-1,0}$ onto the n-fold categories defined from CubB by taking the objects of $(Cub\mathrm{B})^{n-1}$ and $(Cub\mathrm{B})^{n}$ respectively.

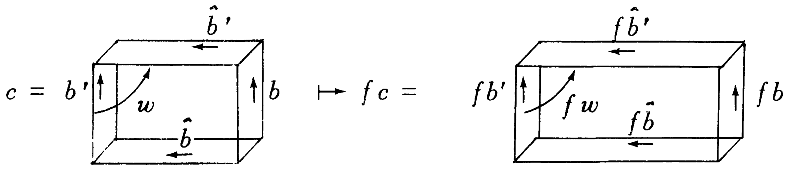
B. The adjoint functors *Cube* **and** *LaxLink*.

If $f: \mathrm{B} \to \mathrm{B}'$ is an n-fold functor, the $(n-2)$-fold functor

$$Hom(\mathrm{M}, f): Hom(\mathrm{M}, \mathrm{B}) \to Hom(\mathrm{M}, \mathrm{B}'): c \mapsto fc$$

underlies an $(n+1)$-fold functor $Cubf: Cub\mathrm{B} \to Cub\mathrm{B}'$ defined by:

$$c = (b', \hat{b}', w', w, \hat{b}, b) \mapsto fc = (fb', f\hat{b}', fw', fw, f\hat{b}, fb).$$



This determines the functor $Cub_{n,n+1}: Cat_n \to Cat_{n+1}$:

$$(f: \mathrm{B} \to \mathrm{B}') \mapsto (Cubf: Cub\mathrm{B} \to Cub\mathrm{B}'),$$

called the *functor Cube from* Cat_n *to* Cat_{n+1}.

PROPOSITION 1. *The functor* $Cub_{n,n+1}: Cat_n \to Cat_{n+1}$ *admits a left adjoint* $LaxLk_{n+1,n}: Cat_{n+1} \to Cat_n$.

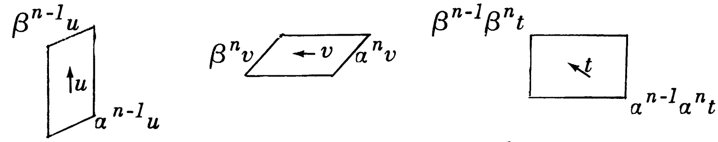
PROOF. Let A be an $(n+1)$-fold category, α^i and β^i the maps source and target of the i-th category A^i.

1° *We define an n-fold category* $\bar{\mathrm{A}}$, which will be the free object generated by A, as follows:

a) Let G be the graph whose vertices are those blocks e of A which are objects for both A^{n-1} and A^n, the arrows ν from e to e' being the objects of either A^n, A^{n-1} or A^{n-2} such that

$$\alpha^n \alpha^{n-1} \nu = e \quad \text{and} \quad \beta^n \beta^{n-1} \nu = e'.$$

Hence the arrows of G are of one of the three forms :



where u, v, t will always denote objects of A^n, A^{n-1}, A^{n-2}, respectively.

b) If K is an n-fold category, we say that $f: G \to K$ *is an admissible morphism* if $f: G \to \underline{K}$ is a map satisfying the 8 following conditions :

(i) If $\nu: e \to e'$ in G, then $f(\nu): f(e) \to f(e')$ in K^0.

(ii) $|A^n|^{n-1} \hookrightarrow G \xrightarrow{f} K^0$ and $|A^{n-1}|^n \hookrightarrow G \xrightarrow{f} K^0$ are functors (where $|A^i|^j$ is the subcategory of A^j formed by the objects of A^i).

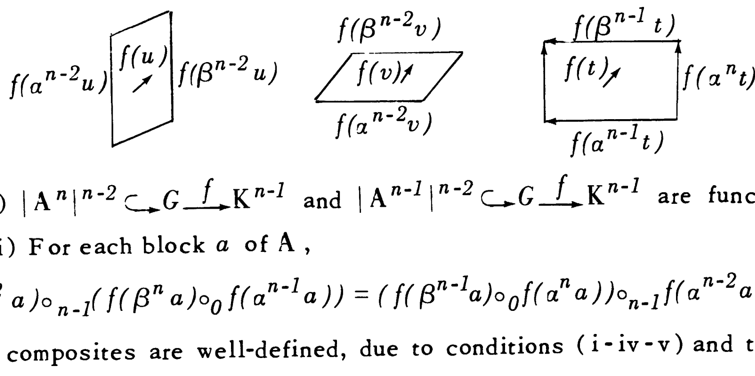
(iii) $|A^j|^i \hookrightarrow G \xrightarrow{f} K^{i+1}$ is a functor, for

$$i < n-2 \quad \text{and} \quad j = n,\ n\text{-}1 \text{ or } n\text{-}2.$$

(iv) For each arrow ν of G,

$$f(\nu): f(\beta^n \alpha^{n-2}\nu) \circ_0 f(\alpha^{n-1}\alpha^{n-2}\nu) \to f(\beta^{n-1}\beta^{n-2}\nu) \circ_0 f(\alpha^n \beta^{n-2}\nu)$$

in the category K^{n-1}.



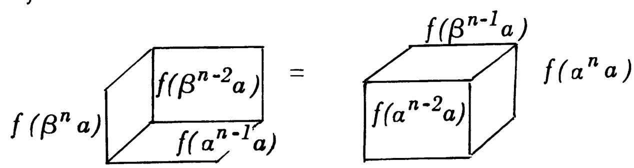
(v) $|A^n|^{n-2} \hookrightarrow G \xrightarrow{f} K^{n-1}$ and $|A^{n-1}|^{n-2} \hookrightarrow G \xrightarrow{f} K^{n-1}$ are functors.

(vi) For each block a of A,

$$f(\beta^{n-2} a) \circ_{n-1} (f(\beta^n a) \circ_0 f(\alpha^{n-1} a)) = (f(\beta^{n-1} a) \circ_0 f(\alpha^n a)) \circ_{n-1} f(\alpha^{n-2} a)$$

(these composites are well-defined, due to conditions (i-iv-v) and to the fact that K is an n-fold category). This condition (vi) is equivalent to:

(vi') $c_f a = (f\beta^n a,\ f\beta^{n-1} a,\ f\beta^{n-2} a,\ f\alpha^{n-2} a,\ f\alpha^{n-1} a,\ f\alpha^n a)$

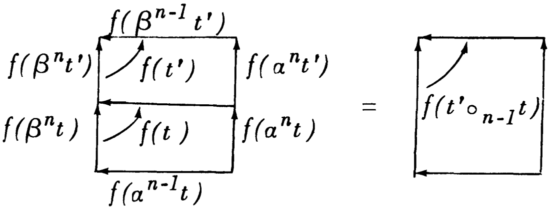


is a cube of K for each block a of A .

(vii) If $t' \circ_{n-1} t$ is defined in $|A^{n-2}|^{n-1}$, then

$$f(t' \circ_{n-1} t) = (f(t') \circ_0 f(a^n t)) \circ_{n-1} (f(\beta^n t') \circ_0 f(t)).$$

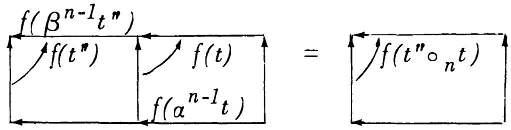
With (iv) this means that $f(t' \circ_{n-1} t)$ is the 2-cell of the vertical composite up-square of $K^{n-1,0}$:



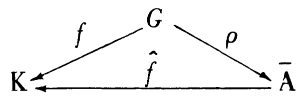
(viii) If $t'' \circ_n t$ is defined, then

$$f(t'' \circ_n t) = (f(\beta^{n-1} t'') \circ_0 f(t)) \circ_{n-1} (f(t'') \circ_0 f(a^{n-1} t)).$$

Hence, with (iv), in the horizontal category of up-squares of $K^{n-1,0}$:



c) By the general existence theorem of « universal solutions» [6], there exist: an n-fold category $\bar{A}$ and an admissible morphism $\rho: G \to \bar{A}$ such that any admissible morphism $f: G \to K$ factors uniquely through ρ into an n-fold functor $\hat{f}: \bar{A} \to K$.



Indeed, if we take the set of all admissible morphisms $\phi: G \to K_\phi$ with K_ϕ a small n-fold category, there exists an n-fold category $\prod_\phi K_\phi$ product in the category of n-fold categories associated to a universe to which belongs the universe of small sets. The factor

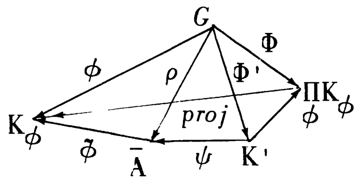
$$\Phi: G \to \prod_\phi K_\phi : \nu \mapsto (\phi(\nu))_\phi$$

of the family of maps ϕ is an admissible morphism, as well as its restriction $\Phi': G \to K'$ to the n-fold subcategory K' of $\prod_\phi K_\phi$ generated by the image

$\Phi(G)$. As $\Phi(G)$ and K' are equipotent (by Proposition 2 [4]) and $\Phi(G)$ is of lesser cardinality than the small set G, it follows that there exists an isomorphism $\psi : \mathrm{K}' \to \bar{\mathrm{A}}$ onto a small n-fold category $\bar{\mathrm{A}}$. Then

$$\rho = (G \xrightarrow{\Phi'} \mathrm{K}' \xrightarrow{\psi} \bar{\mathrm{A}})$$

is a «universal» admissible morphism, since each admissible morphism



$\phi : G \to \mathrm{K}_\phi$ factors uniquely into

$$\phi = (G \xrightarrow{\rho} \bar{\mathrm{A}} \xrightarrow{\bar{\phi}} \mathrm{K}_\phi),$$

where

$$\bar{\phi} = (\bar{\mathrm{A}} \xrightarrow{\psi^{-1}} \mathrm{K}' \hookrightarrow \prod_\phi \mathrm{K}_\phi \xrightarrow{\text{projection}} \mathrm{K}_\phi).$$

Remark that the blocks $\rho(\nu)$, for any arrow ν of G, generate $\bar{\mathrm{A}}$.

d) An explicit construction of the universal admissible morphism $\rho : G \to \bar{\mathrm{A}}$ is sketched now (it will not be used later on).

(i) Let $\mathrm{P}(G)^0$ be the free quasi-category of paths $(\nu_k, \dots, \nu_0)$ of the graph G; an arrow ν is identified to the path (ν). On the same set $\underline{\mathrm{P}}(G)$ of paths, there is a category $\mathrm{P}(G)^{i+1}$, whose composition is deduced pointwise from that of A^i, for each $i < n-2$. If r is the relation on $\underline{\mathrm{P}}(G)$ defined by:

$$(u', u) \sim u' \circ_{n-1} u \quad \text{if } u \text{ and } u' \text{ are objects of } \mathrm{A}^n,$$
$$(v', v) \sim v' \circ_n v \quad \text{if } v \text{ and } v' \text{ are objects of } \mathrm{A}^{n-1},$$

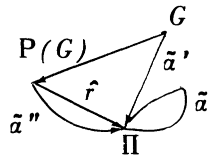
there exists an $(n-1)$-fold category Π quasi-quotient (Proposition 3 [4]) of $\mathrm{P}(G) = (\mathrm{P}(G)^0, \dots, \mathrm{P}(G)^{n-2})$ by r; the canonical morphism is denoted by $\hat{r} : \mathrm{P}(G) \to \Pi$.

(ii) We define a graph on $\underline{\Pi}$: Consider the morphism

$$\tilde{\alpha}' : \nu \mapsto \hat{r}(\beta^n \alpha^{n-2} \nu, \alpha^{n-1} \alpha^{n-2} \nu)$$

from the graph G to the graph (Π, α^0, β^0) underlying the category Π^0.

By the universal property of $P(G)$, $\tilde{\alpha}'$ extends into a quasi-functor $\tilde{\alpha}''$ from $P(G)^0$ to Π^0, and $\tilde{\alpha}'': P(G) \to \Pi$ is also a morphism, due to the pointwise definition of $P(G)^{i+1}$. Moreover, $\tilde{\alpha}''$ is seen to be compatible with r. Hence it factors uniquely into an $(n-1)$-fold functor $\tilde{\alpha} : \Pi \to \Pi$. The



equality $\tilde{\alpha}\tilde{\alpha}\hat{r} = \tilde{\alpha}\hat{r}$ implies $\tilde{\alpha}\tilde{\alpha} = \tilde{\alpha}$. Similarly, there is an $(n-1)$-fold functor $\tilde{\beta} : \Pi \to \Pi$ such that

$$\tilde{\beta}\hat{r}(\nu) = \hat{r}(\beta^{n-1}\beta^{n-2}\nu, \alpha^n\beta^{n-2}\nu)$$

for each arrow ν of G, and we have

$$\tilde{\beta}\tilde{\beta} = \tilde{\beta}, \quad \tilde{\beta}\tilde{\alpha} = \tilde{\alpha}, \quad \tilde{\alpha}\tilde{\beta} = \tilde{\beta}.$$

These equalities mean that $(\underline{\Pi}, \tilde{\alpha}, \tilde{\beta})$ is a graph, in which a block π of Π is an arrow $\pi : \tilde{\alpha}(\pi) \to \tilde{\beta}(\pi)$.

(iii) Let $P(\Pi)^{n-1}$ be the free quasi-category of all paths $<\pi_k, \ldots, \pi_0>$ of the graph $(\underline{\Pi}, \tilde{\alpha}, \tilde{\beta})$ (equipped with the concatenation). A block π of Π is identified to the path $<\pi>$. On the set $\underline{P}(\Pi)$ of these paths, we consider the relation r' defined by:

$<\hat{r}(u'), \hat{r}(u)> \sim \hat{r}(u' \circ_{n-2} u)$, if u and u' are objects of A^n,
$<\hat{r}(v'), \hat{r}(v)> \sim \hat{r}(v' \circ_{n-2} v>$, if v and v' are objects of A^{n-1},
$<\hat{r}(\beta^{n-2}a), \hat{r}(\beta^n a) \circ_0 \hat{r}(\alpha^{n-1}a)> \sim <\hat{r}(\beta^{n-1}a) \circ_0 \hat{r}(\alpha^n a), \hat{r}(\alpha^{n-2}a)>$
for each block a of A,
$\hat{r}(t' \circ_{n-1} t) \sim <\hat{r}(t') \circ_0 \hat{r}(\alpha^n t), \hat{r}(\beta^n t') \circ_0 \hat{r}(t)>$,
if $t' \circ_{n-1} t$ is defined in $|A^{n-2}|^{n-1}$,
$\hat{r}(t'' \circ_n t) \sim <\hat{r}(\beta^{n-1} t'') \circ_0 \hat{r}(t), \hat{r}(t'') \circ_0 \hat{r}(\alpha^{n-1} t)>$,
if $t'' \circ_n t$ is defined in $|A^{n-2}|^n$.

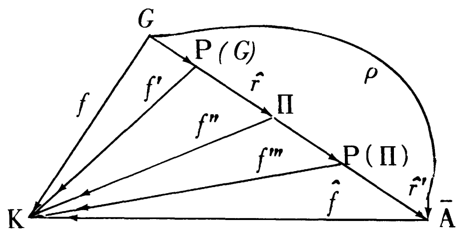
(iv) For $i < n-2$, there is also a category $P(\Pi)^i$ on $P(\Pi)$ whose composition is deduced pointwise from that of Π^i. There exists an n-fold category $\bar{A}$ quasi-quotient of $P(\Pi) = (P(\Pi)^0, \ldots, P(\Pi)^{n-2}, P(\Pi)^{n-1})$

by r', the canonical morphism being $\hat{r}': \mathrm{P}(\Pi) \to \overline{\mathrm{A}}$. The composite map ρ:

$$G \hookrightarrow \underline{\mathrm{P}}(G) \xrightarrow{\hat{r}} \underline{\Pi} \hookrightarrow \underline{\mathrm{P}}(\Pi) \xrightarrow{\hat{r}'} \underline{\overline{\mathrm{A}}}$$

gives an admissible morphism $\rho: G \to \overline{\mathrm{A}}$ due to the construction of $\hat{r}$ and $\hat{r}'$.

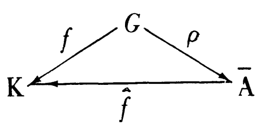
(v) $\rho: G \to \overline{\mathrm{A}}$ is a universal admissible morphism. Indeed, let $f: G \to \mathrm{K}$ be an admissible morphism. As f satisfies (i), it extends into a (quasi-) functor $f': \mathrm{P}(G)^0 \to \mathrm{K}^0$; by (iii), $f': \mathrm{P}(G) \to \mathrm{K}^{0,\dots,n-2}$ is a morphism which is compatible with r (according to (ii)). By the universal property of Π, there exists a factor $f'': \Pi \to \mathrm{K}^{0,\dots,n-2}$ of f' through $\hat{r}$. The con-



dition (iv) implies that f'' is a morphism of graphs

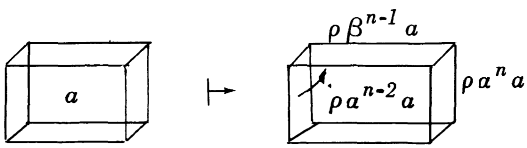
$$f'': (\Pi, \tilde{\alpha}, \tilde{\beta}) \to (\mathrm{K}^{n-1}, \alpha^{n-1}, \beta^{n-1})$$

so that it extends into a (quasi-)functor $f''': \mathrm{P}(\Pi)^{n-1} \to \mathrm{K}^{n-1}$, defining a morphism $f''': \mathrm{P}(\Pi) \to \mathrm{K}$ (the composition of $\mathrm{P}(\Pi)^i$ being deduced pointwise from that of Π^i). The conditions (v, vi, vii, viii) mean that f''' is compatible with r'. Hence f''' factors through $\hat{r}'$ into an n-fold functor $\hat{f}: \overline{\mathrm{A}} \to \mathrm{K}$; and $\hat{f}$ is the unique n-fold functor rendering commutative the diagram



2° *There exists an $(n+1)$-fold functor $l: \mathrm{A} \to Cub\,\overline{\mathrm{A}}$:*

$$a \mapsto l(a) = (\rho\beta^n a, \rho\beta^{n-1} a, \rho\beta^{n-2} a, \rho\alpha^{n-2} a, \rho\alpha^{n-1} a, \rho\alpha^n a)$$



where $\rho: G \to \overline{\mathrm{A}}$ is a fixed universal admissible morphism.

a) As ρ satisfies (vi') and as $l(a)$ is the cube $c_\rho(a)$ considered in this condition, the map l is well-defined.

b) Suppose $i < n-2$. The composition of $(Cub\bar{\mathrm{A}})^i$ being deduced pointwise from that of $\bar{\mathrm{A}}^{i+1}$, for $l: \mathrm{A}^i \to (Cub\bar{\mathrm{A}})^i$ to be a functor, it suffices that the maps

$$\rho\alpha^n,\ \rho\beta^n,\ \rho\alpha^{n-1},\ \rho\beta^{n-1},\ \rho\alpha^{n-2},\ \rho\beta^{n-2}$$

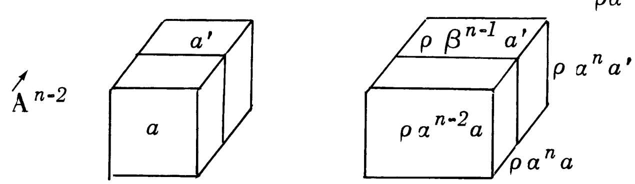
sending a onto each of the six factors of the cube $l(a)$ define functors $\mathrm{A}^i \to \bar{\mathrm{A}}^{i+1}$. Since $\alpha^n: \mathrm{A}^i \to |\mathrm{A}^n|^i$ is a functor and axiom (iii) is satisfied, $\rho\alpha^n$ defines the composite functor:

$$\mathrm{A}^i \xrightarrow{\alpha^n} |\mathrm{A}^n|^i \hookrightarrow G \xrightarrow{\rho} \bar{\mathrm{A}}^{i+1},$$

and similarly for the five other maps.

c) $l: \mathrm{A}^{n-2} \to (Cub\bar{\mathrm{A}})^{n-2}$ is a functor. Indeed, suppose $a' \circ_{n-2} a$ defined in A^{n-2}. The composition of $(Cub\bar{\mathrm{A}})^{n-2}$ being deduced «laterally pointwise» from that of $\bar{\mathrm{A}}^{n-1}$, there exists $l(a')\circ_{n-2} l(a) =$

$$(\rho\beta^n a' \circ_{n-1} \rho\beta^n a,\ \rho\beta^{n-1} a' \circ_{n-1} \rho\beta^{n-1} a,\ \rho\beta^{n-2} a', \rho\alpha^{n-2} a,\ \rho\alpha^{n-1} a' \circ_{n-1} \rho\alpha^{n-1} a,\ \rho\alpha^n a' \circ_{n-1} \rho\alpha^n a).$$



Now, by (v),

$$\rho\alpha^n a' \circ_{n-1} \rho\alpha^n a = \rho(\alpha^n a' \circ_{n-2} \alpha^n a) = \rho\alpha^n(a' \circ_{n-2} a),$$

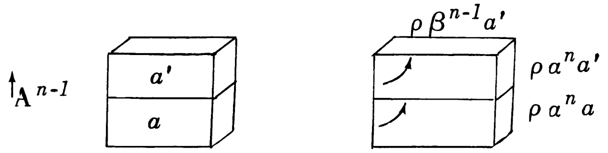
which is also the right lateral face of the cube $l(a' \circ_{n-2} a)$. Same proof for the other lateral faces. Finally,

$$\rho\alpha^{n-2} a = \rho\alpha^{n-2}(a' \circ_{n-2} a)$$

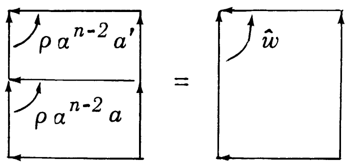
is the front face of both $l(a')\circ_{n-2} l(a)$ and $l(a' \circ_{n-2} a)$, whose back face is $\rho\beta^{n-2} a'$. Hence, $l(a' \circ_{n-2} a) = l(a')\circ_{n-2} l(a)$.

d) $l: \mathrm{A}^{n-1} \to (Cub\bar{\mathrm{A}})^{n-1}$ is a functor. Indeed, suppose $a' \circ_{n-1} a$ defined. The composition of $(Cub\bar{\mathrm{A}})^{n-1}$ being the «vertical» composition, the composite

$$l(a')\boxminus l(a)=(\rho\,\beta^n a'\circ_0\rho\,\beta^n a,\ \rho\,\beta^{n-1}a',\ \hat{w}',\ \hat{w},\ \rho\,\alpha^{n-1}a\ ,\ \rho\,\alpha^n a'\circ_0\rho\,\alpha^n a)$$



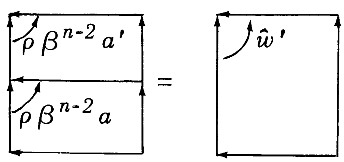
is defined; $\hat{w}$ is the 2-cell of the vertical composite up-square



which, by (vii), is equal to

$$\rho(\alpha^{n-2}a'\circ_{n-1}\alpha^{n-2}a)=\rho\,\alpha^{n-2}(a'\circ_{n-1}a),$$

and this is the 2-cell of the front face of $l(a'\circ_{n-1}a)$. Similarly, $\hat{w}'$ is the 2-cell of the back face of $l(a'\circ_{n-1}a)$.

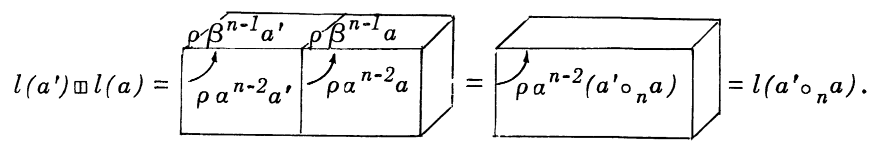


Using (ii), we get

$$\rho\,\alpha^n a'\circ_0\rho\alpha^n a=\rho(\alpha^n a'\circ_{n-1}\alpha^n a)=\rho\alpha^n(a'\circ_{n-1}a)$$

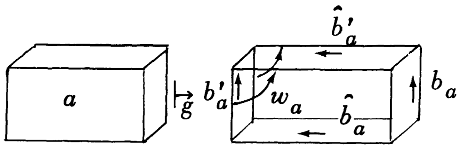
and idem with β instead of α. Hence $l(a')\boxminus l(a)=l(a'\circ_{n-1}a)$.

e) The same proof (using (viii) instead of (vii)) shows that l defines the functor $l: A^n\to(Cub\bar{A})^n$: if $a'\circ_n a$ is defined,



3° *$l: A\to Cub\bar{A}$ is the liberty morphism defining $\bar{A}$ as a free object generated by* A : Let B be an n-fold category and $g: A\to Cub B$ an $(n+1)$-fold functor. The cube $g(a)$ of B, for any block a of A, is written

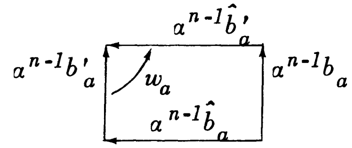
$$g(a)=(b'_a,\hat{b}'_a,w'_a,w_a,\hat{b}_a,b_a).$$



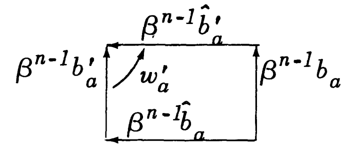
In particular,

$$g(\alpha^n a) = b_a^{\boxbar},\quad g(\alpha^{n-1} a) = \hat{b}_a^{\boxminus},\quad g(\beta^n a) = b_a'^{\boxbar},\quad g(\beta^{n-1} a) = \hat{b}_a'^{\boxminus},$$

$g(\alpha^{n-2} a)$ and $g(\beta^{n-2} a)$ are the degenerate cubes determined by



and



.

a) There is an admissible morphism $f: G \to B$ mapping ν onto the diagonal $\partial g(\nu)$ of the cube $g(\nu)$.

(i) As $\partial g(a) = (\hat{b}_a' \circ_0 b_a) \circ_{n-1} w_a$, we have

$$f(\alpha^n a) = \partial g(\alpha^n a) = b_a,\quad f(\alpha^{n-1} a) = \hat{b}_a,\quad f(\alpha^{n-2} a) = w_a,$$

$$f(\beta^n a) = b_a',\quad f(\beta^{n-1} a) = \hat{b}_a',\quad f(\beta^{n-2} a) = w_a',$$

so that

$$c_f(a) = (f\beta^n a, f\beta^{n-1} a, f\beta^{n-2} a, f\alpha^{n-2} a, f\alpha^{n-1} a, f\alpha^n a) =$$
$$= (b_a', \hat{b}_a', w_a', w_a, \hat{b}_a, b_a) = g(a)$$

is a cube, and f satisfies (vi). It also satisfies (i) and (iv), because it is more precisely defined by

$$f(u) = b_u,\quad f(v) = \hat{b}_v,\quad f(t) = w_t,$$

where u, v, t always denote objects of A^n, A^{n-1}, A^{n-2} respectively.

(ii) $|A^n|^{n-1} \hookrightarrow G \xrightarrow{f} B^0$ is a functor. Indeed, if $u' \circ_{n-1} u$ is defined,

$$g(u' \circ_{n-1} u) = g(u') \boxminus g(u) = b_{u'}^{\boxbar} \boxminus b_u^{\boxbar} = (b_{u'} \circ_0 b_u)^{\boxbar}$$

so that

$$f(u' \circ_{n-1} u) = \partial g(u' \circ_{n-1} u) = b_{u'} \circ_0 b_u = f(u') \circ_0 f(u).$$

Similarly, $|A^{n-1}|^n \hookrightarrow G \xrightarrow{f} B^0$ is a functor, since

$$g(v' \circ_n v) = g(v') \boxbar g(v) = (\hat{b}_{v'} \circ_0 \hat{b}_v)^{\boxminus},$$

so $$f(v' \circ_n v) = \hat{b}_{v'} \circ_0 \hat{b}_v = f(v') \circ_0 f(v).$$

That $|A^n|^{n-2} \hookrightarrow G \xrightarrow{f} B^{n-1}$ and $|A^{n-1}|^{n-2} \hookrightarrow G \xrightarrow{f} B^{n-1}$ are functors is deduced from the equalities

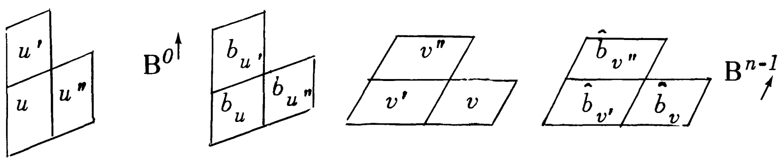
$$g(u''\circ_{n-2}u) = g(u'')\circ_{n-2}g(u) = b^{\boxbar}_{u''}\circ_{n-2}b^{\boxbar}_{u} = (b_{u''}\circ_{n-1}b_u)^{\boxbar},$$

$$g(v''\circ_{n-2}v) = g(v'')\circ_{n-2}g(v) = \hat{b}^{\boxminus}_{v''}\circ_{n-2}\hat{b}^{\boxminus}_{v} = (\hat{b}_{v''}\circ_{n-1}\hat{b}_v)^{\boxminus},$$

giving

$$f(u''\circ_{n-2}u) = b_{u''}\circ_{n-1}b_u = f(u'')\circ_{n-1}f(u)$$

and $f(v''\circ_{n-2}v) = f(v'')\circ_{n-1}f(v)$. Hence, f verifies (ii) and (v).



(iii) For $i < n-2$, there is a functor $\partial : (Cub\mathrm{B})^i \to \mathrm{B}^{i+1}$, since the pointwise deduction of the composition of $(Cub\mathrm{B})^i$ from that of B^{i+1}, and the permutability axiom in B imply:

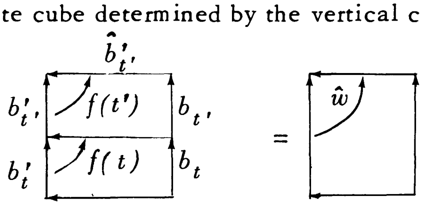
$$\partial(c_1\circ_i c) = ((\hat{b}'_1\circ_{i+1}\hat{b}')\circ_0(b_1\circ_{i+1}b))\circ_{n-1}(w_1\circ_{i+1}w) =$$
$$= ((\hat{b}'_1\circ_0 b_1)\circ_{n-1}w_1)\circ_{i+1}((\hat{b}'\circ_0 b)\circ_{n-1}w) = \partial c_1\circ_{i+1}\partial c,$$

if $c_1\circ_i c$ is defined in $(Cub\mathrm{B})^i$, with $c = (b', \hat{b}', w', w, \hat{b}, b)$ and idem for c_1 with indices. The composite functor

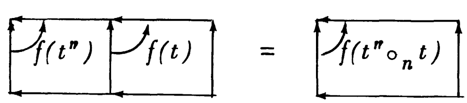
$$|A^j|^i \xrightarrow{g} (Cub\mathrm{B})^i \xrightarrow{\partial} \mathrm{B}^{i+1}$$

is defined by a restriction of f, for $j = n$, $n-1$ or $n-2$. So f satisfies (iii).

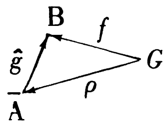
(iv) If $t'\circ_{n-1}t$ is defined in $|A^{n-2}|^{n-1}$, then $g(t'\circ_{n-1}t) = g(t')\boxminus g(t)$ is the degenerate cube determined by the vertical composite up-square

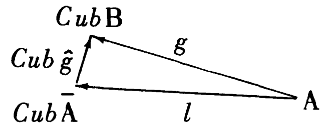


so that its diagonal $f(t'\circ_{n-1}t)$ is the 2-cell $\hat{w}$ of this composite. Therefore f satisfies (vii) and (by a similar proof) (viii).



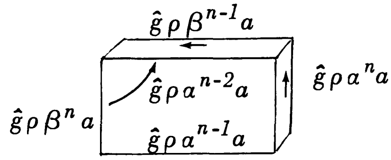
b) This proves that $f: G \to \mathrm{B}$ is an admissible morphism; so it factors uniquely through the universal admissible morphism $\rho: G \to \bar{\mathrm{A}}$ into an n-fold functor $\hat{g}: \bar{\mathrm{A}} \to \mathrm{B}$.





(i) For each block a of A, the cube

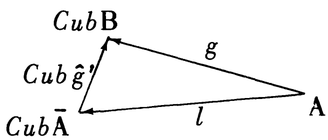
$\hat{g}l(a) =$



is identical to $c_f(a) = g(a)$ (see a), since $f = \hat{g}\rho$; so

$$(g: \mathrm{A} \to Cub\mathrm{B}) = (\mathrm{A} \xrightarrow{l} Cub\bar{\mathrm{A}} \xrightarrow{Cub\hat{g}} Cub\mathrm{B}).$$

(ii) Let $\hat{g}': \bar{\mathrm{A}} \to \mathrm{B}$ be an n-fold functor, rendering commutative the diagram



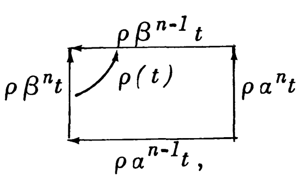
We are going to prove that $\hat{g}'\rho = f$; the unicity of the factor of f through ρ then implies $\hat{g}' = \hat{g}$. Indeed, for an object u of A^n, from the equalities

$$l(u) = \rho(u)^{\boxplus} \text{ and } g(u) = \hat{g}'l(u) = (\hat{g}'\rho(u))^{\boxplus}$$

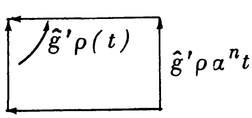
we deduce $f(u) = \partial g(u) = \hat{g}'\rho(u)$. If v is an object of A^{n-1}, then

$$l(v) = \rho(v)^{\boxminus},\ g(v) = (\hat{g}'\rho(v))^{\boxminus} \text{ and } f(v) = \hat{g}'\rho(v).$$

If t is an object of A^{n-2}, the degenerate cubes $l(t)$ and $g(t) = \hat{g}'l(t)$ are determined by the up-squares



and



so that $f(t) = \partial g(t) = \hat{g}'\rho(t)$. Hence, $\hat{g}'\rho = f$, and $\hat{g}' = \hat{g}$.

REMARK. To prove that $\hat{g}'\rho = f$, we could have used the relations

$$\partial \, Cub \, \hat{g}' = \hat{g}'\bar{\partial} \quad \text{and} \quad \bar{\partial} \, l(\nu) = \rho(\nu) \quad \text{for each } \nu \text{ in } G,$$

where $\bar{\partial}$ is the diagonal map from $Cub\bar{\mathrm{A}}$ to $\bar{\mathrm{A}}$.

4° For each $(n+1)$-fold small category A, we choose a universal admissible morphism $\rho_{\mathrm{A}} : G_{\mathrm{A}} \to \bar{\mathrm{A}}$ (where G_{A} is the graph G above), for example the canonical one constructed in 1-c; by the preceding proof, $\bar{\mathrm{A}}$ is a free object generated by A with respect to the *Cube* functor. $\bar{\mathrm{A}}$ will be called the *multiple category of lax links of* A, denoted by $LaxLk\mathrm{A}$. The corresponding left adjoint

$$LaxLk_{n+1,n} : Cat_{n+1} \to Cat_n \quad \text{of} \quad Cub_{n,n+1} : Cat_n \to Cat_{n+1}$$

maps $h: \mathrm{A} \to \mathrm{A}'$ onto the unique n-fold functor

$$LaxLkh : LaxLk\mathrm{A} \to LaxLk\mathrm{A}'$$

satisfying

$$LaxLkh(\rho_{\mathrm{A}}\nu) = \rho_{\mathrm{A}'}h(\nu)$$

for each object ν of A^n, A^{n-1} or A^{n-2}. ∇

By iteration, for each integer $m > n$, we define the functors

$$Cub_{n,m} = (\, Cat_n \xrightarrow{Cub_{n,n+1}} Cat_{n+1} \to \dots \to Cat_{m-1} \xrightarrow{Cub_{m-1,m}} Cat_m \,),$$

$$LaxLk_{m,n} = (\, Cat_m \xrightarrow{LaxLk_{m,m-1}} Cat_{m-1} \to \dots \to Cat_{n+1} \xrightarrow{LaxLk_{n+1,n}} Cat_n \,).$$

DEFINITION. $Cub_{n,m}$ is called the *Cube* functor from Cat_n to Cat_m and $LaxLk_{m,n}$ the *LaxLink functor from* Cat_m *to* Cat_n.

COROLLARY. *The Cube functor from* Cat_n *to* Cat_m *admits as a left adjoint the LaxLink functor from* Cat_m *to* Cat_n *for any integer* $m > n > 1$.

This results from Proposition 1, since a composite of left adjoint functors is a left adjoint functor of the composite. ∇

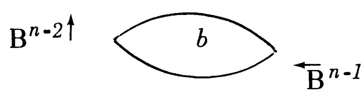
REMARK. If B is an n-fold category, in the $2n$-fold category $Cub_{n,2n}\mathrm{B}$ the $2i$-th and $(2i+1)$-th compositions are deduced respectively «verticaly» and «horizontaly» from the composition of B^i.

C. Cylinders of a multiple category.

We recall that an *n-category* is an n-fold category K whose objects for the last category K^{n-1} are also objects for K^{n-2}.

The full subcategory of Cat_n whose objects are the (small) n-categories is denoted by n-Cat. It is reflective and coreflective in Cat_n. More precisely, the insertion functor $n\text{-}Cat \hookrightarrow Cat_n$ admits:

- A right adjoint $\mu_n: Cat_n \to n\text{-}Cat$ mapping the n-fold category B onto *the greatest n-category included in* B, which is the n-fold subcategory of B formed by those blocks b of B such that $\alpha^{n-1} b$ and $\beta^{n-1} b$ are also objects of B^{n-2} (those blocks are called *n-cells of* B).



- A left adjoint $\lambda_n: Cat_n \to n\text{-}Cat$, whose existence follows from the general existence Theorem of free objects [6] (its hypotheses are satisfied, n-Cat being complete and each infinite subcategory of an n-category K generating an equipotent sub-n-category of K). In fact, $\lambda_n(B)$ is the n-category quasi-quotient of B by the relation:

$$u \sim \alpha^{n-2} u \quad \text{for each object } u \text{ of } B^{n-1}.$$

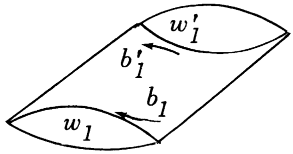
1° The multiple category CylB.

Let n be an integer, $n \geqslant 2$, and B be an n-fold category.

DEFINITION. The greatest $(n+1)$-category included in the $(n+1)$-fold category CubB of cubes of B is called the *$(n+1)$-category of cylinders of* B, denoted by Cyl B.

So a cylinder of B is a cube of the form

$$q_1 = (\beta^0 b_1', b_1', w_1', w_1, b_1, \alpha^0 b_1)$$



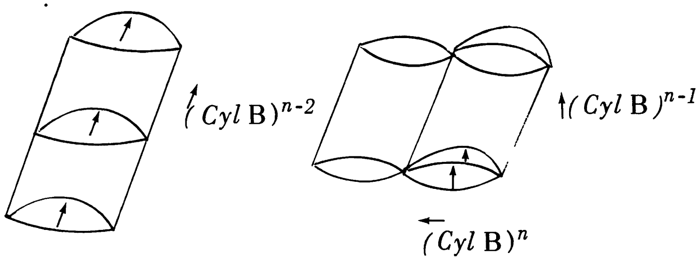
its front and back faces «reduce» to the 2-cells w_1 and w_1' of the double category $B^{n-1,0}$. We will write more briefly

$$q_1 = [\, b'_1,\, w'_1,\, w_1,\, b_1\,].$$

The composition of $(Cyl\,B)^i$, for $i < n-2$, is deduced pointwise from that of B^{i+1}. The $(n-2)$-th composition of $Cyl\,B$ is:

$$q_2 \circ_{n-2} q_1 = [\, b'_2 \circ_{n-1} b_1,\, w'_2,\, w_1,\, b_2 \circ_{n-1} b_1] \quad \text{iff} \quad w'_1 = w_2,$$

so that the objects of $(Cyl\,B)^{n-2}$ are the degenerate cylinders «reduced to their front face» $[\beta^{n-1} w, w, w, \alpha^{n-1} w]$, denoted by w°, for any 2-cell w of $B^{n-1,0}$.



The $(n-1)$-th composition of $Cyl\,B$ is the vertical one:

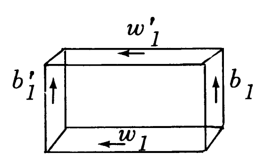
$$q_3 \boxminus q_1 = [\, b'_3,\, w'_3 \circ_{n-1} w'_1,\, w_3 \circ_{n-1} w_1,\, b_1] \quad \text{iff} \quad b'_1 = b_3,$$

and its objects are the degenerate squares $b^{\boxminus}$, for any block b of B. The n-th composition of $Cyl\,B$ is the horizontal one:

$$q_4 \boxplus q_1 = [\, b'_4 \circ_0 b'_1,\, w'_4 \circ_0 w'_1,\, w_4 \circ_0 w_1,\, b_4 \circ_0 b_1] \quad \text{iff} \quad \beta^0 b'_1 = \alpha^0 b_4$$

(which is deduced pointwise from the composition of B^0); its objects are the degenerate squares $e^{\boxminus}$, for any object e of B^0.

REMARKS. 1° The cylinder q_1 of B may be identified with the square



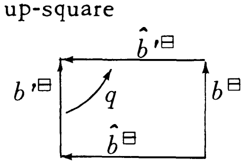
of B^{n-1}, in which w_1 and w'_1 are 2-cells of $B^{n-1,0}$; in this way, $Cyl\,B$ is identified with the greatest $(n+1)$-category included in

$$Sq(B^{n-1,1,\ldots,n-2,0})^{0,\ldots,n-3,n-1,n,n-2}.$$

2° $(Cub\,B)^{n-1,n}$ is identified with the double category of up-squares of the 2-category $(Cyl\,\mathrm{B})^{n-1,n}$ by identifying the cube

$$c = (b', \hat{b}', w', w, \hat{b}, b) \quad \text{of } \mathrm{B}$$

with the up-square



where $q = [\, b' \circ_0 \hat{b},\, w',\, w,\, \hat{b}' \circ_0 b \,]$.

2° The functor Cylinder.

If $f: \mathrm{B} \to \mathrm{B}'$ is an n-fold functor, there is an $(n+1)$-fold functor

$$Cylf: Cyl\mathrm{B} \to Cyl\mathrm{B}': [\, b'_1, w'_1, w_1, b_1 \,] \mapsto [\, fb'_1, fw'_1, fw_1, fb_1 \,]$$

restriction of $Cub\, f$. This determines a functor

$$Cyl_{n,n+1}: Cat_n \to Cat_{n+1}: f \mapsto Cyl\,f,$$

called the *Cylinder functor from* Cat_n *to* Cat_{n+1}. Remark that this functor is equal to the composite

$$Cat_n \xrightarrow{Cub_{n,n+1}} Cat_{n+1} \xrightarrow{\mu_{n+1}} (n+1)\text{-}Cat \hookrightarrow Cat_{n+1}$$

where μ_{n+1} is the right adjoint of the insertion.

PROPOSITION 2. *The functor* $LaxLk_{n+1,n}: Cat_{n+1} \to Cat_n$ *is equivalent to a left inverse of* $Cyl_{n,n+1}: Cat_n \to Cat_{n+1}$.

PROOF. We are going to prove that, *for each n-fold category* B, *the n-fold category* $LaxLk(Cyl\,\mathrm{B})$ *is canonically isomorphic with* B. It follows that, in the construction of the *LaxLink* functor (Proof, Proposition 1), we may choose B as the free object generated by $Cyl\mathrm{B}$, for each n-fold category B (remark that $Cyl\,\mathrm{B}$ determines uniquely B); in this way, we obtain the identity as the composite

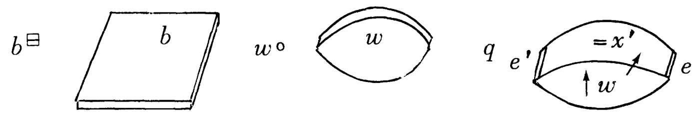
$$Cat_n \xrightarrow{Cyl_{n,n+1}} Cat_{n+1} \xrightarrow{LaxLk_{n+1,n}} Cat_n.$$

To prove the assertion, we take up the notations of Proposition 1, Proof, with $\mathrm{A} = Cyl\,\mathrm{B}$, $\bar{\mathrm{A}} = LaxLk\,\mathrm{A}$ and $\rho: G \to \bar{\mathrm{A}}$ the universal admissible morphism.

1° $\bar{\mathrm{A}}$ *is generated by the blocks* $\rho(b^{\boxminus})$, *for any block* b *of* B. Indeed, the arrows of the graph G are the objects $b^{\boxminus}$ of the vertical category of cylinders A^{n-1} and the objects w° of the category A^{n-2} (each object of A^n being also an object of A^{n-1}); the n-fold category $\bar{A}$ is generated by the blocks

$\rho(b^{\boxminus})$ for any block b of B and

$\rho(w^\circ)$ for any 2-cell w of the double category $B^{n-1,0}$.



Now, given the 2-cell w, there is a cylinder $q = [x', x', w, w]$ of B, where $x' = \beta^{n-1} w : e \to e'$ in B^0. Applying to q (considered as a cube) the axiom (vi) satisfied by the admissible morphism ρ, we get

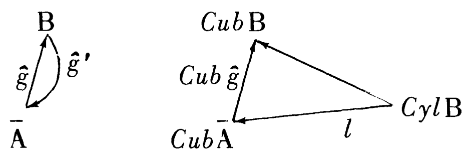
$$(\rho(x'^{\boxminus}) \circ_0 \rho(e^{\boxminus})) \circ_{n-1} \rho(w^\circ) = (\rho(e'^{\boxminus}) \circ_0 \rho(w^{\boxminus})) \circ_{n-1} \rho(x'^\circ) ;$$

as $\rho(x'^{\boxminus})$ is an object of $\bar{A}^{n-1}$ and $\rho(e^{\boxminus})$ an object of $\bar{A}^0$ (axioms (i) and (iv)), this equality gives $\rho(w^\circ) = \rho(w^{\boxminus})$. Hence $\bar{A}$ is generated by the sole blocks $\rho(b^{\boxminus})$.

2° a) To the insertion $Cyl\,B \hookrightarrow Cub\,B$ is associated (by the adjunction between the *Cube* and *Lax Link* functors, Proposition 1) the n-fold functor $\hat{g} : \bar{A} \to B$ such that

$$\hat{g}\,\rho(b^{\boxminus}) = \partial b^{\boxminus} = b \quad \text{for each block } b \text{ of } B$$

(this determines uniquely $\hat{g}$ by 1).



b) There is also an n-fold functor

$$\hat{g}' : B \to \bar{A} : b \mapsto \rho(b^{\boxminus}).$$

Indeed, $\hat{g}'$ is the composite functor

$$B \xrightarrow{-^{\boxminus}} |(Cyl\,B)^{n-1}|^{n,0,\dots,n-2} \xrightarrow{\rho'} \bar{A}, \qquad \rho' = \rho \circ G$$

where $-^{\boxminus}$ is the canonical isomorphism $b \mapsto b^{\boxminus}$ onto the n-fold category of objects of $(Cub\mathrm{B})^{n-1}$ (Section A-3) and where ρ' is a functor according to the axioms (ii, iii, v) satisfied by ρ.

c) $\hat{g}'$ is the inverse of $\hat{g}$. Indeed, for each block b of B we have

$$\hat{g}\hat{g}'(b) = \hat{g}\,\rho(b^{\boxminus}) = b \quad \text{and} \quad \hat{g}'\hat{g}(\rho(b^{\boxminus})) = \hat{g}'(b) = \rho(b^{\boxminus}).$$

These equalities mean that $\hat{g}\hat{g}'$ is an identity, as well as $\hat{g}'\hat{g}$, since the blocks $\rho(b^{\boxminus})$ generate $\bar{\mathrm{A}}$ by 1. So $\hat{g}' = \hat{g}^{-1}$.

3° Let $f: \mathrm{B} \to \mathrm{B}'$ be an n-fold functor, and

$$\hat{g}'_{\mathrm{B}'}: \mathrm{B}' \to LaxLk(Cyl\mathrm{B}'): b' \mapsto \rho_{\mathrm{B}'}(b'^{\boxminus})$$

the isomorphism similar to $\hat{g}'$. The square

$$\begin{array}{ccc} LaxLk(Cyl\mathrm{B}') & \xleftarrow{\hat{g}'_{\mathrm{B}'}} & \mathrm{B}' \\ {\scriptstyle LaxLkf}\uparrow & & \uparrow{\scriptstyle f} \\ \bar{\mathrm{A}} & \xleftarrow{\hat{g}'} & \mathrm{B} \end{array}$$

is commutative, since, for each block b of B,

$$LaxLkf(\hat{g}'(b)) = LaxLkf(\rho(b^{\boxminus})) = \rho_{\mathrm{B}'}(f(b)^{\boxminus}) = \hat{g}'_{\mathrm{B}'}f(b)$$

(by the construction of *LaxLink*, Proposition 1). This proves that the functor

$$Cat_n \xrightarrow{Cyl_{n,n+1}} Cat_{n+1} \xrightarrow{LaxLk_{n+1,n}} Cat_n$$

is equivalent to an identity. ∇

COROLLARY 1. *If $h: Cyl\mathrm{B} \to Cyl\mathrm{B}'$ is an $(n+1)$-fold functor, there exists a unique n-fold functor $f: \mathrm{B} \to \mathrm{B}'$ such that $h = Cylf$.*

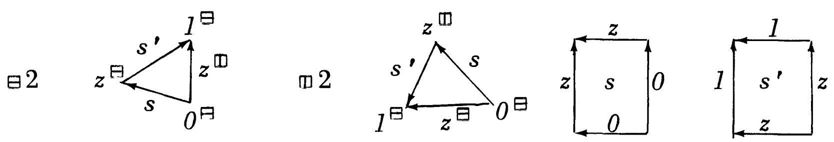
Indeed, this expresses the fact that B is a free object generated by $Cyl\mathrm{B}$ (Proof above) with respect to the *LaxLink* functor. ∇

COROLLARY 2. *For each integer $m > n > 1$, the LaxLink functor from Cat_n to Cat_m is equivalent to a left inverse of the functor $Cyl_{n,m}$ =*

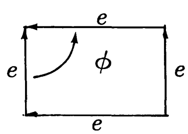
$$(Cat_n \xrightarrow{Cyl_{n,n+1}} Cat_{n+1} \to \dots \to Cat_{m-1} \xrightarrow{Cyl_{m-1,m}} Cat_m). \quad \nabla$$

REMARK. Proposition 1 may be compared with the fact that the *Link* functor is equivalent to a left inverse of the *Square* functor (Proposition 5

[5]). However the *LaxLink* functor is not equivalent to a left inverse of the *Cube* functor. Indeed, B and *LaxLk(CubB)* are isomorphic iff each *(n+1)*-fold functor $h: Cub\mathrm{B} \to Cub\mathrm{B}'$ is of the form *Cubf*. A counter example is obtained as follows. Let B be the double category $(2, \underline{2}^{dis})$ so that $Cub\mathrm{B} = (\boxminus 2, ◫2, \underline{\square 2}^{dis})$, where $2 = 1 \xleftarrow{z} 0$,



Let B' be the 2-category (Z_2, Z_2), where Z_2 is the group $\{e, \phi\}$ of unit e. The unique triple functor $h: Cub\mathrm{B} \to Cub\mathrm{B}'$ mapping s and s' onto the degenerate cube



is not of the form $Cubf: Cub\mathrm{B} \to Cub\mathrm{B}'$ for any double functor $f: \mathrm{B} \to \mathrm{B}'$.

3° The functor *n-Cyl*.

The *Cylinder* functor from Cat_n to Cat_{n+1} taking its values in *(n+1)-Cat*, it admits as a restriction a functor

$$n\text{-}Cyl: n\text{-}Cat \to (n+1)\text{-}Cat .$$

PROPOSITION 3. *The functor n-Cyl: n-Cat → (n+1)-Cat admits a left adjoint which is equivalent to a left inverse of n-Cyl.*

PROOF. By definition of the *Cylinder* functor, *n-Cyl* is equal to the composite functor

$$n\text{-}Cat \hookrightarrow Cat_n \xrightarrow{Cub_{n,n+1}} Cat_{n+1} \xrightarrow{\mu_{n+1}} (n+1)\text{-}Cat,$$

where μ_{n+1} is the right adjoint of the insertion. So this functor admits as a left adjoint the composite functor

$$(n+1)\text{-}Cat \hookrightarrow Cat_{n+1} \xrightarrow{LaxLk_{n+1,n}} Cat_n \xrightarrow{\lambda_n} n\text{-}Cat,$$

where λ_n is a left adjoint of the insertion (which exists, as seen above). The free object $\overline{\mathrm{K}}$ generated by an *(n+1)*-category K with respect to

n-Cyl is the n-category reflection of the n-fold category $Lax Lk \mathrm{K}$. In particular, if $\mathrm{K} = Cyl\mathrm{B}$ for some n-category B, then $Lax Lk\mathrm{K}$ is isomorphic with B (by Proposition 2), hence is an n-category, and $\overline{\mathrm{K}}$ is also isomorphic with B. ∇

COROLLARY. *The composite functor* (n, m)-Cyl =

$$(\, n\text{-}Cat \xrightarrow{n\text{-}Cyl} (n+1)\text{-}Cat \to \dots \to (m-1)\text{-}Cat \xrightarrow{(m-1)\text{-}Cyl} m\text{-}Cat\,)$$

admits a left adjoint equivalent to a left inverse of (n,m)-Cyl. ∇

D. Some applications.

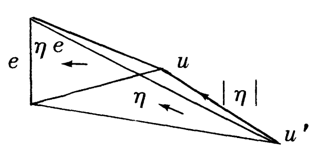
1° Existence of generalized limits.

An $(n+1)$-fold category H is *representable* (Section C-2 [4]) if the insertion functor $|\mathrm{H}|^n \hookrightarrow \mathrm{H}^n$ admits a right adjoint, where $|\mathrm{H}|^n$ is the subcategory of H^n formed by those blocks of H which are objects for the n first categories H^i; in this case, the greatest $(n+1)$-category included in H is also representable.

Remark that the order of the n first compositions of H does not intervene: H is representable iff so is $\mathrm{H}^{\gamma(0),\dots,\gamma(n-1),n}$ for any permutation γ of $\{0,\dots,n-1\}$. More generally:

DEFINITION. For each $i < n$, we denote by $\mathrm{H}^{\dots,i}$ the $(n+1)$-fold category $\mathrm{H}^{0,\dots,i-1,i+1,\dots,n,i}$ obtained by «putting the i-th composition at the last place», by $|\mathrm{H}|^i$ the subcategory of H^i formed by the blocks of H which are objects for each H^j, $i \neq j \leqslant n$. We say that H *is representable for the i-th composition* if the insertion functor $|\mathrm{H}|^i \hookrightarrow \mathrm{H}^i$ admits a right adjoint (i.e., if $\mathrm{H}^{\dots,i}$ is representable).

So, H is representable for the i-th composition iff, for each object e of H^i, there exists a morphism $\eta e : u \to e$ in H^i with u a vertex



of H, through which factors uniquely any morphism $\eta : u' \to e$ of H^i with

u' a vertex of H , so that

$$\eta = \eta e \circ_i /\eta/ , \text{ where } /\eta/ : u' \to u \text{ in } |\mathrm{H}|^i .$$

ηe is called *an i-representing block for e*.

From Proposition 11 [4], we deduce that, if H is representable for the i-th composition and if $|\mathrm{H}|^i$ is (finitely) complete, then the n-fold category $|\mathrm{H}^i|^{0,\dots,i-1,i+1,\dots,n}$ formed by the objects of H^i is $\mathrm{H}^{\dots,i}$-wise (finitely) complete.

Let B be an n-fold category, for an integer $n > 1$. The three following propositions are concerned with the representability of $Sq\mathrm{B}$, $Cyl\mathrm{B}$ and $Cub\mathrm{B}$ for the three last compositions. From the isomorphism

$$\mathrm{B}^{\dots,0} \xrightarrow{\boxminus} |(Cub\mathrm{B})^{n-1}|^{0,\dots,n-2,n} = |(Sq\mathrm{B})^{n-1}|^{0,\dots,n-2,n} : b \mapsto b^{\boxminus}$$

it follows that:

- $|Cub\mathrm{B}|^n = |Sq\mathrm{B}|^n$ is isomorphic with $|\mathrm{B}|^0$,
- $|Cub\mathrm{B}|^{n-2}$ and $|Sq\mathrm{B}|^{n-2}$ are isomorphic with $|\mathrm{B}|^{n-1}$,
- the vertices of $Cub\mathrm{B}$, $Sq\mathrm{B}$ and $Cyl\mathrm{B}$ are the degenerate cubes $u^{\boxminus}$, where u is a vertex of B .

PROPOSITION 4. *1° If* B *is representable for the 0-th composition, then* $Sq\mathrm{B}$ *is representable for the n-th and (n-1)-th compositions.*

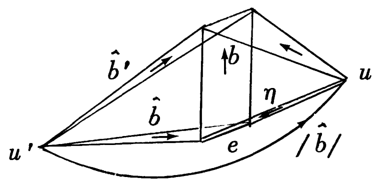
2° If B *is representable, then* $Cyl\mathrm{B}$ *is representable for the (n-2)-th composition.*

PROOF. 1° As the categories

$$(Sq\mathrm{B})^n = \square\mathrm{B}^0 \quad \text{and} \quad (Sq\mathrm{B})^{n-1} = \boxminus\mathrm{B}^0$$

are isomorphic as well as $|Sq\mathrm{B}|^n$ and $|Sq\mathrm{B}|^{n-1}$ (isomorphic with $|\mathrm{B}|^0$), the $(n+1)$-fold categories $Sq\mathrm{B}$ and $(Sq\mathrm{B})^{\dots,n-1}$ are simultaneously representable. Suppose that $\mathrm{B}^{\dots,0}$ is representable and that $b^{\square}$ is an object of $(Sq\mathrm{B})^n$; let $\eta : u \to e$ be the 0-representing block for $e = a^0 b$. Then $sb = (b, b\circ_0 \eta, \eta, u)$ is a square and $a^{\square}(sb) = u^{\square} = u^{\boxminus}$ is a vertex of $Sq\mathrm{B}$. If $s = (b, \hat{b}', \hat{b}, u')$ is a square of B with u' a vertex of B , and if $/\hat{b}/$ is the unique factor of $\hat{b}$ through η, then $/\hat{b}/^{\boxminus} : u'^{\boxminus} \to u^{\boxminus}$ is the unique morphism of $|Sq\mathrm{B}|^n$ such that $sb \,\square\, /\hat{b}/^{\boxminus} = s$, since

$$\hat{b}' = b \circ_0 \hat{b} = b \circ_0 \eta \circ_0 /\hat{b}/ .$$

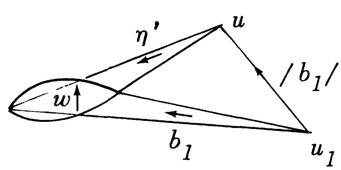


Hence sb is an n-representing square for $b^{\boxplus}$.

2° Suppose that B is representable and that w° is an object of the category $(Cyl\mathrm{B})^{n-2}$ (so that w is a 2-cell of $\mathrm{B}^{n-1,0}$). The same method proves that there exists an $(n-2)$-representing cylinder for w°, which is

$$qw = [\, w \circ_0 \eta', \; w, u, \; \eta' \,] ,$$

where $\eta' : u \to e$ is the n-representing block for $e = a^{n-1} w$. (This can



also be deduced from 1 using Remark 1-C, by a proof similar to that which will be used in Proposition 6.) ∇

COROLLARY. *1° If* $\mathrm{B}^{\cdots,0}$ *is representable and if* $|\mathrm{B}|^0$ *admits (finite) limits, then* $\mathrm{B}^{\cdots,0}$ *admits* $Sq\mathrm{B}$*-wise (finite) limits.*

2° If B *is representable and if* $|\mathrm{B}|^{n-1}$ *admits (finite) limits, then the greatest n-category included in* $\mathrm{B}^{\cdots,0}$ *is* $(Cyl\mathrm{B})^{\cdots,n-2}$*-wise (finitely) complete.*

PROOF. The first assertion comes from Proposition 4, and the remarks preceding it. The second one uses the fact that $|Cyl\mathrm{B}|^{n-2}$ is isomorphic with $|\mathrm{B}|^{n-1}$ and that $|(Cyl\mathrm{B})^{n-2}|^{0,\ldots,n-3,n-1,n}$ is isomorphic with the greatest n-category included in $\mathrm{B}^{\cdots,0}$. ∇

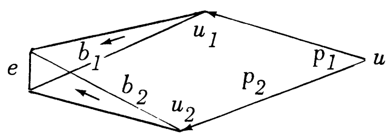
REMARKS. 1° $Cyl\mathrm{B}$ is not representable for the $(n-1)$-th composition.

2° If C is a representable 2-category, the double category $Q(\mathrm{C})$ of its up-squares is also representable [3] and Part 2 of the preceding corollary applied to $\mathrm{B} = Q(\mathrm{C})$ gives Bourn's Proposition 7 [2], since a $(Cyl\mathrm{B})^{\dots, n-2}$-wise limit is an analimit in the sense of Bourn, $|\mathrm{B}|^1$ «is» the category of 1-morphisms of C and C «is» the greatest 2-category included in $Q(\mathrm{C})^{\boxplus, \boxminus}$.

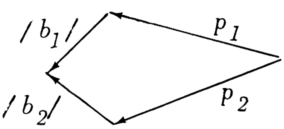
PROPOSITION 5. *If* B *is representable and if* $|\mathrm{B}|^{n-1}$ *admits pullbacks, then* $Cub\mathrm{B}$ *and* $Sq\mathrm{B}$ *are representable for the* $(n-2)$*-th composition.*

PROOF. For each object e of B^{n-1}, we denote by $\eta e : re \to e$ an $(n-1)$-representing block for e.

1° If $b_1 : u_1 \to e$ and $b_2 : u_2 \to e$ are morphisms of B^{n-1} with u_1, u_2 vertices of B, there exists a «universal» square



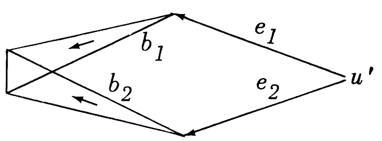
of B^{n-1} with p_1 and p_2 in $|\mathrm{B}|^{n-1}$ (called a $|\mathrm{B}|^{n-1}$*-pullback*). Indeed, by hypothesis, there exists in $|\mathrm{B}|^{n-1}$ a pullback



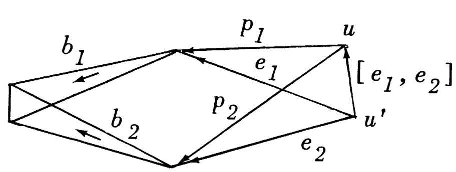
of the factors $/b_i/$ of b_i through ηe, and

$$b_2 \circ_{n-1} p_2 = \eta e \circ_{n-1} /b_2/ \circ_{n-1} p_2 = b_1 \circ_{n-1} p_1 .$$

If



is a square of B^{n-1} with e_1 and e_2 in $|\mathrm{B}|^{n-1}$, then $/b_1/ \circ_{n-1} e_1$ and $/b_2/ \circ_{n-1} e_2$ are both equal to the factor of $b_1 \circ_{n-1} e_1 = b_2 \circ_{n-1} e_2$ through ηe, so that there exists a unique

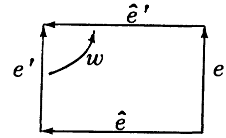


$$[e_1, e_2] : u' \to u \quad \text{in } |B|^{n-1}$$

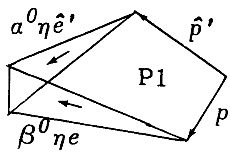
factorizing (e_1, e_2) through the pullback, i.e. satisfying

$$p_1 \circ_{n-1} [e_1, e_2] = e_1 \quad \text{and} \quad p_2 \circ_{n-1} [e_1, e_2] = e_2 .$$

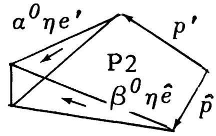
2° Let κ be an object of $(Cub\,B)^{n-2}$, which is a degenerate cube «reduced to its front face»



a) Construction of the $(n-2)$-representing cube for κ. By 1, there exist $|B|^{n-1}$-pullbacks

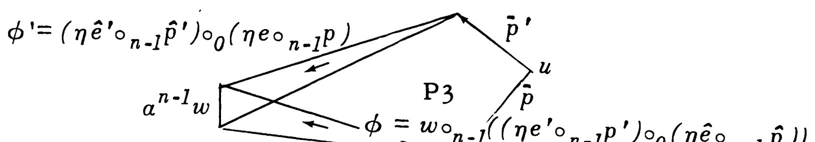


and



.

As $p, p', \hat{p}, \hat{p}'$ are in particular objects of B^0, the composites ϕ and ϕ' are defined and admit a $|B|^{n-1}$-pullback



.

The construction has been done so that $c\kappa =$

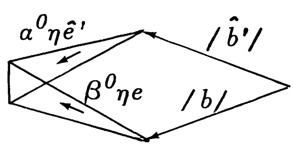
$$(\eta e' \circ_{n-1} p' \circ_{n-1} \bar{p}, \eta \hat{e}' \circ_{n-1} \hat{p}' \circ_{n-1} \bar{p}', w, u, \eta \hat{e} \circ_{n-1} \hat{p} \circ_{n-1} \bar{p}, \eta e \circ_{n-1} p \circ_{n-1} \bar{p}')$$

be a cube of B.

b) Universal property of $c\kappa$. Let $c = (b', \hat{b}', w, u', \hat{b}, b)$ be a cube with u' a vertex of B and $\beta^{n-2} c = \kappa$ (this means :

$$e = \beta^{n-1} b, \quad e' = \beta^{n-1} b', \quad \hat{e} = \beta^{n-1} \hat{b}, \quad \hat{e}' = \beta^{n-1} \hat{b}').$$

If $/b/$ and $/\hat{b}'/$ are the factors of b and $\hat{b}'$ through ηe and $\eta \hat{e}'$ there is a square



whose diagonal is

$$a^0 \eta \hat{e}' \circ_{n-1} / \hat{b}'/ = a^0 (\eta \hat{e}' \circ_{n-1} / \hat{b}'/) = a^0 \hat{b}' = \beta^0 b = \beta^0 \eta e \circ_{n-1} / b/ .$$

By the universal property of the $|\mathrm{B}|^{n-1}$-pullback P1, there is a unique

$$[\hat{b}', b] : u' \to a^{n-1} p \quad \text{in} \quad |\mathrm{B}|^{n-1}$$

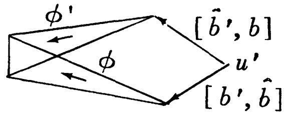
such that

$$\hat{p}' \circ_{n-1} [\hat{b}', b] = / \hat{b}'/ \quad \text{and} \quad p \circ_{n-1} [\hat{b}', b] = / b/ .$$

In the same way, using the equality $a^0 b' = \beta^0 \hat{b}$, the factors $/ b'/$ of b' through $\eta e'$ and $/ \hat{b}/$ of $\hat{b}$ through $\eta \hat{e}$ factorize through the $|\mathrm{B}|^{n-1}$-pullback P2 into a unique

$$[b', \hat{b}] : u' \to a^0 \hat{p} \quad \text{in} \quad |\mathrm{B}|^{n-1} .$$

Using the permutability axiom in B and the fact that p' and $\hat{p}$ are objects of B^0, we find the square



since

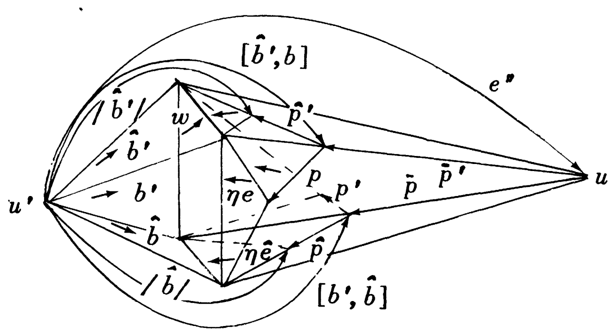
$$\phi' \circ_{n-1} [\hat{b}', b] = ((\eta \hat{e}' \circ_{n-1} \hat{p}') \circ_0 (\eta e \circ_{n-1} p)) \circ_{n-1} [\hat{b}', b] =$$
$$= (\eta \hat{e}' \circ_{n-1} \hat{p}' \circ_{n-1} [\hat{b}', b]) \circ_0 ((\eta e \circ_{n-1} p) \circ_{n-1} [\hat{b}', b]) =$$
$$= (\eta \hat{e}' \circ_{n-1} / \hat{b}'/) \circ_0 (\eta e \circ_{n-1} / b/) = \hat{b}' \circ_0 b ,$$

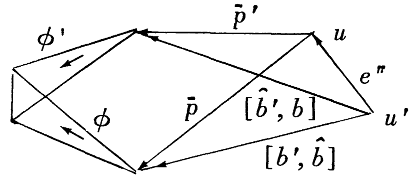
and similarly

$$\phi \circ_{n-1} [b', \hat{b}] = w \circ_{n-1} (b' \circ_0 \hat{b}) = \partial c = \hat{b}' \circ_0 b .$$

This square factorizes through the $|\mathrm{B}|^{n-1}$-pullback P3 into a unique

$$e'' : u' \to u \quad \text{in} \quad |\mathrm{B}|^{n-1} .$$





We have $c\kappa \circ_{n-2} e''^{\boxminus} = c$, since

$$\eta e \circ_{n-1} p \circ_{n-1} \bar{p}' \circ_{n-1} e'' = \eta e \circ_{n-1} p \circ_{n-1} [\hat{b}', b] = \eta e \circ_{n-1} /b/ = b,$$

and idem for the other lateral faces. The unicity of the different factors implies that $e''^{\boxminus}$ is the unique cube $/c/ : u'^{\boxminus} \to u^{\boxminus}$ in $|\,Cub\mathrm{B}\,|^{n-2}$ (isomorphic with $|\,\mathrm{B}\,|^{n-1}$) such that $c\kappa \circ_{n-2} /c/ = c$. This proves that $c\kappa$ is an $(n-2)$-representing cube for κ .

3° Let κ be an object of $(Sq\mathrm{B})^{n-2}$. Then κ is of the form considered in 2 except that now

$$w = \hat{e}' \circ_0 e = e' \circ_0 \hat{e} .$$

The $(n-2)$-representing cube $c\kappa$ «reduces» to a square (w being an object of B^{n-1}), and it is also the $(n-2)$-representing square for κ . ∇

COROLLARY. *If* B *is representable and if* $|\,\mathrm{B}\,|^{n-1}$ *admits (finite) limits, then* $|\,(Cub\mathrm{B})^{n-2}\,|^{0,\dots,n-3,n-1,n}$ *and* $Sq(\,|\,\mathrm{B}^{n-1}\,|^{0,\dots,n-2})$ *admit respectively* $(Cub\mathrm{B})^{\dots,n-2}$*-wise and* $(Sq\mathrm{B})^{\dots,n-2}$*-wise (finite) limits.*

PROOF. This results from Proposition 5 and the remarks preceding Proposition 4. In fact, $|\,(Cub\mathrm{B})^{n-2}\,|^{0,\dots,n-3,n-1,n}$ «is formed» by the up-squares of the greatest 2-category included in $\mathrm{B}^{n-1,0}$, its two last compositions are the vertical and horizontal compositions of up-squares, and its i-th composition, for $i < n-2$, is deduced pointwise from that of B^{i+1} .

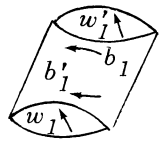
PROPOSITION 6. *If* B *is representable for the 0-th composition and if* $|\,\mathrm{B}\,|^{0}$ *admits pullbacks, then* $Cyl\,\mathrm{B}$, $Cub\mathrm{B}$ *and* $(Cub\mathrm{B})^{\dots,n-1}$ *are representable.*

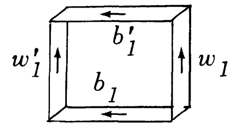
PROOF. 1° Let B' denote the n-fold category $\mathrm{B}^{n-1,1,\dots,n-2,0}$ deduced from $\mathrm{B}^{\dots,0}$ by the permutation

$$(1,\dots,n-1,0) \mapsto (n-1,1,\dots,n-2,0)$$

on the order of compositions. There is a canonical isomorphism

$$f\colon Cyl\,\mathrm{B} \tilde{\to} \mathrm{K} : [\,b'_1, w'_1, w_1, b_1\,] \to (w'_1, b'_1, b_1, w_1)$$





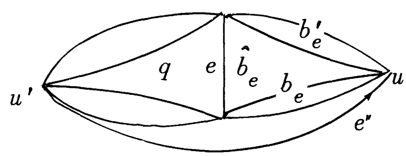
(Remark 1-1-C) onto the greatest $(n+1)$-category K included in the $(n+1)$-fold category $(Sq\mathrm{B}')^{\dots,n-2}$. As $\mathrm{B}^{\dots,0}$ is representable, so is B', and $|\mathrm{B}'|^{n-1} = |\mathrm{B}|^0$ admits pullbacks. By Proposition 5, $Sq\mathrm{B}'$ is representable for the $(n-2)$-th composition, as well as its greatest $(n+1)$-category K, and also the isomorphic $(n+1)$-category $Cyl\,\mathrm{B}$. More precisely, let $e^{\boxplus}$ be an object of $(Cyl\,\mathrm{B})^n$ (so that e is an object of B^0); then $e^{\boxminus} = f(e^{\boxplus})$ is an object of $(Sq\mathrm{B}')^{n-2}$ which admits an $(n-2)$-representing square

$$c\,e^{\boxminus} = (b'_e, \hat{b}'_e, \hat{b}_e, b_e) \colon u^{\boxminus} \to e^{\boxminus} \quad \text{in } \mathrm{K}^{n-2};$$

the cylinder of B :

$$f^{-1}(ce^{\boxminus}) = [\,\hat{b}'_e, b'_e, b_e, \hat{b}_e\,]$$

is the n-representing cylinder qe for $e^{\boxplus}$. If $q\colon u'^{\boxplus} \to e^{\boxplus}$ in $(Cyl\,\mathrm{B})^n$ with u' a vertex of B, its unique factor $e''^{\boxminus}$ through qe is such that $e''^{\boxplus}$ be the factor of $f(q)$ through $ce^{\boxminus}$.

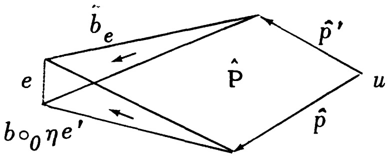


2° Let $b^{\boxplus}$ be an object of $(Cub\mathrm{B})^n$, $b \in \mathrm{B}$. We are going to construct an n-representing cube for $b^{\boxplus}$. Suppose $b\colon e' \to e$ in B^0.

a) By 1, there exists an n-representing cylinder

$$qe = [\,\hat{b}'_e, w'_e, w_e, \hat{b}_e\,] \quad \text{for } e^{\boxplus}.$$

Applying Part 1 of the proof of Proposition 5 to $\mathrm{B}^{\dots,0}$ instead of B (we interchange the 0-th and $(n-1)$-th compositions), there exists a $|\mathrm{B}|^0$-pullback $\hat{\mathrm{P}}$ of the following form, where $\eta e'$ denotes the 0-representing block for e':

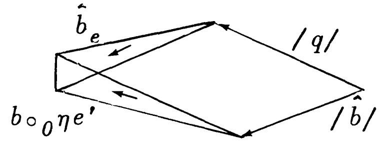


$$cb = (b, \hat{b}'_e \circ_0 \hat{p}', w'_e \circ_0 \hat{p}', w_e \circ_0 \hat{p}', \eta e' \circ_0 \hat{p}, u)$$

is a cube, since its diagonal ∂cb is :

$$(\hat{b}'_e \circ_0 \hat{p}') \circ_{n-1} (w_e \circ_0 \hat{p}') = (\hat{b}'_e \circ_{n-1} w_e) \circ_0 \hat{p}' = \partial qe \circ \hat{p}' =$$
$$= (w'_e \circ_{n-1} \hat{b}_e) \circ_0 \hat{p}' = (w'_e \circ_0 \hat{p}') \circ_{n-1} (b \circ_0 \eta e' \circ_0 \hat{p}).$$

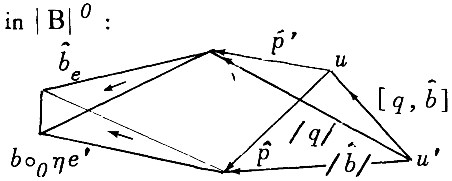
b) Let $c = (b, \hat{b}', w', w, \hat{b}, u')$ be a cube with u' a vertex of B. Then the factor $/q/$ of the cylinder $q = [\hat{b}', w', w, b \circ_0 \hat{b}]$ through qe and the factor $/\hat{b}/$ of $\hat{b}$ through $\eta e'$ determine the square



because

$$b \circ_0 \eta e' \circ_0 /\hat{b}/ = b \circ_0 \hat{b} = \hat{b}_e \circ_0 /q/ .$$

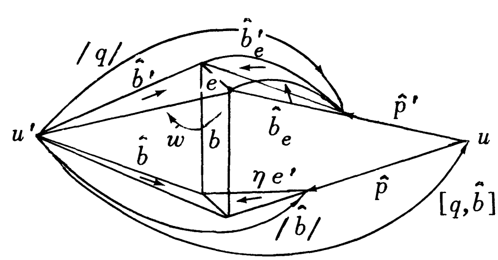
This square factors uniquely through the $|B|^0$-pullback $\hat{P}$ into a morphism $[q, \hat{b}] : u' \to u$ in $|B|^0$:



It follows from the construction that $cb \boxplus [q, \hat{b}]^{\boxminus} = c$, since

$$\eta e' \circ_0 \hat{p} \circ_0 [q, \hat{b}] = \eta e' \circ_0 /\hat{b}/ = \hat{b}, \quad w_e \circ_0 \hat{p}' \circ_0 [q, \hat{b}] = w_e \circ_0 /q/ = w,$$

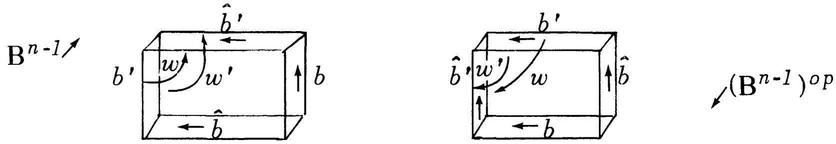
and idem for the other terms of c. Moreover, the unicity of the successive factors implies that $[q, \hat{b}]^{\boxminus}$ is the unique morphism $/c/$ of $|CubB|^n$ sa-



tisfying $cb \boxdot / c / = c$. Hence cb is a representing cube for $b^{\boxdot}$.

c) $(Cub\mathrm{B})^{\dots,n-1}$ is representable. Indeed, let B^{op}_{n-1} be the n-fold category obtained from B by replacing the $(n-1)$-th category B^{n-1} by its opposite. B^{op}_{n-1} and B being simultaneously representable for the 0-th composition (B^{n-1} and $(\mathrm{B}^{n-1})^{op}$ have the same objects), $Cub(\mathrm{B}^{op}_{n-1})$ is representable by Part 2. There is a canonical isomorphism «reversing the cubes» $F : (Cub\,\mathrm{B})^{n-1} \to (Cub\,\mathrm{B}^{op}_{n-1})^{n}$:

$$(b', \hat{b}', w', w, \hat{b}, b) \mapsto (\hat{b}', b', w, w', b, \hat{b}),$$



which maps $|\,Cub\mathrm{B}\,|^{n-1}$ onto $|\,Cub\mathrm{B}^{op}_{n-1}\,|^{n}$. Hence $(Cub\mathrm{B})^{\dots,n-1}$ is also representable. ∇

REMARK. F defines an isomorphism $(Cub\mathrm{B})^{\dots,n-1} \to (Cub\,\mathrm{B}^{op}_{n-1})^{op}_{n-2}$. The $(n+1)$-fold category $Cub(\mathrm{B}^{op}_{n-1})$ might be called *the multiple category of down-cubes* of B (by analogy with the notion of a down-square of a 2-category), denoted by $Cub^{\downarrow}\mathrm{B}$.

COROLLARY. *If* B *is representable for the 0-th composition and if* $|\,\mathrm{B}\,|^{0}$ *admits (finite) limits, then* $\mathrm{B}^{\dots,0}$ *admits* $Cub\mathrm{B}$*-wise (finite) limits.*

This results from Proposition 6, since $|\,(Cub\mathrm{B})^{n}\,|^{0,\dots,n-1}$ is isomorphic with $\mathrm{B}^{\dots,0}$. ∇

2° A laxified internal Hom on Cat_n.

Imitating the construction of the cartesian closed structure on Cat_n given in Section C [5], we define a «closure» functor on Cat_n by replacing the *Square* functor and the *Link* functor respectively by the *Cube* functor and by the *LaxLink* functor.

Let $LaxHom_n : Cat_n^{op} \times Cat_n \to Cat_n$ be the composite functor

$$Cat_n^{op} \times Cat_n \xrightarrow{id \times Cub_{n,2n}} Cat_n^{op} \times Cat_{2n} \xrightarrow{id \times \tilde{\gamma}} Cat_n^{op} \times Cat_{2n} \xrightarrow{Hom(-,-)} Cat_n$$

where :

- $\tilde{\gamma}: Cat_{2n} \to Cat_{2n}$ is the isomorphism «permutation of the compositions» associated to the permutation

$$\gamma : (0, \dots, 2n-1) \mapsto (0, 2, \dots, 2n-2, 1, 3, \dots, 2n-1),$$

which associates to the $2n$-fold category H the $2n$-fold category H^γ in which the i-th category is H^{2i} and the $(i+n)$-th category is H^{2i+1}, for each $i < n$.

- $Hom(-,-)$ is the restriction of the internal Hom functor of the monoidal closed category $(\coprod_n Cat_n, \blacksquare, Hom)$ (defined in [4]); it maps the couple (A, H) of an n-fold category A and a $2n$-fold category H onto the n-fold category $Hom(A, H)$ formed by the n-fold functors $f: A \to H^{0,\dots,n-1}$, the i-th composition being deduced pointwise from that of H^{n+i}, for $i < n$.

DEFINITION. The functor $LaxHom_n : Cat_n^{op} \times Cat_n \to Cat_n$ is called the *laxified internal Hom on* Cat_n.

If A and B are n-fold categories, then

$$LaxHom_n(A, B) = Hom(A, (Cub B)^\gamma)$$

is formed by the n-fold functors

$$h: A \to (Cub_{n,2n} B)^{0,2,\dots,2n-2},$$

the i-th composition being deduced pointwise from the $(2i+1)$-th composition of $Cub_{n,2n}B$ (itself deduced «horizontaly» from the composition of B^i, as remarked at the end of Section B).

PROPOSITION 7. *For each n-fold category* A, *the partial functor*

$$LaxHom_n(A, -): Cat_n \to Cat_n$$

admits a left adjoint $-\otimes A: Cat_n \to Cat_n$. *The corresponding tensor product functor* $\otimes: Cat_n \times Cat_n \to Cat_n$ *admits as a unit the n-fold category* 1_n *on the set* $1 = \{0\}$.

PROOF. 1° a) Since $LaxHom_n(A, -)$ is equal to the composite

$$Cat_n \xrightarrow{Cub_{n,2n}} Cat_{2n} \xrightarrow{\tilde{\gamma}} Cat_{2n} \xrightarrow{Hom(A,-)} Cat_n,$$

it admits as a left adjoint, denoted by $-\otimes A : Cat_n \to Cat_n$, the composite functor

$$Cat_n \xrightarrow{-\blacksquare A} Cat_{2n} \xrightarrow{\tilde{\gamma}^{-1}} Cat_{2n} \xrightarrow{LaxLk_{2n,n}} Cat_n ,$$

where $-\blacksquare A$ is the partial square product functor, left adjoint of $Hom(A, -)$ (see [4]) and $LaxLk_{2n,n}$ is the left adjoint of $Cub_{n,2n}$ (Proposition 1, Corollary 1). So, if B is an n-fold category, we have

$$B \otimes A = LaxLk_{2n,n}(B \blacksquare A)^{\gamma^{-1}} ,$$

where $(B \blacksquare A)^{\gamma^{-1}}$ is the $2n$-fold category in which

- the $2i$-th category is $\underline{B}^{dis} \times A^i$,
- the $(2i+1)$-th category is $B^i \times \underline{A}^{dis}$, for $i < n$.

b) There exists a functor

$$\otimes : Cat_n \times Cat_n \to Cat_n$$

extending the functors $-\otimes A$, for any n-fold category A. This comes from the fact that the right adjoints $LaxHom_n(A, -)$ of $-\otimes A$ are all restrictions of the functor $LaxHom_n$. The functor $\otimes$ maps the couple

$$(f: A \to A',\ g: B \to B')$$

of n-fold functors onto the n-fold functor $g\otimes f: B\otimes A \to B'\otimes A'$ corresponding by adjunction to the composite n-fold functor:

$$B \xrightarrow{g} B' \xrightarrow{l} Hom(A', B'\otimes A') \xrightarrow{Hom(f, B'\otimes A')} Hom(A, B'\otimes A')$$

where l is the liberty morphism defining $B'\otimes A'$ as a free object generated by B' with respect to $Hom(A', -)$.

2° $\otimes$ *admits* 1_n *as a unit* (up to isomorphisms): We have to construct, for each n-fold category A, canonical isomorphisms

$$1_n \otimes A \xrightarrow{\sim} A \xrightarrow{\sim} A \otimes 1_n ,$$

where

$$1_n \otimes A = LaxLk_{2n,n}(1_n \blacksquare A)^{\gamma^{-1}} \quad \text{and} \quad A\otimes 1_n = LaxLk_{2n,n}(A \blacksquare 1_n)^{\gamma^{-1}} .$$

Now, there are isomorphisms:

- $(0, a) \mapsto a$ from $(1_n \blacksquare A)^{\gamma^{-1}}$ onto the $2n$-fold category

$$\tilde{A} = (A^0, \underline{A}^{dis}, \dots, A^{n-1}, \underline{A}^{dis})$$

such that $\tilde{A}^{2i} = A^i$ and $\tilde{A}^{2i+1} = \underline{A}^{dis}$, for $i < n$,

- $(a, 0) \mapsto a$ from $(A \blacksquare 1_n)^{\gamma^{-1}}$ onto the $2n$-fold category

$$\tilde{\tilde{A}} = (\underline{A}^{dis}, A^0, \dots, \underline{A}^{dis}, A^{n-1})$$

such that $\tilde{\tilde{A}}^{2i} = \underline{A}^{dis}$ and $\tilde{\tilde{A}}^{2i+1} = A^i$, for $i < n$.

Hence, it suffices to construct isomorphisms

$$A \xrightarrow{\sim} LaxLk_{2n,n}\tilde{A} \quad \text{and} \quad A \xrightarrow{\sim} LaxLk_{2n,n}\tilde{\tilde{A}}.$$

For this, we first prove the assertions a and b:

a) *If* H *is an* $(m+1)$*-fold category such that* H^m *is the discrete category on* $\underline{H}$, *then* $LaxLk H \approx H^{m-1, 0, \dots, m-2}$.

Indeed, an $(m+1)$-fold functor $g: H \to Cub K$, where K is an m-fold category, takes its values into the objects of $(Cub K)^m$ (we use that H^m is discrete), so that it admits a restriction

$$g': H^{0,\dots,m-1} \to |(Cub K)^m|^{0,\dots,m-1}.$$

Then,

$$\hat{g} = (H^{0,\dots,m-1} \xrightarrow{g'} |(Cub K)^m|^{0,\dots,m-1} \xrightarrow{(-^{\square})^{-1}} K^{1,\dots,m-1,0})$$

is an m-fold functor, as well as

$$\hat{g}: H^{m-1,0,\dots,m-2} \to K : \eta \mapsto k \quad \text{if } g(\eta) = k^{\square}.$$

This determines a 1-1 correspondence $g \mapsto \hat{g}$ from the set of $(m+1)$-fold functors $g: H \to Cub K$ onto the set of m-fold functors $H^{m-1,0,\dots,m-2} \to K$. It follows that $H^{m-1,0,\dots,m-2}$ is a free object generated by H with respect to the functor $Cub_{m,m+1}: Cat_m \to Cat_{m+1}$, and we can choose it as $LaxLk H$ (Proposition 1).

b) *If* H *is an* $(m+1)$*-fold category such that* H^{m-1} *is discrete, then* $LaxLk H \approx H^{m,0,\dots,m-2}$. The proof is similar, using the isomorphism

$$|(Cub K)^{m-1}|^{0,\dots,m-2,m} \xrightarrow{(-^{\boxminus})} K^{1,\dots,m-1,0}.$$

c) Applying a to the $2n$-fold category $\tilde{A}$ whose last composition is the discrete one, we find an isomorphism

$$LaxLk\tilde{A} \approx (A^{n-1}, A^0, \underline{A}^{dis}, \dots, A^{n-2}, \underline{A}^{dis}),$$

and by iteration, $1_n \otimes A \approx LaxLk_{2n,n}\tilde{A}$ may be identified with A. Similarly, we deduce from b that

$$LaxLk\tilde{\tilde{A}} \approx (A^{n-1}, \underline{A}^{dis}, A^{0}, \dots, \underline{A}^{dis}, A^{n-2}),$$

and by iteration $A \otimes 1_n \approx LaxLk_{2n,n}\tilde{\tilde{A}}$ may be identified with A. ∇

COROLLARY. *The vertices of* $LaxHom_n(A,B)$ *are identified with the n-fold functors from* A *to* B.

PROOF. These vertices are identified [4] with the n-fold functors

$$f: 1_n \to LaxHom_n(A,B),$$

which by adjunction (Proposition 7) are in 1-1 correspondence with the n-fold functors $A \tilde{\to} 1_n \otimes A \to B$. ∇

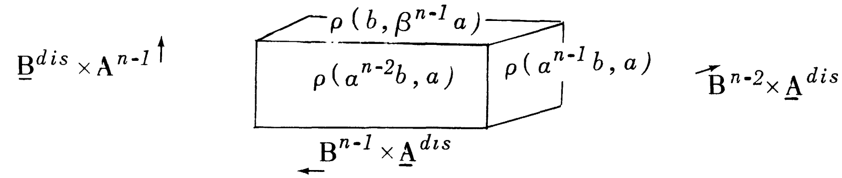
EXAMPLES.

1° Let A and B be n-fold categories. Then $L = LaxLk((B \blacksquare A)^{\gamma^{-1}})$ is generated by the blocks

$$\rho(u,a), \quad \rho(b,v), \quad \rho(t,a),$$

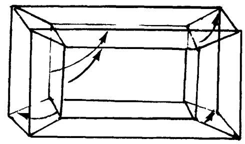
where a and b are blocks of A and B, where u, v, t are objects of B^{n-1}, A^{n-1} and B^{n-2} respectively, and where ρ is the universal admissible morphism used in the construction of *LaxLink* (Proof, Proposition 1). In particular, for any couple (b,a), there exist blocks of L

$$\rho(\alpha^{n-2}b, a), \quad \rho(\beta^{n-2}b, a), \quad \langle b, a\rangle = \rho(b, \beta^{n-1}a) \circ_0 \rho(\alpha^{n-1}b, a).$$



So L may be seen as an «enrichment» of $B \times A$ by the blocks $\rho(t,a)$, for each object t of B^{n-2}. By iteration, $B \otimes A$ is an «enrichment», or a «laxification» of $B \times A$.

2° For $n = 2$, the 4-fold category $(Cub_{2,4}A)^{\gamma}$ is defined in a similar way as the 4-fold category of frames $(Sq_{2,4}A)^{\gamma}$ (Example, Section C [5]), by replacing the frames, which are «squares of squares» by «full frames», which are «cubes of cubes». Then $LaxHom_2(A,B)$ has a description analogous to that given for $Hom_2(A, B)$, except that frames



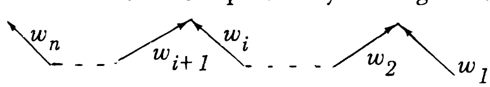
are replaced by full frames; the vertices remain the double functors $A \to B$ (Corollary, Proposition 7). In particular, if A and B are 2-categories, the greatest 2-category included in $LaxHom_2(A, B)$ is the 2-category $Fun(A, B)$ introduced by Gray [7], and the tensor product $B \otimes A$ admits as a reflection the 2-category tensor product constructed by Gray [8].

COMPLEMENTS. *Other closure functors.*

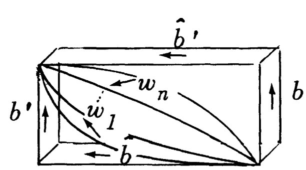
1° *A closure functor on the category n-Cat of n-categories* is defined by the same method as above, replacing the *Cube* functor $Cub_{n,2n}$ by the *Cylinder* functor *(n, 2n)-Cyl* (Section C), and there is also associated a tensor product on *n-Cat*.

2° In the last Remark of 1-D, we have defined the *(n+1)*-fold category of down-cubes of B; it gives rise to a functor «*Down-cube*» $Cub^{\downarrow}_{n,2n}$ from Cat_n to Cat_{2n}, and as above to a «*laxified*» *internal Hom functor on* Cat_n, denoted by $Lax\,Hom^{\downarrow}_n$, *for which Proposition 7 is also valid*, with a tensor product functor $\otimes^{\downarrow}$ having 1_n as unit.

3° The tensor product functors $\otimes$ and $\otimes^{\downarrow}$ on Cat_n are not symmetric, one being in some sense the symmetric of the other. More generally, we may replace the cubes by «laxified cubes» in which the 2-cells w' and w of $B^{n-1,0}$ would be replaced by «strings of 2-cells of $B^{n-1,0}$»



(with respect to the category B^{n-1}).



This gives rise to an *(n+1)*-fold category $Lax\,Cub\,B$, containing both

CubB and $Cub^{\downarrow}$B as $(n+1)$-fold subcategories. *The constructions of this paper may be generalized in this setting.*

4° «*Less-laxified*» *internal Hom functors on* Cat_n are defined by replacing in Proposition 7 the composite $Cub_{n,2n}$ of *Cube* functors by a composite in which at some steps $Cub_{m,m+1}$ is replaced by $Sq_{m,m+1}$. Then *Proposition 7 remains valid*, so that we obtain different tensor products of the couple (B,A) of n-fold categories, the «smallest» one being the cartesian product $B \times A$ (corresponding to the internal Hom functor constructed in [5], where only *Square* functors are taken), the «greatest» one being $B \otimes A$ (where only *Cube* functors are used); all admit 1_n as a unit up to isomorphisms. In Part III, we have constructed an $(n+1)$-category Nat_n «gluing together» the n-fold categories $Hom_n(A,B)$, for any n-fold categories A and B. If $\hat{H}$ is an internal Hom functor other than the «cartesian closure functor» Hom_n, there is no $(n+1)$-fold category on the n-fold category coproduct of the multiple categories $\hat{H}(A,B)$, the canonical composition functor

$$\hat{\kappa}: \hat{H}(A,B) \otimes \hat{H}(B,K) \to \hat{H}(A,K)$$

admitting as its domain a tensor product and not a cartesian product.

5° *The constructions of Square, Link, Cube, LaxLink, and so the results given in Parts III and IV may be «internalized» (without essential changes) for multiple categories in(ternal to) a category* V *with commuting coproducts* (see Penon [8] and Part III, Appendix) *and cokernels.* Indeed there exist then free categories in V generated by a graph in V and quasi-quotient categories in V.

3° Characterization of multiple categories in terms of 2-categories.

The construction of $LaxLink$ will be used now to prove that each double category «is» a double sub-category of a double category of squares of a 2-category.

PROPOSITION 8. *Let Q: 2-Cat → Cat_2 be the functor mapping a 2-category* C *onto the double category $Q(C)$ of its (up-)squares. Then Q admits a left adjoint String: Cat_2 → 2-Cat.*

PROOF. Q may be seen as the composite of the four functors

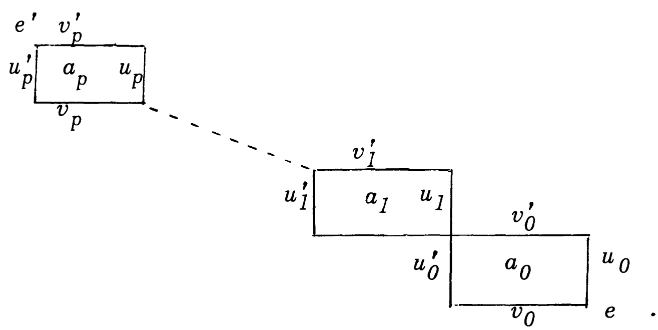
$$2\text{-}Cat \hookrightarrow Cat_2 \xrightarrow{\tilde{y}^{1,0}} Cat_2 \xrightarrow{Cub} Cat_3 \xrightarrow{|-|^{1,2}} Cat_2 ,$$

where $\tilde{y}^{1,0}$ is the isomorphism «interchanging the two compositions» and where $|-|^{1,2}$ is the functor mapping a triple category T onto the double category formed by the objects of the 0-th category T^0. These four functors admitting left adjoints, their composite Q admits a left adjoint, constructed as follows:

Let A be a double category and $\bar{A}$ be the triple category with the same blocks $(\underline{A}^{dis}, A^0, A^1)$ whose 0-th category is the discrete category on $\underline{A}$ (it is the free object generated by A with respect to $|-|^{1,2}$, by Proposition 9, Part II). The free object $(LaxLk\bar{A})^{1,0}$ generated by $\bar{A}$ with respect to

$$Cat_2 \xrightarrow{\tilde{y}^{1,0}} Cat_2' \xrightarrow{Cub} Cat_3$$

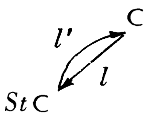
is a 2-category whose 1-morphisms are equivalence classes of strings of objects of alternately A^0 and A^1, and whose 2-cells from e to e' are classes of strings of blocks of A :

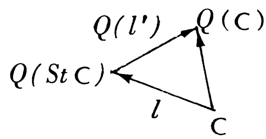


This 2-category is the free object generated by A with respect to Q. It will be called the *2-category of strings of* A, denoted by $St\,A$. ∇

COROLLARY. *The functor String: $Cat_2 \to 2\text{-}Cat$ is equivalent to a left inverse of the inclusion: $2\text{-}Cat \hookrightarrow Cat_2$.*

PROOF. It suffices to prove that, if C is a 2-category, $St\,C$ is isomorphic to C. Indeed, let $l: C \to Q(St\,C)$ be the liberty double functor. As C is a 2-category, l takes its values into the greatest sub-2-category $St\,C$ of $Q(St\,C)$, and its restriction $l: C \to St\,C$ admits as an inverse the 2-func-





tor $l': St\,C \to C$ associated by adjunction to the inclusion $C \hookrightarrow Q(C)$. ∇

REMARK. If $\mathbb{A}$ is the double category $Q(C)$ of squares of a 2-category C then C is not isomorphic to $St\,\mathbb{A}$; counter example: C is the 2-category $(\underline{2}^{dis}, 2)$.

PROPOSITION 9. *If* $\mathbb{A}$ *is a double category, then it is canonically isomorphic to a double sub-category of the double category* $Q(St\mathbb{A})$ *of squares of the 2-category* $St\mathbb{A}$.

PROOF. The liberty double functor $l: \mathbb{A} \to Q(St\mathbb{A})$ is injective. Indeed, let a and a' be blocks of $\mathbb{A}$ such that $l(a) = l(a')$. By definition of the equivalence relation used to define $Lax\,Lk\bar{\mathbb{A}}$ (and therefore $St\mathbb{A}$), there exists a family (b_i) of «smaller» blocks of $\mathbb{A}$ admitting both a and a' as double composites. More precisely, let Λ be the free double non-associative category generated by the double graph underlying $\mathbb{A}$, and $\lambda: \Lambda \to \mathbb{A}$ be the canonical non-associative double functor (for its existence, see [6]); then there exist blocks η and η' of Λ constructed on the family (b_i) and such that

$$a = \lambda(\eta) = \lambda(\eta') = a'.$$

(Example:

<table>
<tr><td>b_5</td><td colspan="2">b_4</td></tr>
<tr><td>b_3</td><td>b_2</td><td>b_1</td></tr>
</table>

$$a = (b_5 \circ_0 b_4) \circ_1 (b_3 \circ_0 b_2 \circ_0 b_1) = (b_5 \circ_1 b_3) \circ_0 (b_4 \circ_1 (b_2 \circ_0 b_1)) = a'.)$$

So l is injective, and its image $l(\mathbb{A})$ is isomorphic to $\mathbb{A}$. ∇

Hence all double categories «are» double sub-categories of double categories of squares of a 2-category. This explains why it was difficult to find natural examples of double categories other than 2-categories and their squares! (Spencer [9] has characterized double categories of squares of a 2-category as those double categories admitting a special connection in the sense of Brown.)

It follows that, if $\mathbb{A}$ is a double category and $f: \mathrm{K} \to |\mathbb{A}^0|^1$ a functor, an $\mathbb{A}$-wise limit of f is simply a lax-limit (in the sense of Gray-Bourn-Street) of f considered as a 2-functor from $(\underline{\mathrm{K}}^{dis}, \mathrm{K})$ into the greatest 2-category included in $\mathbb{A}$, such that the 2-cells projections of the lax-limit take their values in $\mathbb{A}$; this is a restrictive condition, since $\mathbb{A}$ is only a double sub-category of $Q(St\mathbb{A})$. Hence generalized limits (defined in Part II) are just lax-limits «relativized to a double sub-category».

From Proposition 9, we deduce :

PROPOSITION 10. *Let* $\mathbb{A}$ *be an n-fold category, with* $n > 2$. *Then there exists a canonical embedding from* $\mathbb{A}$ *into an n-fold category of the form* $Cub_{2,n}Q(\mathrm{C})$, *where* C *is a 2-category.*

PROOF. The functor

$$2\text{-}Cat \xrightarrow{Q} Cat_2 \xrightarrow{Cub_{2,n}} Cat_n$$

admits a left adjoint which associates to $\mathbb{A}$ the 2-category

$$\mathrm{C} = St(Lax\,Lk_{n,2}\mathbb{A}).$$

Remark that the corresponding liberty morphism $L: \mathbb{A} \to Cub_{2,n}Q(\mathrm{C})$ is generally not injective, since it factors through the liberty morphism l from $\mathbb{A}$ to $Cub(LaxLk\mathbb{A})$ which identifies (Proof, Proposition 1) two blocks of $\mathbb{A}$ having the same sources and targets for the last three compositions. ∇

COMPLEMENT. Proposition 10 does not give a complete characterization of n-fold categories, for $n > 2$, in terms of 2-categories, since the embedding L is generally not injective. However there is such a characterization (which will be given elsewhere), obtained by laxifying at each step the construction of the functor *Cube*, in a way similar to that used to proceed from the functor *Square* to the functor *Cube*.