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CAHIERS DE TOPOLOGIE
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MULTIPLE FUNCTORS
III. THE CARTESIAN CLOSED CATEGORY Cat_n

by Andrée and Charles EHRESMANN

INTRODUCTION.

This paper is Part III of our work on multiple functors [4, 5] and it is a direct continuation of Part II. It is devoted to an explicit description of the cartesian closed structure on Cat_n (= category of n-fold categories) which will be «laxified», in the Part IV [6] (this is a much more general result than that announced in Part I). The existence of such structures might be deduced from general theorems on sketched structures [7,14], but this does not lead to concrete definitions. Here the construction uses the monoidal closed category $(\coprod_n Cat_n, \blacksquare, Hom)$ of multiple categories defined in Part II.

In the cartesian closed category Cat, the internal Hom functor maps (A, C) onto the category of natural transformations from A to C, which is identified with the category $Hom(A, \square C)$, where $\square C$ is the double category of squares of C.

To generalize this situation, the idea is to construct a functor $\square_n$ from Cat_n to Cat_{2n} (which reduces for $n = 1$ to the functor $\square : Cat \to Cat_2$), whose composite with the functor $Hom(A, -): Cat_{2n} \to Cat_n$ gives, for each n-fold category A, the partial internal Hom functor of the cartesian closed structure of Cat_n. In fact, we first define a pair of adjoint functors *Square* and *Link* between Cat_n and Cat_{n+1}, which has also some interest of its own; iteration of this process leads to a functor $\square_n : Cat_n \to Cat_{2n}$ whose left adjoint maps $B \blacksquare A$ onto the product $B \times A$, for each n-fold category B. Hence the functor

$$Hom(A, \square_n -): Cat_n \to Cat_n$$

is a right adjoint of the product functor $- \times A$, as desired.

The delicate point is the explicit construction of *Link*, which «is» a left inverse of *Square*. The category of components of a 2-category, as well as the crossed product category associated to the action [8] of a category on a category, appear as examples of $Link\,\mathrm{A}$.

Finally Cat_n is «embedded» as the category of 1-morphisms in the $(n+1)$-category Nat_n of hypertransformations (or «natural transformation between natural transformations, between...»), whose n first categories form the n-fold category coproduct of $Hom_n(\mathrm{A},\mathrm{B})$, for any n-fold categories A, B. The construction of Nat_n uses the equivalence (see Appendix) between categories enriched in a category V with commuting coproducts (in the sense of [21]) and categories internal to V whose object of objects is a coproduct of copies of the final object.

NOTATIONS.

The notations are those introduced in Part II. In particular, if B is an n-fold category, B^i denotes its i-th category for each integer $i < n$, and $\mathrm{B}^{i_0,\dots,i_{p-1}}$, for each sequence $(i_0,\dots,i_{p-1})$ of distinct integers $i_j < n$, is the p-fold category whose j-th category is B^{i_j}.

Let A be an m-fold category. The square product $\mathrm{B} \blacksquare \mathrm{A}$ is the $(n+m)$-fold category on the product set $\underline{\mathrm{B}} \times \underline{\mathrm{A}}$ (where $\underline{\mathrm{B}}$ always denotes the set of blocks of B) whose i-th category is:

$$\underline{\mathrm{B}}^{dis} \times \mathrm{A}^i \text{ for } i < m, \quad \mathrm{B}^{i-m} \times \underline{\mathrm{A}}^{dis} \text{ for } m \leqslant i < n+m$$

($\underline{\mathrm{B}}^{dis}$ is the discrete category on $\underline{\mathrm{B}}$).

If $m < n$, then $Hom(\mathrm{A},\mathrm{B})$ is the $(n-m)$-fold category on the set of multiple functors $f: \mathrm{A} \to \mathrm{B}$ (i.e., on the set of m-fold functors f from A to $\mathrm{B}^{0,\dots,m-1}$) whose j-th composition is deduced «pointwise» from that of B^{m+j}, for each integer $j < n-m$.

The category $\coprod_n Cat_n$ of (all small) multiple categories, equipped with $\blacksquare$ and Hom is monoidal closed (Proposition 7 [5]), i.e., the partial functor $Hom(\mathrm{A},-): Cat_{n+m} \to Cat_n$ is right adjoint to $- \blacksquare \mathrm{A}: Cat_n \to Cat_{n+m}$.

A. The adjoint functors Square and Link.

This Section is devoted to the construction of the functor *Square* from Cat_n to Cat_{n+q}, and of its left adjoint, the functor *Link*. For $n = 1$, the functor *Square* reduces to the functor $\square : Cat \to Cat_2$, whose definition is first recalled to fix the notations.

2 is always the category

$$1 \xleftarrow{\quad (1,0) \quad} 0$$

so that 2×2 is represented by the commutative diagram:

$$\begin{tikzcd}
(1,1) & (0,1) \arrow[l, "{(z,1)}"'] \\
(1,0) \arrow[u, "{(1,z)}"] & (0,0) \arrow[l, "{(z,0)}"] \arrow[u, "{(0,z)}"'] \arrow[ul, "{(z,z)}"]
\end{tikzcd} \qquad (\text{ where } z = (0,1)).$$

Let C be a category. A functor $f: 2 \times 2 \to C$:

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, "{(z,1)}"'] \\
\cdot \arrow[u, "{(1,z)}"] & \cdot \arrow[l, "{(z,0)}"] \arrow[u, "{(0,z)}"'] \arrow[ul, "{(z,z)}"]
\end{tikzcd} \quad \longmapsto \quad \begin{tikzcd}
\cdot & \cdot \arrow[l, "{f(z,1)}"'] \\
\cdot \arrow[u, "{f(1,z)}"] & \cdot \arrow[l, "{f(z,0)}"] \arrow[u, "{f(0,z)}"'] \arrow[ul, "{f(z,z)}"]
\end{tikzcd}$$

is entirely determined by the (commutative) square of C :

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, "{f(z,1)}"'] \\
\cdot \arrow[u, "{f(1,z)}"] & \cdot \arrow[l, "{f(z,0)}"] \arrow[u, "{f(0,z)}"'] \arrow[ul, phantom, "="]
\end{tikzcd}$$

(since $f(z,z)$ is the «diagonal» of this square:

$$f(z,1)\, f(0,z) = f(1,z)\, f(z,0) \text{);}$$

and every square $(c', \hat{c}', \hat{c}, c)$

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, "\hat{c}'"'] \\
\cdot \arrow[u, "c'"] & \cdot \arrow[l, "\hat{c}"] \arrow[u, "c"']
\end{tikzcd}$$

of C is obtained in this way. So we shall identify the set $Hom(2 \times 2, C)$ of functors from 2×2 to C with the set of squares of C.

On this set, the «vertical» and the «horizontal» compositions:

$$(\bar c', \hat c'', \hat c', \bar c) \boxminus (c', \hat c', \hat c, c) = (\bar c' c', \hat c'', \hat c, \bar c c),$$
$$(c'', \tilde c', \tilde c, c') \boxplus (c', \hat c', \hat c, c) = (c'', \tilde c' \hat c', \tilde c \hat c, c),$$

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "\hat c''"'] \\
\bullet \arrow[u, "\bar c'"] & \bullet \arrow[l, "\hat c'"'] \arrow[u, "\bar c"'] \\
\bullet \arrow[u, "c'"] & \bullet \arrow[l, "\hat c"] \arrow[u, "c"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet & \bullet \arrow[l, "\tilde c'"'] & \bullet \arrow[l, "\hat c'"'] \\
\bullet \arrow[u, "c''"] & \bullet \arrow[l, "\tilde c"] \arrow[u, "c'"] & \bullet \arrow[l, "\hat c"] \arrow[u, "c"']
\end{tikzcd}$$

define categories $\boxminus C$ and $\boxplus C$ (which are both isomorphic, and also called by some authors category of arrows of C). The couple $(\boxminus C, \boxplus C)$ is the *double category* $\square C$ *of squares of* C.

The functor $\square : Cat \to Cat_2$ maps $g: C \to C'$ onto

$$\square g : \square C \to \square C' : (c', \hat c', \hat c, c) \mapsto (g(c'), g(\hat c'). g(\hat c), g(c)).$$

Now let n be an integer, $n > 1$. Let B be an n-fold category. Taking for C above the 0-th category B^0 of B, we have, on the set of squares of B^0 (to which are identified the functors $2 \times 2 \to B^0$), not only the double category $\square B^0$, but also the $(n-1)$-fold category $Hom(2 \times 2, B)$, whose i-th composition (deduced pointwise from that of B^{i+1}) is written with squares:

$$(b_1', \hat b_1', \hat b_1, b_1) \circ_i (b', \hat b', \hat b, b) = (b_1' \circ_{i+1} b', \hat b_1' \circ_{i+1} \hat b', \hat b_1 \circ_{i+1} \hat b, b_1 \circ_{i+1} b),$$

iff the four composites are defined in B^{i+1}.

DEFINITION. The *multiple category of squares of* B, denoted by SqB, is the $(n+1)$-fold category on the set of squares of B^0 such that:

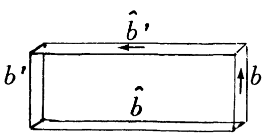
$$(SqB)^{0,\dots,n-2} = Hom(2 \times 2, B), \quad (SqB)^{n-1} = \boxminus B^0, \quad (SqB)^n = \boxplus B^0$$

(the $(n-1)$ first compositions are those of $Hom(2 \times 2, B)$, the two last ones being the vertical and the horizontal compositions of squares).

To «visualize» this multiple category SqB, we shall also represent a square

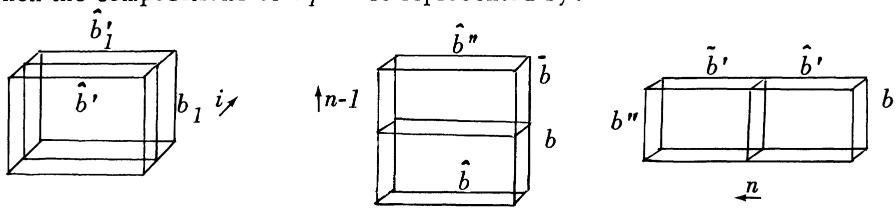
$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "\hat b'"'] \\
\bullet \arrow[u, "b'"] & \bullet \arrow[l, "\hat b"] \arrow[u, "b"']
\end{tikzcd}$$

of B^0 by



;

then the compositions of SqB are represented by:



REMARK (not used afterwards). The construction of SqB may be interpreted in terms of sketched structures. To each category $\phi : \sigma \to V$ internal to a category V with pullbacks, it is associated a category $\partial\phi : \sigma \to V$ internal to V^σ (Proposition 28 [7]). If $\phi : \sigma \to Cat_{n-1}$ is the category in Cat_{n-1} canonically associated to $B^{1,\dots,n-1,0}$ (Appendix, Part II [5]), then

$$\sigma \xrightarrow{\partial\phi} Cat^{\sigma}_{n-1} \xrightarrow{\ \sim\ } Cat_n$$

is the category in Cat_n associated to SqB.

There is a functor from Cat_n to Cat_{n+1}, called the *functor Square*, and denoted by

$$Sq_{n,n+1} : Cat_n \to Cat_{n+1},$$

which maps an n-fold functor $g: B \to B'$ onto the $(n+1)$-fold functor

$$Sqg: SqB \to SqB' : (b', \hat{b}', \hat{b}, b) \mapsto (g(b'), g(\hat{b}'), g(\hat{b}), g(b))$$

(defined by $\square g: \square B^0 \to \square B'^0$).

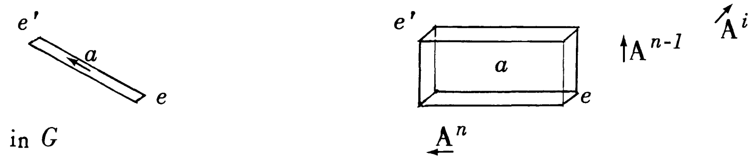
PROPOSITION 1. *The functor $Sq_{n,n+1} : Cat_n \to Cat_{n+1}$ admits a left adjoint $Lk_{n+1,n} : Cat_{n+1} \to Cat_n$.*

PROOF. The proof, quite long, will be decomposed in several steps. Let A be an $(n+1)$-fold category, α^i and β^i the maps source and target of A^i for each integer $i \leq n$.

1° *We define an n-fold category, called the multiple category of $(n-1, n)$-links of* A, denoted by LkA (later on, it will be proved that LkA is the free object generated by A with respect to the functor *Square*).

a) Consider the graph G whose vertices are those blocks e of A which are objects for the two last categories A^{n-1} and A^n, and whose edges $a: e \to e'$ from e to e' are the blocks a of A such that:

$$\alpha^n \alpha^{n-1}(a) = e \quad \text{and} \quad \beta^n \beta^{n-1}(a) = e'.$$



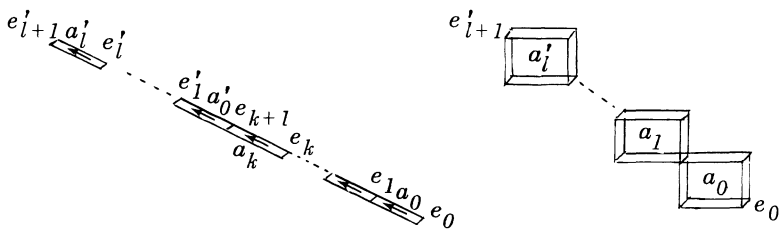
b) Let P^0 be the set $\underline{P}$ of all paths of the graph G (i. e. sequences

$$(a_k, \dots, a_0), \quad \text{where} \quad a_i : e_i \to e_{i+1} \quad \text{in } G \text{),}$$

equipped with the concatenation :

$$(a'_l, \dots, a'_0) \circ_0 (a_k, \dots, a_0) = (a'_l, \dots, a'_0, a_k, \dots, a_0)$$

$$\text{iff } \alpha^n \alpha^{n-1}(a'_0) = \beta^n \beta^{n-1}(a_k).$$

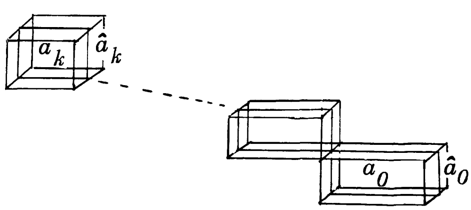


P^0 is an associative but non-unitary category (called a quasi-category in [10], where P^0 is shown to be the free quasi-category generated by G).

c) For each integer i with $0 \leqslant i < n-1$, there is a category P^{i+1} on $\underline{P}$ whose composition is deduced «pointwise» from that of A^i, which means :

$$(\hat{a}_l, \dots, \hat{a}_0) \circ_{i+1} (a_k, \dots, a_0) = (\hat{a}_k \circ_i a_k, \dots, \hat{a}_0 \circ_i a_0)$$

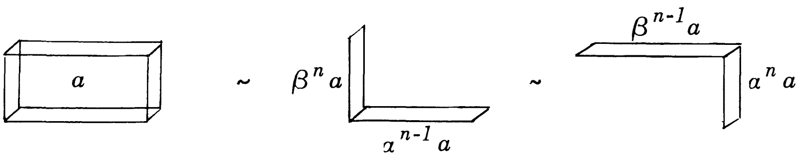
iff $l = k$ and the composites $\hat{a}_j \circ_i a_j$ are defined in A^i, for $j \leqslant k$.



REMARK. It is to be able to define P^i that we had to take all the paths of G, and not only the reduced ones (i. e., those without objects) which form the free category generated by G.

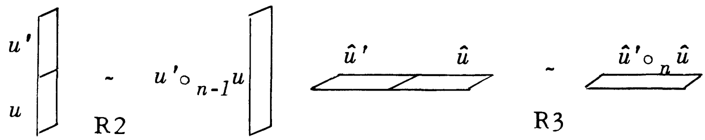
d) Consider on the set $\underline{P}$ of all the paths of G the relation r defined as follows:

(R 1) $(a) \sim (\beta^n a, \alpha^{n-1} a) \sim (\beta^{n-1} a, \alpha^n a)$ for each block a of $\mathbb{A}$.



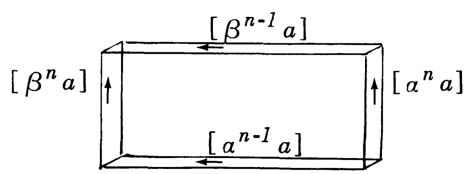
(R 2) $(u', u) \sim (u' \circ_{n-1} u)$ iff (u', u) is a couple of objects of $\mathbb{A}^n$ whose composite exists in the category $\mathbb{A}^{n-1}$.

(R 3) $(\hat{u}', \hat{u}) \sim (\hat{u}' \circ_n \hat{u})$ iff $(\hat{u}', \hat{u})$ is a couple of objects of $\mathbb{A}^{n-1}$ whose composite exists in the category $\mathbb{A}^n$.



e) According to the proof of Proposition 3 [5], there exists an n-fold category (called the *multiple category of $(n\text{-}1, n)$-links of* $\mathbb{A}$, denoted by $Lk\mathbb{A}$) quasi-quotient of $\mathrm{P} = (\mathrm{P}^0, \mathrm{P}^1, \dots, \mathrm{P}^{n-1})$ by r and such that the canonical morphism $\hat{r}: \mathrm{P} \to Lk\mathbb{A}$ defines a quasi-functor $\hat{r}: \mathrm{P}^0 \to Lk\mathbb{A}^0$ and a functor $\hat{r}: \mathrm{P}^i \to Lk\mathbb{A}^i$ for $1 \leqslant i < n$. The image $\hat{r}(a_k, \dots, a_0)$ is denoted by $[a_k, \dots, a_0]$; those blocks generate $Lk\mathbb{A}$ ($\hat{r}$ may not be onto).

2° *There is an $(n+1)$-fold functor $l: \mathbb{A} \to Sq(Lk\mathbb{A})$ which maps a block a of $\mathbb{A}$ onto the square $l(a)$ of $(Lk\mathbb{A})^0$ such that*

$l(a) =$ 

(intuitively, $l(a)$ «is» the frame of a in the double category $(\mathbb{A}^{n-1}, \mathbb{A}^n)$).

a) The map l is well-defined: The relation (R1) has been introduced so that $l(a)$ be a commutative square of $(Lk\mathbb{A})^0$, since

$$[\beta^{n-1} a] \circ_0 [\alpha^n a] = [(\beta^{n-1} a, \alpha^n a)] \underset{\mathrm{R1}}{=} [(\beta^n a, \alpha^{n-1} a)] = [\beta^n a] \circ_0 [\alpha^{n-1} a].$$

b) For $0 \leqslant i < n\text{-}1$ the map l defines a functor $l : \mathbf{A}^i \to Sq(Lk\mathbf{A})^i$: The i-th composition of $Sq(Lk\mathbf{A})$ is deduced «pointwise» from the $(i+1)$-th composition of $Lk\mathbf{A}$, which is itself deduced «pointwise» from the composition of $\mathbf{A}^i$. Suppose the composite $a' \circ_i a$ defined in $\mathbf{A}^i$; as $\alpha^n : \mathbf{A}^i \to \mathbf{A}^i$ is a functor, we have

$$[\alpha^n(a' \circ_i a)] = [(\alpha^n a') \circ_i (\alpha^n a)] = [\alpha^n a'] \circ_{i+1} [\alpha^n a] ;$$

similar equalities are valid if we replace α^n by β^n, by α^{n-1} or by β^{n-1}. Hence:

$$l(a' \circ_i a) = [\beta^n(a'\circ_i a)]\ \underset{[\alpha^{n-1}(a'\circ_i a)]}{\overset{[\beta^{n-1}(a'\circ_i a)]}{\boxed{}}}\ [\alpha^n(a'\circ_i a)] =$$

$$= [\beta^n a']\ \underset{[\alpha^{n-1} a']}{\overset{[\beta^{n-1} a']}{\boxed{}}}\ [\alpha^n a'] \ \circ_i\ [\beta^n a]\ \underset{[\alpha^{n-1} a]}{\overset{[\beta^{n-1} a]}{\boxed{}}}\ [\alpha^n a]$$

$$= \quad l(a') \circ_i l(a).$$

c) The relation (R2) implies that $l : \mathbf{A}^{n-1} \to (Sq(Lk\mathbf{A}))^{n-1}$ is a functor: By definition,

$$(Sq(Lk\mathbf{A}))^{n-1} = \boxminus(Lk\mathbf{A})^0 .$$

Suppose $a'' \circ_{n-1} a$ defined in $\mathbf{A}^{n-1}$. As $\alpha^n : \mathbf{A}^{n-1} \to \mathbf{A}^{n-1}$ is a functor,

$$[\alpha^n(a'' \circ_{n-1} a)] = [\alpha^n(a'') \circ_{n-1} \alpha^n(a)] \underset{R2}{=} [(\alpha^n a'', \alpha^n a)] =$$

$$= [\alpha^n a''] \circ_0 [\alpha^n a] ;$$

and similarly with α^n replaced by β^n. Moreover:

$$[\alpha^{n-1}(a'' \circ_{n-1} a)] = [\alpha^{n-1} a], \quad [\beta^{n-1}(a'' \circ_{n-1} a)] = [\beta^{n-1} a''] .$$

$$\begin{array}{|c|}\hline a'' \\ \hline a \\ \hline\end{array}\ \mathbf{A}^{n-1} \qquad \begin{array}{r} [\beta^n a''] \\ [\beta^n a] \end{array} \overset{[\beta^{n-1} a'']}{\begin{array}{|c|}\hline l(a'') \\ \hline l(a) \\ \hline\end{array}} \begin{array}{l} [\alpha^n a''] \\ [\alpha^n a] \end{array}$$

It follows that

$$l(a''\circ_{n-1} a) = l(a'') \boxminus l(a).$$

d) Using the relation (R 3) instead of (R 2) it is proved analogously that $l: \mathbb{A}^n \to (Sq(Lk\mathbb{A}))^n = \boxdot Lk\mathbb{A}$ is a functor.

3° *$l: \mathbb{A} \to Sq(Lk\mathbb{A})$ is the liberty morphism defining $Lk\mathbb{A}$ as the free object generated by $\mathbb{A}$ with respect to $Sq_{n,n+1}: Cat_n \to Cat_{n+1}$.*

Indeed, let $\mathbb{B}$ be an n-fold category and $g: \mathbb{A} \to Sq\mathbb{B}$ an $(n+1)$-fold functor.

a) The «diagonal map» d sending a square s of $\mathbb{B}^0$ onto its diagonal defines an $(n-1)$-fold functor

$$d: (Sq\mathbb{B})^{0,\dots,n-2} \to \mathbb{B}^{1,\dots,n-1}:$$

This map d sends the square

$$s = \quad \begin{tikzcd} \cdot & \cdot \arrow[l, "\hat{b}'"'] \\ \cdot \arrow[u, "b'"] & \cdot \arrow[u, "b"'] \arrow[l, "\hat{b}"] \arrow[ul, phantom, "="] \end{tikzcd}$$

of $\mathbb{B}^0$ onto

$$d(s) = \hat{b}'\circ_0 b = b'\circ_0 \hat{b}.$$

For each integer $i < n-1$, the composition of $(Sq\mathbb{B})^i$ is deduced pointwise from that of $\mathbb{B}^{i+1}$. As $\mathbb{B}$ is an n-fold category, the 0-th and $(i+1)$-th compositions of $\mathbb{B}$ satisfy the permutability axiom (P). Hence, if $s_1\circ_i s$ is defined in $(Sq\mathbb{B})^i$, then

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "\hat{b}'_1\circ_{i+1}\hat{b}'"'] \\ \cdot \arrow[u, "b'_1\circ_{i+1} b"] & \cdot \arrow[u, "b_1\circ_{i+1} b"'] \arrow[l, "\hat{b}_1\circ_{i+1}\hat{b}"] \end{tikzcd} \quad = s_1\circ_i s,$$

$$d(s_1\circ_i s) = (\hat{b}'_1\circ_{i+1}\hat{b}')\circ_0(b_1\circ_{i+1} b) \underset{\mathrm{P}}{=} (\hat{b}'_1\circ_0 b_1)\circ_{i+1}(\hat{b}'\circ_0 b)$$
$$= d(s_1)\circ_{i+1} d(s).$$

b) There is a unique morphism $h: \mathbb{P} \to \mathbb{B}$ extending the composite

$(n\text{-}1)$-fold functor

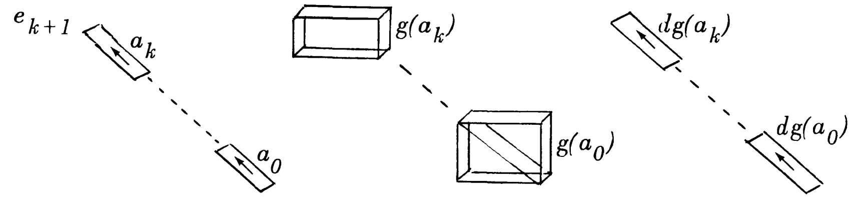
$$A^{0,\dots,n-2} \xrightarrow{g} (SqB)^{0,\dots,n-2} \xrightarrow{d} B^{1,\dots,n-1} :$$

The edge $a: e \to e'$ of the graph G is mapped by dg onto the morphism

$$dg(a): dg(e) \to dg(e') \text{ of } B^0 .$$

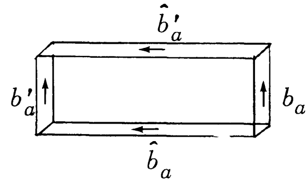
There is a unique quasi-functor $h: P^0 \to B^0$ extending dg (by the universal property of P^0) and h sends the path $p = (a_k, \dots, a_0)$ onto the composite:

$$h(p) = dg(a_k) \circ_0 \dots \circ_0 dg(a_0).$$



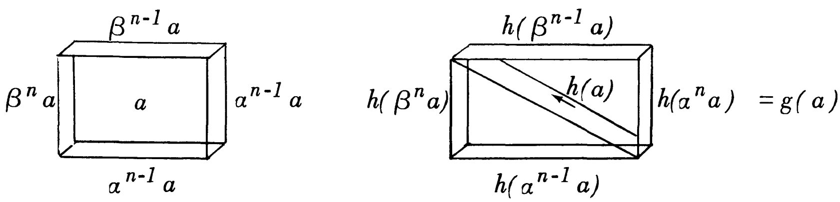
For $0 \leqslant i < n-1$, the composition of P^{i+1} is deduced pointwise from that of A^i and $dg: A^i \to B^{i+1}$ is a functor; it follows that $h: P^{i+1} \to B^{i+1}$ is a functor. Hence, $h: P \to B$ is a morphism.

c) $h: P \to B$ is compatible with the relation r used to define LkA : If a is a block of A, the square $g(a)$ of B^0 will be denoted by:

$g(a) =$ 

- As $g: A^{n-1} \to \boxminus B^0$ is a functor, $g(\alpha^{n-1} a)$ is the vertical source of the square $g(a)$, and its diagonal $h(\alpha^{n-1} a)$ is equal to $\hat{b}_a$. Similarly, $h(\beta^n a) = b'_a$, since $g: A^n \to \boxplus B^0$ is a functor. Therefore,

$$h(a) = b'_a \circ_0 \hat{b}_a = h(\beta^n a) \circ_0 h(\alpha^{n-1} a) = h(\beta^n a, \alpha^{n-1} a).$$



In an analogous way, we get

$$h(a) = \hat{b}'_a \circ_0 b_a = h(\beta^{n-1} a, \alpha^n a).$$

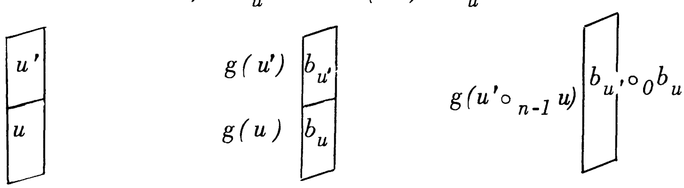
This proves that h is compatible with (R 1).

- Let the composite $u' \circ_{n-1} u$ be defined in A^{n-1}, with u and u' objects of A^n. Applying the functor $g: A^{n-1} \to \boxminus B^0$, we have

$$g(u' \circ_{n-1} u) = g(u') \boxminus g(u).$$

As $g: A^n \to \boxbar B^0$ is a functor, it maps the objects u and u' of A^n onto objects of $\boxbar B^0$ whose diagonals are

$$h(u) = b_u \quad \text{and} \quad h(u') = b_{u'}.$$



The composite $g(u' \circ_{n-1} u) = g(u') \boxminus g(u)$ is also an object of $\boxbar B^0$ whose diagonal is $b_{u'} \circ_0 b_u$. It follows that

$$h(u' \circ_{n-1} u) = d(g(u') \boxminus g(u)) = b_{u'} \circ_0 b_u = h(u') \circ_0 h(u) = h(u', u).$$

Hence h is compatible with (R 2). The compatibility with (R 3) is proved by a similar method.

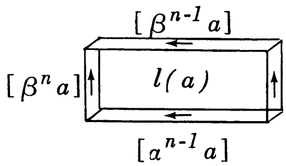
d) By the universal property of the quasi-quotient LkA of P by r, there exists a unique n-fold functor $\hat{g}: LkA \to B$ factorizing the morphism $h: P \to B$ compatible with r through the canonical morphism $\hat{r}: P \to LkA$:

$$\begin{tikzcd} & B & \\ LkA \arrow[ur, "\hat{g}"] & & P \arrow[ul, "h"'] \arrow[ll, "\hat{r}"] \end{tikzcd}$$

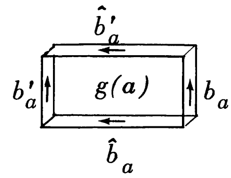
It maps the block $[a_k, \ldots, a_0]$ of LkA onto $h(a_k) \circ_0 \cdots \circ_0 h(a_0)$. In particular, for each block a of A, we have

$$\hat{g}([\alpha^n a]) = h(\alpha^n a) = b_a, \quad \hat{g}([\beta^n a]) = b'_a,$$
$$\hat{g}([\alpha^{n-1} a]) = \hat{b}_a, \quad \hat{g}([\beta^{n-1} a]) = \hat{b}'_a.$$

These equalities imply that $Sq\,\hat{g}: Sq(LkA) \to SqB$ maps



onto



Therefore

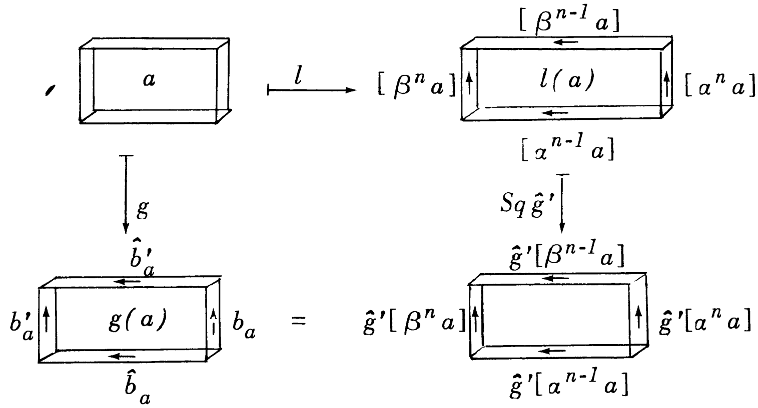
$$(\, g\colon \mathrm{A} \to Sq\mathrm{B}\,) = (\, \mathrm{A} \xrightarrow{\;l\;} Sq(Lk\mathrm{A}) \xrightarrow{\;Sq\hat{g}\;} Sq\mathrm{B}\,).$$

$$\begin{tikzcd} & \mathrm{B} \\ Lk\mathrm{A} \arrow[ur, "\hat{g}"] & \end{tikzcd} \qquad \begin{tikzcd} Sq\mathrm{B} & \\ Sq(Lk\mathrm{A}) \arrow[u, "Sq\hat{g}"] & \mathrm{A} \arrow[l, "l"] \arrow[ul, "g"'] \end{tikzcd}$$

$$Cat_n \xrightarrow[\;Sq_{n,n+1}\;]{} Cat_{n+1}$$

e) Suppose that $\hat{g}'\colon Lk\mathrm{A} \to \mathrm{B}$ is an n-fold functor such that

$$(Sq\hat{g}')(l(a)) = g(a) \quad \text{for each block } a \text{ of } \mathrm{A}.$$



In particular, this implies that $\hat{g}'(u) = b_u$ for each object u of A^n, and $\hat{g}'(\hat{u}) = \hat{b}_{\hat{u}}$ for each object $\hat{u}$ of A^{n-1}. Then:

$$\hat{g}'\hat{r}(a) \underset{\mathrm{R1}}{=} \hat{g}'[\beta^n a, \alpha^{n-1} a] = \hat{g}'[\beta^n a] \circ_0 \hat{g}'[\alpha^{n-1} a] = \hat{b}'_a \circ_0 b_a = h(a),$$

i.e., the two morphisms

$$h\colon \mathrm{P} \to \mathrm{B} \quad \text{and} \quad (\, \mathrm{P} \xrightarrow{\;\hat{r}\;} Lk\mathrm{A} \xrightarrow{\;\hat{g}'\;} \mathrm{B}\,)$$

have the same «restriction» to the graph G. By the unicity of h (see b), it follows that they are equal, and $\hat{g}\colon Lk\mathrm{A} \to \mathrm{B}$ is their unique factor through

$\hat{r}$. Hence, $\hat{g}' = \hat{g}$.

f) This proves that $Lk\mathbf{A}$ is the free object generated by $\mathbf{A}$. The corresponding left adjoint of $Sq_{n,n+1}: Cat_n \to Cat_{n+1}$, denoted by

$$Lk_{n+1,n}: Cat_{n+1} \to Cat_n ,$$

maps the $(n+1)$-fold functor $f: \mathbf{A} \to \mathbf{A}'$ onto $Lkf: Lk\mathbf{A} \to Lk\mathbf{A}'$ such that

$$(Lkf)[a_k, \dots, a_0] = [f(a_k), \dots, f(a_0)]. \qquad \nabla$$

DEFINITION. The functor $Lk_{n+1,n}: Cat_{n+1} \to Cat_n$ defined above is called the *Link functor from* Cat_{n+1} *to* Cat_n.

COROLLARY 1. *The functor* $\square: Cat \to Cat_2$ *admits as a left adjoint the Link functor from* Cat_2 *to* Cat. ∇

By iteration, for each integer m, we define the functor $Sq_{n,n+m} =$

$$(Cat_n \xrightarrow{Sq_{n,n+1}} Cat_{n+1} \;\dots\; Cat_{n+m-1} \xrightarrow{Sq_{n+m-1,\,n+m}} Cat_{n+m}).$$

COROLLARY 2. *The functor* $Sq_{n,n+m}$ *admits as a left adjoint the functor* $Lk_{n+m,n} =$

$$(Cat_{n+m} \xrightarrow{Lk_{n+m,\,n+m-1}} Cat_{n+m-1} \;\dots\; Cat_{n+1} \xrightarrow{Lk_{n+1,n}} Cat_n). \quad \nabla$$

DEFINITION. $Sq_{n,n+m}$ will be called the *Square functor, from* Cat_n *to* Cat_{n+m}, and $Lk_{n+m,n}$ the *Link functor from* Cat_{n+m} *to* Cat_n.

These functors (for $n = m$) will be used as essential tools in Section C to describe the cartesian closed structure on Cat_n.

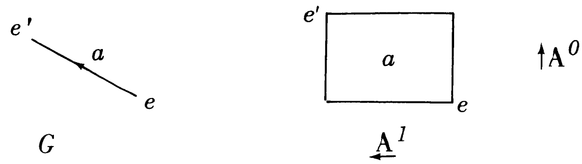
B. Some examples concerning double categories.

1° The category of links of a double category.

By Corollary 1, Proposition 1, the functor $\square: Cat \to Cat_2$ admits as a left adjoint the functor *Link* from Cat_2 to Cat. If $\mathbf{A}$ is a double category $(\mathbf{A}^0, \mathbf{A}^1)$, the category of its links $Lk\mathbf{A}$ may also be described as follows:

Let G be the graph associated to $\mathbf{A}$ in Proposition 1, whose vertices are the vertices e of $\mathbf{A}$ and whose edges $a: e \to e'$ are the blocks of A such that

$$\alpha^1 \alpha^0 a = e \quad \text{and} \quad \beta^1 \beta^0 a = e'.$$



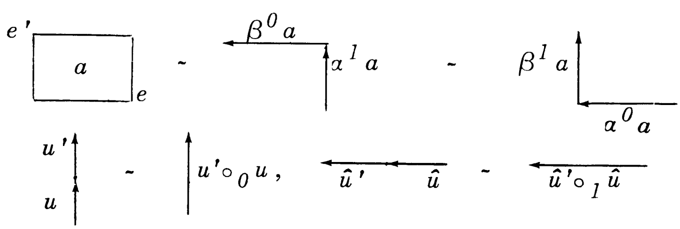
Let L be the free category generated by this graph; its objects are the vertices of A and its other morphisms are the «reduced» (i. e., with no factor a vertex) paths $(a_k, \dots, a_0)$ of G. Let R be the equivalence relation compatible with the composition of L generated by the relation r (introduced in Proposition 1):

$$(a) \sim (\beta^0 a, \alpha^1 a) \sim (\beta^1 a, \alpha^0 a),$$

for each block a of A which is not a vertex,

$$(u', u) \sim u' \circ_0 u, \quad \text{for } u' \text{ and } u \text{ objects of } A^1,$$

$$(\hat{u}', \hat{u}) \sim \hat{u}' \circ_1 \hat{u}, \quad \text{for } \hat{u}' \text{ and } \hat{u} \text{ objects of } A^0.$$

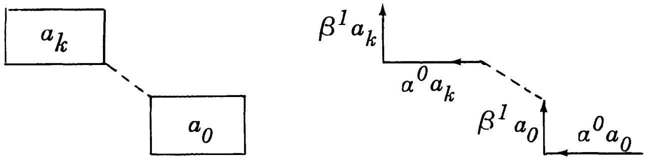


As distinct objects of L are not identified by r, and a fortiori by R, there exists a category L / R, quotient of L by R, whose morphisms are the equivalence classes modulo R, denoted by $[a_k, \dots, a_0]$. *The category* L *may be identified with* LkA.

Indeed, as we have remarked in the proof of Proposition 1, the quasi-category P^0 of all paths of G was introduced to insure that the compositions of A other than the last two ones give rise to categories P^i; here, there are only two compositions on A, so that it is equivalent to consider the «smallest» category L instead of P^0.

A morphism of L will be called a *simple path* if it is of the form $(v_l, \dots, v_0)$, where the factors v_i are objects of one and only one category

A^0 or A^1 and two successive factors are not objects of the same category. Any morphism $(a_k, \dots, a_0)$ of L is equivalent modulo R to at least one simple path. Indeed,



$$(a_k, \dots, a_0) \sim (\beta^1 a_k, \alpha^0 a_k, \dots, \beta^1 a_0, \alpha^0 a_0)$$

if this path is reduced; otherwise, there exist successive factors of this path, $(v_{j+m}, \dots, v_j)$, which are objects of the same category A^i; in this case, we replace $(v_{j+m}, \dots, v_j)$ by its composite $v_{j+m} \circ_i \dots \circ_i v_j$. The sequence thus obtained is a simple path, equivalent to $(a_k, \dots, a_0)$ modulo R. Hence the morphisms of $L \approx LkA$ are of the form $[v_l, \dots, v_0]$, where $(v_l, \dots, v_0)$ is a simple path. Remark that two different simple paths may be equivalent modulo R, as shows the example of the double category $2 \blacksquare 2$ which has only one non-degenerate block a:

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "\beta^0 a"'] \\ \cdot \arrow[u, "\beta^1 a"] \arrow[ur, phantom, "a"] & \cdot \arrow[l, "\alpha^0 a"] \arrow[u, "\alpha^1 a"'] \end{tikzcd}$$

and in which

$$(\beta^1 a, \alpha^0 a) \text{ and } (\beta^0 a, \alpha^1 a)$$

are two simple paths which are equivalent modulo R.

REMARK. With the general hypotheses of Proposition 1, to each path p of G is also associated a «simple path» defined as above (with A^0 and A^1 replaced by A^{n-1} and A^n), and which is mapped by $\hat{r}: P \to LkA$ onto the same block than p. But the compositions of LkA other than the 0-th one are not expressed easily on these simple paths.

2° Fibrations as categories of links.

Let $F: C \to Cat$ be a functor, where C is a small category (F is also called «une espèce de morphismes» [8]).

a) F *determines an action* κ' *of the category* C *on the category* S *coproduct of the categories* $F(u)$, *for all objects* u *of* C, defined by:

$$\kappa'(c,s) = F(c)(s) \quad (\text{written } cs)$$

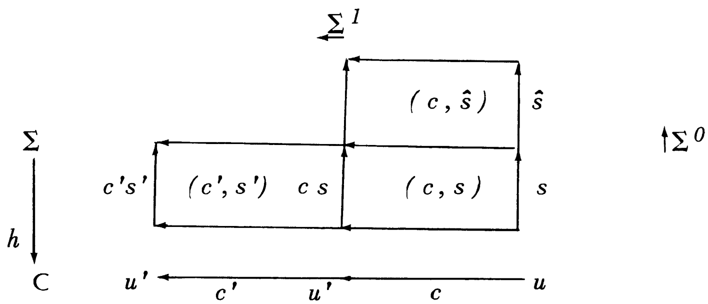
iff $c: u \to u'$ in C and s in $F(u)$.

Conversely, each action of a (small) category on a (small) category corresponds in this way to a functor toward *Cat* (see Chapter II [8]).

b) *To* F (or to the action κ' of C on S) *is also associated a double functor* $h: \Sigma \to (\underline{C}^{dis}, C)$ defined as follows:

- Let $h: \Sigma^1 \to C$ be the discrete fibration (or «foncteur d'hypermorphisme» in the terminology of [8]) associated to the action κ' of C on the set $\underline{S}$ of morphisms of S: the morphisms of Σ^1 are the couples (c, s) such that the composite $\kappa'(c,s) = cs$ is defined; the composition of Σ^1 is:

$$(c', s') \circ_1 (c, s) = (c'c, s) \quad \text{iff} \quad s' = cs.$$



The object (u, s) of Σ^1 is identified with the morphism s of S. The functor $h: \Sigma^1 \to C$ maps (c, s) onto c.

- There is another category Σ^0 with the same set $\underline{\Sigma}$ of morphisms than Σ^1, whose composition is:

$$(\hat{c}, \hat{s}) \circ_0 (c, s) = (c, \hat{s}s) \quad \text{iff} \quad c = \hat{c} \text{ and } \hat{s}s \text{ defined in S}.$$

The couple (Σ^0, Σ^1) is a double category Σ, and $h: \Sigma \to (\underline{C}^{dis}, C)$ is a double functor.

c) By the construction of b, we obtain every double functor $f: T \to K$ satisfying the two conditions:

(F 1) The 0-th category of K is discrete;

(F 2) The functor $f\colon T^1 \to K^1$ is a discrete fibration.

A double functor $f\colon T \to K$ satisfies (F 2) iff it is a *discrete fibration internal to Cat* (i.e., a realization in *Cat* of the sketch of discrete fibrations given in 0-D [4]), and then it is in 1-1 correspondence with a *category action in Cat* (in the sense of [4], page 22).

The category actions in *Cat* have been introduced in 1963 [9] under the name «catégories $\mathcal{F}$-structurées d'opérateurs» or «$\mathcal{F}$-espèces de morphismes»; in this Note, it was also indicated that the actions of a category on a category (or the functors toward *Cat*) are in 1-1 correspondence with the discrete fibrations internal to *Cat* over a double category whose 0-th category is discrete.

d) *To* F (or to the action κ' of C on S) *is also associated the (non-discrete) fibration* $h'\colon X \to C$, where X is the *crossed product category* defined as follows (see Chapter II [8]);

- The morphisms of X are the triples (s, c, e) such that e is an object of S, the composite $ce = \kappa'(c, e)$ is defined and $s\colon ce \to e'$ is a morphism of S. The composition of X is:

$$(s', c', e').(s, c, e) = (s'(c's), c'c, e) \quad \text{iff} \quad s\colon ce \to e'.$$

$$\begin{tikzcd}
{} & & & \\
{} \arrow[u, "s'"] & e' \arrow[l, "(c',e')"'] & & X \arrow[dd, "h'"] \\
{} \arrow[u, "c's"] & ce \arrow[u, "s"'] \arrow[l, "(c',ce)"] & e \arrow[l, "(c,e)"] & \\
u'' & u' \arrow[l, "c'"] & u \arrow[l, "c"] & C
\end{tikzcd}$$

- The category X is generated by the morphisms of one of the forms:
$(e', s, \hat{e})$, where $s\colon \hat{e} \to e'$ in S, identified with s,
(ce, c, e), denoted by (c, e).
The functor $h'\colon X \to C$ maps (s, c, e) onto c.

Different characterizations of X have been indicated [15,16,17], and fibrations are of a great actuality [20,2]. Another characterization of X is given now:

PROPOSITION 2. *Let* $h : \Sigma \to (\underline{C}^{dis}, C)$ *be the discrete fibration internal to Cat associated (in* b*) to the action* κ' *of* C *on the category* S. *Then* $Lk\Sigma$ *is isomorphic with the crossed product category* X.

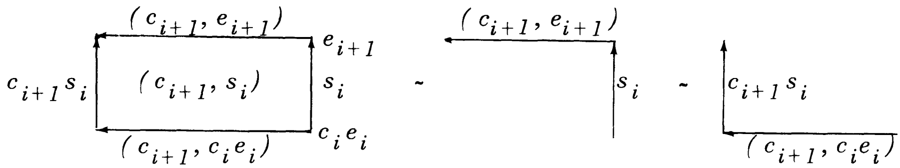
PROOF. 1° *Each morphism of the category* $Lk\Sigma$ *is of the form* $[s, (c, e)]$, *where* (s, c, e) *is a morphism of* X :

Indeed, the objects of Σ^1 are the morphisms of S, those of Σ^0 are the couples (c, e), where e is an object of S. So a simple path p is of the form

$$p = (s_k, (c_k, e_k), \dots, s_0, (c_0, e_0)),$$

where $s_i : c_i e_i \to e_{i+1}$ in S, for each $i \leqslant k$. We have

$$((c_{i+1}, e_{i+1}), s_i) \sim (c_{i+1}, s_i) \sim (c_{i+1} s_i, (c_{i+1}, c_i e_i))$$

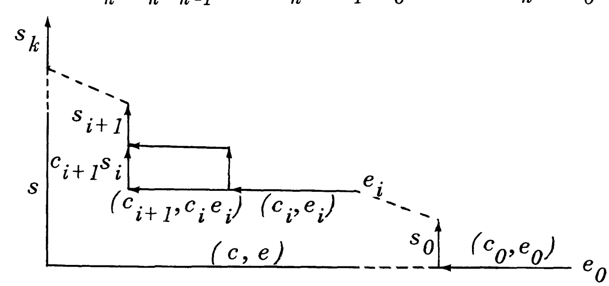


in the equivalence relation R defining $Lk\Sigma$ as a quotient of the category of paths (we use the «simplified» construction of $Lk\Sigma$ given in 1-B above). Moreover, in R, we have also :

$$(s_{i+1}, c_{i+1} s_i, (c_{i+1}, c_i e_i), (c_i, e_i)) \sim (s_{i+1}(c_{i+1} s_i), (c_{i+1} c_i, e_i)).$$

By iteration it follows that $p \sim (s, (c, e_0))$ where

$$s = s_k(c_k s_{k-1}) \dots (c_k \dots c_1) s_0, \quad c = c_k \dots c_0 .$$



Since each morphism of $Lk\Sigma$ is of the form $[p]$ for some simple path p, it is also of the form $[s, (c, e)]$, as announced.

2° There is a double functor $g : \Sigma \to \square\, X$ mapping (c, s) onto the square

$$\begin{tikzcd} {} & e' \arrow[l, "{(c,e')}"'] \\ {} \arrow[u, "cs"] & e \arrow[l, "{(c,e)}"] \arrow[u, "s"'] \end{tikzcd} \qquad \text{of } X,$$

whose diagonal $d(g(c,s))$ is (cs, c, e). Since $Lk\Sigma$ is a free object generated by Σ with respect to $\square : Cat \to Cat_2$, there corresponds to g a unique functor $\hat{g}: Lk\Sigma \to X$ which maps $[s,(c,e)]$ onto

$$d(g(s)).d(g(c,e)) = s.(c,e) = (s,c,e):$$

$$\begin{tikzcd} X \\ Lk\Sigma \arrow[u, "\hat{g}"] \end{tikzcd} \qquad \begin{tikzcd} \square X & \\ \square Lk\Sigma \arrow[u, "\square\hat{g}"] & & \Sigma \arrow[ull, "g"'] \arrow[ll, "l"] \end{tikzcd}$$

This functor is 1-1 and onto, hence it is an isomorphism, whose inverse $\hat{g}^{-1}: X \to Lk\Sigma$ maps (s,c,e) onto $[s,(c,e)]$. ∇

COROLLARY. *With the hypotheses of Proposition 2,* X *is a free object generated by* Σ *with respect to* $\square : Cat \to Cat_2$. ∇

REMARK. The category of links of $(\underline{C}^{dis}, C)$ is identified with C, so that

$$Lkh: Lk\Sigma \to Lk(\underline{C}^{dis}, C) \approx C$$

is a fibration isomorphic with $h': X \to C$. This suggests the following generalization of Chapter II [8] : Let $f: T \to K$ be any discrete fibration internal to Cat. The functor $Lkf: Lk\,T \to LkK$ «plays the role» of the fibration associated to the action of a category on a category. In particular, the equivalence classes of the sections of the functor Lkf could be called «classes of cohomology of f of order 1».

3° The multiple category of links of an $(n+1)$-category.

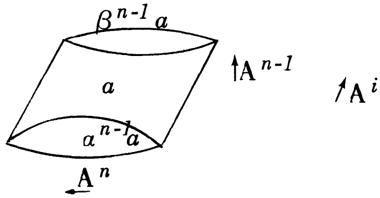
An $(n+1)$-fold category is called an *$(n+1)$-category* A if the objects of A^n are also objects of A^{n-1}. For $n = 1$, this reduces to the usual notion of a 2-category. For $n = 2$, an example of a 3-category is provided by the 3-category of cylinders of a 2-category [1].

Let A be an $(n+1)$-category. Those blocks of A which are objects for A^{n-1} define an n-fold subcategory of $A^{0,\ldots,n-2,n}$, denoted by

$$|A^{n-1}|^{0,\dots,n-2,n}, \text{ or more simply } |A^{n-1}|.$$

There exists (Proposition 3 [5]) an n-fold category quasi-quotient of $|A^{n-1}|$ by the relation:

$$\alpha^{n-1} a \sim \beta^{n-1} a \quad \text{for each block } a \text{ of } A;$$



it will be called the *n-fold category of components of* A, denoted by ΓA. The canonical n-fold functor $\tilde{\rho}: |A^{n-1}| \to \Gamma A$ may not be onto, but its image generates the n-fold category ΓA. Remark that two objects of A^{n-1} which are in the same component of A^{n-1} have the same image by $\tilde{\rho}$.

EXAMPLE. Let A be a 2-category; then $|A^0|^1 = |A^0|$ is the category of 1-morphisms of A; the equivalence relation ρ generated on it by the relation (considered above):

$$\alpha^0 a \sim \beta^0 a \quad \text{for each block (or 2-cell) } a \text{ of } A$$

is defined by:

$$v \sim v' \quad \text{iff } v \text{ and } v' \text{ are in the same component of } A^0.$$

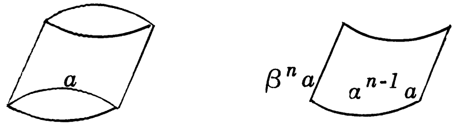
Since ρ is compatible with the composition of $|A^0|^1$, the category ΓA of components of A is then the category quotient of $|A^0|$ by ρ. So its morphisms are the components of A^0, and $\tilde{\rho}: |A^0| \to \Gamma A$ is onto. It is this example which explains the name given to ΓA.

PROPOSITION 3. *Let* A *be an* $(n+1)$*-category,* ΓA *the n-fold category of its components. Then* LkA *is isomorphic to* $(\Gamma A)^{n-1,0,\dots,n-2}$*, which is deduced from* ΓA *by a permutation of the compositions.*

PROOF. 1° *The n-fold category* LkA *is generated by those blocks* $[v]$*, where* v *is an object of* A^{n-1}: With the notations of Proposition 1, Proof, 1°, LkA is generated by the blocks $[a]$, where a is a block of A, and

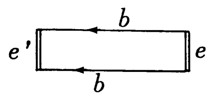
$$(a) \underset{R1}{\sim} (\beta^n a, \alpha^{n-1} a) \underset{R3}{\sim} (\beta^n a \circ_n \alpha^{n-1} a) = (\alpha^{n-1} a),$$

since $\beta^n a$ is also an object of A^{n-1} ; so $[a] = [\alpha^{n-1} a]$.

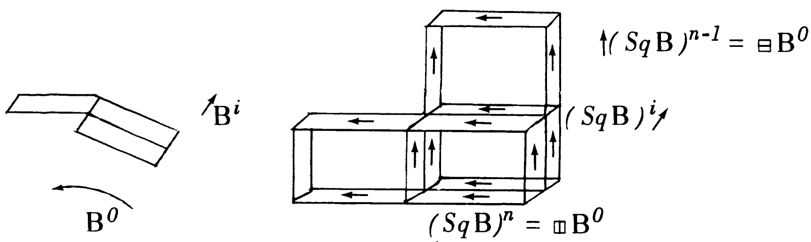


2° *There exists an n-fold functor* $\hat{g}: Lk\mathrm{A} \to (\Gamma \mathrm{A})^{n-1,0,\ldots,n-2}$ *such that* $\hat{g}[v] = \tilde{\rho}(v)$ *for each object* v *of* A^{n-1}, where $\tilde{\rho} : |\mathrm{A}^{n-1}| \to \Gamma \mathrm{A}$ is the canonical n-fold functor.

a) For each n-fold category B , the n-fold subcategory of the n-fold category $(Sq\mathrm{B})^{0,\ldots,n-2,n}$ formed by the objects of $(Sq\mathrm{B})^{n-1} = \boxminus \mathrm{B}^0$ (which are degenerate squares) is isomorphic with $\mathrm{B}^{1,\ldots,n-1,0}$, by the isomorphism mapping $b: e \to e'$ in B^0 onto the degenerate square

$b^{\boxminus} =$ 

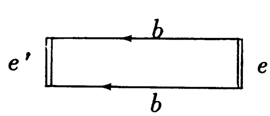
(since the composition of $(Sq\mathrm{B})^i$, for $i < n-1$,is deduced pointwise from that of B^{i+1} and $(Sq\mathrm{B})^n = \boxplus \mathrm{B}^0$).



In particular, let B be the n-fold category $(\Gamma \mathrm{A})^{n-1,0,\ldots,n-2}$; then

$$\mathrm{B}^{1,\ldots,n-1,0} = \Gamma \mathrm{A} \quad \text{and} \quad \mathrm{B}^0 = (\Gamma \mathrm{A})^{n-1},$$

so that the map

$b \mapsto b^{\boxminus} =$ 

(where $b: e \to e'$ in $(\Gamma \mathrm{A})^{n-1}$) defines an n-fold functor

$$-^{\boxminus}: \Gamma \mathrm{A} \to (Sq\mathrm{B})^{0,\ldots,n-2,n} .$$

b) There is an $(n+1)$-fold functor $g: \mathrm{A} \to Sq\mathrm{B} : a \mapsto \tilde{\rho}(\alpha^{n-1} a)^{\boxminus}$: Indeed, the composite n-fold functor

$$A^{0,\dots,n-2,n} \xrightarrow{\alpha^{n-1}} |A^{n-1}|^{0,\dots,n-2,n} \xrightarrow{\tilde{\rho}} \Gamma A \xrightarrow{-^{\boxminus}} (SqB)^{0,\dots,n-2,n}$$

is defined by the map $g\colon a \mapsto \tilde{\rho}(\alpha^{n-1}a)^{\boxminus}$. The map $\tilde{\rho}\,\alpha^{n-1}$ is constant on each component of A^{n-1} (by definition of $\tilde{\rho}$) and $-^{\boxminus}$ takes its values in the set of objects of $(SqB)^{n-1}$ $(=\boxminus(\Gamma A)^{n-1})$; whence the functor

$$g\colon A^{n-1} \to (SqB)^{n-1}.$$

c) To $g\colon A \to SqB$ is canonically associated (by the adjunction between the *Square* and *Link* functors) the n-fold functor $\hat{g}\colon LkA \to B$ which maps $[v]$ onto the diagonal $\tilde{\rho}(v)$ of $g(v) = \tilde{\rho}(v)^{\boxminus}$ for each object v of A^{n-1} (Proof, Proposition 1).

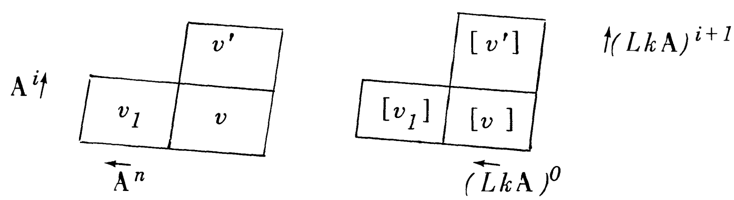
$$\begin{tikzcd} B & & SqB & \\ LkA \arrow[u, "\hat{g}"] & & Sq(LkA) \arrow[u, "Sq\hat{g}"] & & A \arrow[ull, "g"'] \arrow[ll, "l"] \end{tikzcd}$$

3° $\hat{g}\colon LkA \to B$ *is an isomorphism* and its inverse is constructed as follows, using the universal property of ΓA :

a) There is an n-fold functor

$$g'\colon |A^{n-1}|^{0,\dots,n-2,n} \to (LkA)^{1,\dots,n-1,0}\colon v \mapsto [v] :$$

the $(i+1)$-th composition of LkA being deduced pointwise from that of A^i, the map $g'\colon v \mapsto [v]$ defines a functor from the i-th category $|A^{n-1}|^i$ of $|A^{n-1}|^{0,\dots,n-2,n}$ to $(LkA)^{i+1}$ for $i \leqslant n-2$; it defines also a functor from $|A^{n-1}|^n$ to $(LkA)^0$, since



$$(v_1, v) \underset{R3}{\sim} (v_1 \circ_n v) \quad \text{for } v \text{ and } v_1 \text{ objects of } A^{n-1}$$

implies

$$g'(v_1 \circ_n v) = [v_1 \circ_n v] = [v_1] \circ_0 [v] = g'(v_1) \circ_0 g'(v).$$

b) There is an n-fold functor $\hat{g}'\colon B \to LkA$ such that $\hat{g}'\tilde{\rho}(v) = [v]$

for each object v of A^{n-1} : For each block a of A , we have

$$g'(a^{n-1}a) = [a^{n-1}a] \underset{\mathrm{R1}}{=} [\beta^{n-1}a] = g'(\beta^{n-1}a).$$

So g' is compatible with the relation by which $\Gamma\mathrm{A}$ is the quasi-quotient of $|\mathrm{A}^{n-1}|$. It follows that g' factors uniquely through $\tilde{\rho}$ into an n-fold functor

$$\begin{tikzcd} & (Lk\mathrm{A})^{1,\dots,n-1,0} & \\ \Gamma\mathrm{A} \arrow[ur, "\hat{g}'"] & & {|\mathrm{A}^{n-1}|} \arrow[ul, "g'"'] \arrow[ll, "\tilde{\rho}"] \end{tikzcd}$$

$\hat{g}'\colon \Gamma\mathrm{A} \to (Lk\mathrm{A})^{1,\dots,n-1,0}$. After permutation of the compositions, we have also the n-fold functor

$$\hat{g}'\colon \mathrm{B} = (\Gamma\mathrm{A})^{n-1,0,\dots,n-2} \to Lk\mathrm{A}\,.$$

c) $\hat{g}'$ is the inverse of $\hat{g}$: For each object v of A^{n-1},

$$\hat{g}'\hat{g}[v] = \hat{g}'\tilde{\rho}(v) = g'(v) = [v]$$

and

$$\hat{g}\hat{g}'\tilde{\rho}(v) = \hat{g}[v] = \tilde{\rho}(v).$$

As the blocks $[v]$ generate $Lk\mathrm{A}$, it follows that $\hat{g}'\hat{g}$ is an identity; similarly, $\hat{g}\hat{g}'$ is an identity, since the image of $\tilde{\rho}$ generates $\Gamma\mathrm{A}$ (and a fortiori B). Hence $\hat{g}' = \hat{g}^{-1}$. ∇

COROLLARY. *If* A *is a 2-category,* $Lk\mathrm{A}$ *is isomorphic to the category* $\Gamma\mathrm{A}$ *of components of* A .

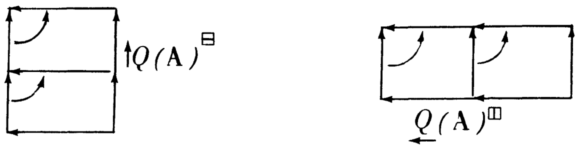
In this case, the preceding proof may be simplified: as $\Gamma\mathrm{A}$ is the quotient category of $|\mathrm{A}^0|^1$ by the relation «in the same component», it follows directly that $\hat{g}$ is 1-1 and onto (hence an isomorphism). ∇

4° The category of links of a multiple category of squares.

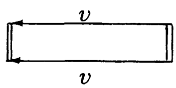
If A is a 2-category, $Q(\mathrm{A})$ denotes the double category of its up-squares (Section 2 [4])

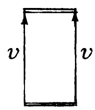
$$s = \begin{tikzcd} \cdot & \cdot \arrow[l, "v'"'] \\ \cdot \arrow[u, "u'"] \arrow[ur, phantom, "a"] & \cdot \arrow[l, "v"] \arrow[u, "u"'] \end{tikzcd} \qquad (a\colon u'\circ_1 v \to v'\circ_1 u \text{ in } \mathrm{A}^0),$$

the *0*-th and *1*-st compositions being the vertical and horizontal compositions of up-squares:



The objects of $Q(\mathrm{A})^{\boxminus}$ and $Q(\mathrm{A})^{\boxbar}$ are respectively the degenerate squares

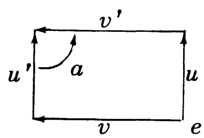
$v^{\boxminus} =$ 

and $v^{\boxbar} =$ 

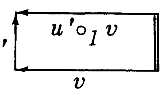
where v is a 1-morphism of A .

PROPOSITION 4. *Let* A *be a 2-category; then* $LkQ(\mathrm{A})$ *is isomorphic to the category* $\Gamma\mathrm{A}$ *of components of* A .

PROOF. 1° *Each morphism of* $LkQ(\mathrm{A})$ *is of the form* $[v^{\boxminus}]$, *where* v *is a 1-morphism of* A . Indeed, $Q(\mathrm{A})$ being a double category, $LkQ(\mathrm{A})$ may be constructed by the method of 1-B as the quotient category L/R of a category L of paths by the equivalence R defined in 1-B. For each up-square

$s =$ 

we have successively:

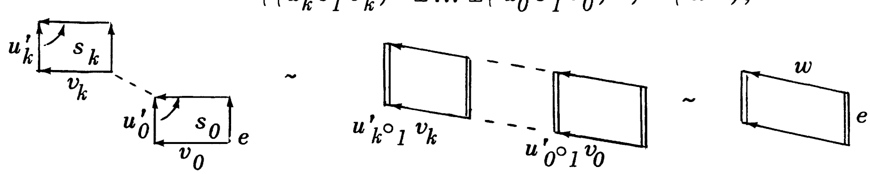
$(s) \sim (u'^{\boxbar}, v^{\boxminus}) \sim (u'$ 

$e) \sim ((u'\circ_1 v)^{\boxminus}, e^{\boxbar}) \sim ((u'\circ_1 v)^{\boxminus})$

modulo R, since $e^{\boxbar} = e^{\boxminus}$ is an object for the two categories of $Q(\mathrm{A})$. Each morphism of $LkQ(\mathrm{A}) = \mathrm{L}/R$ is of the form $[s_k, \dots, s_0]$, and

$$(s_k, \dots, s_0) \sim ((u'_k \circ_1 v_k)^{\boxminus}, \dots, (u'_0 \circ_1 v_0)^{\boxminus}) \sim$$

$$\sim ((u'_k \circ_1 v_k)^{\boxminus} \boxbar \dots \boxbar (u'_0 \circ_1 v_0)^{\boxminus}) = (w^{\boxminus}),$$



where $w = u'_k \circ_1 v_k \circ_1 \dots \circ_1 u'_0 \circ_1 v_0$; hence $[s_k, \dots, s_0] = [w^{\boxminus}]$.

2° a) There is a functor $g: Q(\mathrm{A}) \to \Box\Gamma\mathrm{A}$:

$$s = \begin{tikzcd} \cdot & \cdot \arrow[l, "v'"'] \\ \cdot \arrow[u, "u'"] \arrow[u, phantom, bend left, "a"] & \cdot \arrow[l, "v"] \arrow[u, "u"'] \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} \cdot & \cdot \arrow[l, "<v'>"'] \\ \cdot \arrow[u, "<u'>"] \arrow[ur, phantom, "="] & \cdot \arrow[l, "<v>"] \arrow[u, "<u>"'] \end{tikzcd}$$

where $<u>$ denotes the component of u in A^0 ; indeed, $u' \circ_1 v$ and $v' \circ_1 u$, being the source and target of a in A^0 , are in the same component of A^0 ,

$$\begin{tikzcd} \cdot \arrow[r, bend right, "v' \circ_1 u"'] \arrow[r, phantom, "a"] & \cdot \arrow[l, bend right, "u' \circ_1 v"'] \end{tikzcd} \quad \big\downarrow \mathrm{A}^0$$

so that, in $\Gamma\mathrm{A}$,

$$<u'><v> = <u' \circ_1 v> = <v' \circ_1 u> = <v'><u> .$$

b) To $g: Q(\mathrm{A}) \to \Box\Gamma\mathrm{A}$ corresponds (by the adjunction between the functors *Link* and $\Box$) the functor

$$\hat{g}: LkQ(\mathrm{A}) \to \Gamma\mathrm{A} : [v^{\boxminus}] \mapsto <v> .$$

$$\begin{tikzcd} \Gamma\mathrm{A} \\ LkQ(\mathrm{A}) \arrow[u, "\hat{g}"] \end{tikzcd} \qquad \begin{tikzcd} \Box\Gamma\mathrm{A} & \\ \Box LkQ(\mathrm{A}) \arrow[u, "\Box\hat{g}"] & Q(\mathrm{A}) \arrow[l, "l"] \arrow[ul, "g"'] \end{tikzcd}$$

This functor is onto, each morphism of $\Gamma\mathrm{A}$ being of the form $<v>$ for some 1-morphism v of A (by Example 3-B). It is also 1-1, since $\hat{g}[v^{\boxminus}] = \hat{g}[v'^{\boxminus}]$ means $<v> = <v'>$, which implies $(v^{\boxminus}) \sim (v'^{\boxminus})$ modulo R , hence $[v^{\boxminus}] = [v'^{\boxminus}]$. This proves that $\hat{g}: LkQ(\mathrm{A}) \to \Gamma\mathrm{A}$ is an isomorphism. ∇

COROLLARY. *If* C *is a category,* $Lk(\Box\mathrm{C})$ *is isomorphic to* C .

PROOF. $\Box\mathrm{C}$ is the double category of up-squares of the (trivial) 2-category $(\underline{\mathrm{C}}^{dis}, \mathrm{C})$, whose category of components is (identified with) C . So, the corollary is a particular case of the Proposition 3. ∇

This Corollary means that each double functor $g: \Box\mathrm{C} \to \Box\mathrm{C}'$, where C and C' are categories, is of the form $\Box f$, for a unique functor $f: \mathrm{C} \to \mathrm{C}'$. We use this result to generalize the Corollary as follows :

PROPOSITION 4. *Let* B *be an n-fold category; then* $Lk(Sq\mathrm{B})$ *is isomorphic to* B.

PROOF. It suffices to prove that B is also a free object generated by $Sq\mathrm{B}$ with respect to the functor

$$Sq_{n,n+1} : Cat_n \to Cat_{n+1},$$

the liberty morphism being $id: Sq\mathrm{B} \to Sq\mathrm{B}$. For this, let H be an n-fold category and $g: Sq\mathrm{B} \to Sq\mathrm{H}$ an $(n+1)$-fold functor.

a) As g defines a double functor

$$g: (Sq\mathrm{B})^{n-1,n} = \square\mathrm{B}^0 \to (Sq\mathrm{H})^{n-1,n} = \square\mathrm{H}^0,$$

by the Corollary there exists a unique functor $f: \mathrm{B}^0 \to \mathrm{H}^0$ such that

$$g = \square f : \square\mathrm{B}^0 \to \square\mathrm{H}^0.$$

$$\begin{tikzcd} \cdot \arrow[d, leftarrow, "\hat{b}'"'] & \cdot \arrow[l, "b'"'] \arrow[dl, phantom, "s"] \\ \cdot & \cdot \arrow[l, "b"] \arrow[u, "\hat{b}"'] \end{tikzcd} \quad \overset{g}{\longmapsto} \quad \begin{tikzcd} \cdot \arrow[d, leftarrow, "f(\hat{b}')"'] & \cdot \arrow[l, "f(b')"'] \arrow[dl, phantom, "g(s)"] \\ \cdot & \cdot \arrow[l, "f(b)"] \arrow[u, "f(\hat{b})"'] \end{tikzcd}$$

In particular, $g(b^{\boxminus}) = f(b)^{\boxminus}$ for each block b of B.

b) Let us prove that $f: \mathrm{B} \to \mathrm{H}$ is an n-fold functor. Indeed, denote by $|(Sq\mathrm{B})^{n-1}|$ the n-fold subcategory of $(Sq\mathrm{B})^{0,\dots,n-2,n}$ formed by the objects of $(Sq\mathrm{B})^{n-1} = \boxminus\mathrm{B}^0$ (i.e., formed by the degenerate squares $b^{\boxminus}$). There is an isomorphism

$$(-^{\boxminus})_{\mathrm{B}} : \mathrm{B}^{1,\dots,n-1,0} \xrightarrow{\sim} |(Sq\mathrm{B})^{n-1}| : b \mapsto b^{\boxminus}$$

(see Proof, Proposition 3). The composite functor

$$\mathrm{B}^{1,\dots,n-1,0} \xrightarrow{(-^{\boxminus})_{\mathrm{B}}} |(Sq\mathrm{B})^{n-1}| \xrightarrow{|g|} |(Sq\mathrm{H})^{n-1}| \xrightarrow{(-^{\boxminus})_{\mathrm{H}}^{-1}} \mathrm{H}^{1,\dots,n-1,0}$$

where $|g|$ is a restriction of g, maps b onto $f(b)$, since $g(b)^{\boxminus} = f(b)^{\boxminus}$. Hence it is defined by f, and this implies (after a permutation of compositions) that $f: \mathrm{B} \to \mathrm{H}$ is an n-fold functor. It is the unique n-fold functor such

$$\begin{tikzcd} \mathrm{H} \\ \mathrm{B} \arrow[u, "f"] \end{tikzcd} \qquad \begin{tikzcd} Sq\mathrm{H} & \\ Sq\mathrm{B} \arrow[u, "Sq f"] \arrow[r, equal] & Sq\mathrm{B} \arrow[ul, "g"'] \end{tikzcd}$$

that $Sqf = g: Sq\mathrm{B} \to Sq\mathrm{H}$. ∇

COROLLARY. *Whatever be the integers* n *and* m, *the Link functor from* Cat_{n+m} *to* Cat_n *is equivalent to a left inverse of* $Sq_{n,n+m}: Cat_n \to Cat_{n+m}$.

PROOF. Proposition 4 implies that the composite functor

$$Cat_n \xrightarrow{Sq_{n,n+1}} Cat_{n+1} \xrightarrow{Lk_{n+1,n}} Cat_n$$

is equivalent to the identity. By iteration, the same result is valid for the functors $Lk_{n+m,n}$ and $Sq_{n,n+m}$, due to their definition (end of Section A) as composites of functors $Lk_{p+1,p}$ and $Sq_{p,p+1}$ respectively. ∇

C. The cartesian closed structure of Cat_n.

Let n be an integer, $n > 1$. In this section we are going to show that the category Cat_n of n-fold categories is cartesian closed, by constructing the partial internal Hom functor $Hom_n(\mathrm{A}, -)$, for an n-fold category A, as the composite

$$Cat_n \xrightarrow{Sq_{n,2n}} Cat_{2n} \xrightarrow{\tilde{\gamma}} Cat_{2n} \xrightarrow{Hom(\mathrm{A},-)} Cat_n ,$$

where $Hom(\mathrm{A}, -)$ is the Hom functor associated to the partial monoidal closed structure of $\mathcal{M}\,Cat$ (defined in [5] and recalled on page 2) and where $\tilde{\gamma}$ is the isomorphism «permutation of compositions» associated to the permutation γ:

$$(0, \dots, 2n-1) \mapsto (0, 2, \dots, 2n-2, 1, 3, \dots, 2n-1)$$

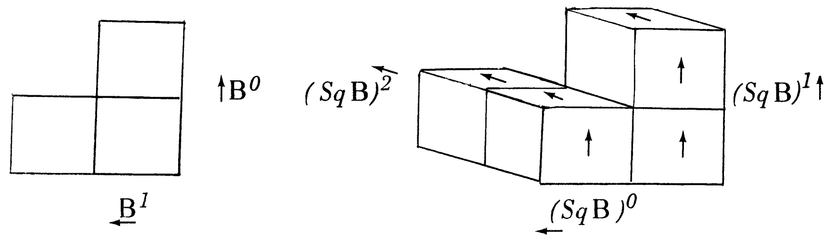
(which maps $g: \mathrm{H} \to \mathrm{K}$ onto $g: \mathrm{H}^\gamma \to \mathrm{K}^\gamma$, where

$$\mathrm{H}^\gamma = \mathrm{H}^{0,2,\dots,2n-2,1,3,\dots,2n-1}).$$

The necessity of introducing this isomorphism $\tilde{\gamma}$ is best understood on the Example here after and on the following Proposition.

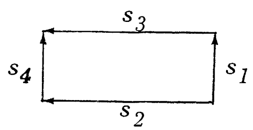
EXAMPLE: *The 4-fold category* $SqSq\mathrm{B}$, *where* B *is a double category.*

By definition, $Sq\mathrm{B}$ is the 3-fold category whose 1-st and 2-nd categories are the vertical and horizontal categories $\boxminus\mathrm{B}^0$ and $\boxplus\mathrm{B}^0$ of squares of the 0-th category B^0 of B, and whose 0-th composition is «deduced pointwise» from that of B^1.



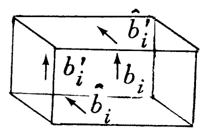
The 4-fold category $Sq_{2,4}(\mathrm{B})$ is constructed as follows:

- The set of its blocks is $\square(\underline{Sq\,\mathrm{B}})^0$, i.e., the blocks are the squares

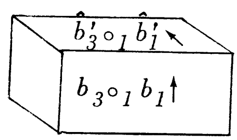


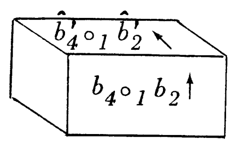
of $(Sq\,\mathrm{B})^0$,

so that

$s_i =$ 

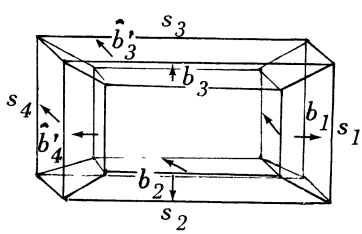
is a square of B^0 for $i = 1, 2, 3, 4$, and

$s_3 \circ_0 s_1 =$ 

$=$ 

$= s_4 \circ_0 s_2 .$

Such a block will be represented by the «frame»



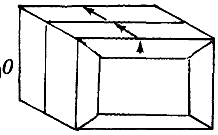
- The 0-th and 1-st compositions are deduced «pointwise» from that of

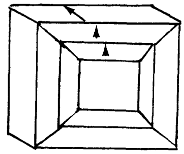
$$(Sq\,\mathrm{B})^1 = \boxminus\,\mathrm{B}^0 \quad \text{and} \quad (Sq\,\mathrm{B})^2 = \boxbar\,\mathrm{B}^0 ,$$

so that they consist in putting «one frame behind the other» and «one frame inside the other».

- The 2-nd and 3-rd compositions are the vertical and horizontal compositions of squares of $(Sq\mathrm{B})^0$ (whose composition is deduced from that of B^1) so that they consist in putting «one frame above the other» and «one frame beside the other» (the common border being «erased»).

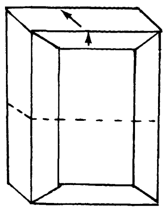
$(Sq\,Sq\mathrm{B})^0$

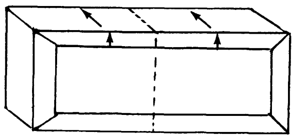




$(Sq\,Sq\mathrm{B})^1$

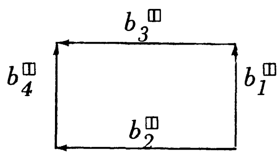
$(Sq\,Sq\mathrm{B})^2$



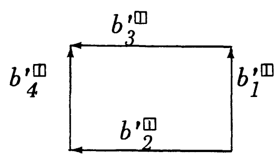


$(Sq\,Sq\mathrm{B})^3$

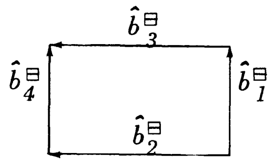
- The sources and targets of (s_4, s_3, s_2, s_1) are respectively the degenerate frames:



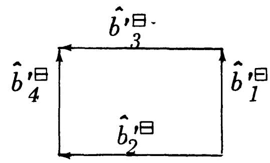
and



for the 0-th category,



and



for the 1-st category,

$s_2^{\boxminus}$ and $s_3^{\boxminus}$ for the 2-nd category,

$s_1^{◫}$ and $s_4^{◫}$ for the 3-rd category.

Hence, the two first compositions are deduced from that of B^0, the two last ones being deduced from that of B^1.

More generally, if we consider $Sq_{n,2n}(\mathrm{B})$ for an n-fold category B, its $(2i)$-th and $(2i+1)$-th compositions are deduced from that of B^i, for each $i < n$. Therefore, $Sq_{n,2n}(\mathrm{B})^\gamma$ has its compositions deduced respectively of that of $\mathrm{B}^0, \dots, \mathrm{B}^{n-1}, \mathrm{B}^0, \dots, \mathrm{B}^{n-1}$.

The following proposition will be an essential tool to describe the cartesian closed structure of Cat_n.

PROPOSITION 6. *Let* A *and* B *be two n-fold categories; then the n-fold category product* $B \times A$ *is isomorphic to* $Lk((B \blacksquare A)^{\gamma^{-1}})$, *where* $B \blacksquare A$ *is the square product and* γ^{-1} *the permutation*

$$(0, \ldots, 2n-1) \mapsto (0, n, \ldots, i, n+i, \ldots, n-1, 2n-1).$$

PROOF. Remark firstly that γ^{-1} is the permutation inverse of the permutation (considered above) γ:

$$(0, \ldots, 2n-1) \mapsto (0, 2, \ldots, 2n-2, 1, 3, \ldots, 2n-1).$$

We denote by H the $(2n)$-fold category $(B \blacksquare A)^{\gamma^{-1}}$, so that:

$$H^{2i} = \underline{B}^{dis} \times A^i \quad \text{and} \quad H^{2i+1} = \underline{B}^i \times \underline{A}^{dis}, \quad \text{for each } i < n.$$

1° LkH *is isomorphic to the* $(2n-1)$*-fold category* K *on* $\underline{B} \times \underline{A}$ *such that*

$$K^0 = B^{n-1} \times A^{n-1} \quad and \quad K^{j+1} = H^j \quad for \ 0 \leqslant j < 2n-2.$$

(hence $K = (B^{n-1} \times A^{n-1}, \underline{B}^{dis} \times A^0, B^0 \times \underline{A}^{dis}, \ldots, \underline{B}^{dis} \times A^{n-2}, B^{n-2} \times \underline{A}^{dis}))$.

a) There exists a $(2n)$-fold functor $g: H \to SqK$:

$$(b, a) \mapsto \begin{array}{ccc} & (b, \beta_A^{n-1} a) & \\ (\beta_B^{n-1} b, a) & \boxed{} & (\alpha_B^{n-1} b, a) \\ & (b, \alpha_A^{n-1} a) & \end{array}$$

where α_A^{n-1} and β_A^{n-1} denote the source and target maps of A^{n-1}:

(i) $g(b, a)$ is a square of $K^0 = B^{n-1} \times A^{n-1}$, for any blocks a of A and b of B.

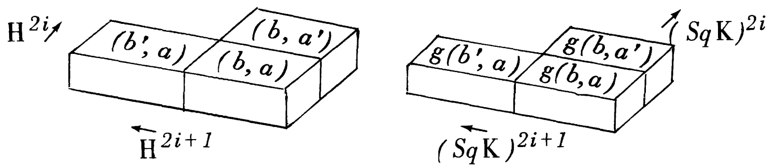
(ii) For $0 \leqslant j < 2n-2$, the j-th composition of SqK is deduced pointwise from that of $K^{j+1} = H^j$; to prove that $g: H^j \to (SqK)^j$ is a functor, it suffices to show that the four maps:

$$\beta_B^{n-1} \times id_A: (b, a) \mapsto (\alpha_B^{n-1} b, a), \quad \beta_B^{n-1} \times id_A: (b, a) \mapsto (\beta_B^{n-1} b, a),$$

$$id_B \times \alpha_A^{n-1}: (b, a) \mapsto (b, \alpha_A^{n-1} a), \quad id_B \times \beta_A^{n-1}: (b, a) \mapsto (b, \beta_A^{n-1} a)$$

define functors from H^j to H^j. This comes from the following facts:

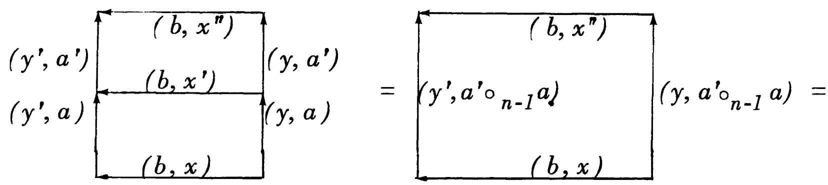
- $H^j = \underline{B}^{dis} \times A^i$ if $j = 2i$ and $H^j = B^i \times \underline{A}^{dis}$ if $j = 2i+1$,
- α_A^{n-1} and β_A^{n-1} define functors $A^j \to A^j$ and $\underline{A}^{dis} \to \underline{A}^{dis}$,
- α_B^{n-1} and β_B^{n-1} define functors $B^j \to B^j$ and $\underline{B}^{dis} \times \underline{B}^{dis}$.



(iii) $g: H^{2n-2} = \underline{B}^{dis} \times A^{n-1} \to (SqK)^{2n-2} = \boxminus K^0$ is a functor. Indeed, if $a: x \to x'$ and $a': x' \to x''$ in A^{n-1} and $b: y \to y'$ in B^{n-1}, then

$$(b, a') \circ_{2n-2} (b, a) = (b, a' \circ_{n-1} a)$$

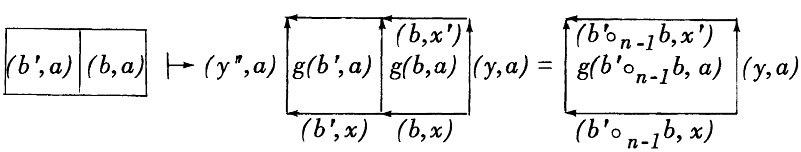
and $g(b, a') \boxminus g(b, a) =$



$$= g(b, a' \circ_{n-1} a).$$

(iv) A similar method gives the functor

$$g: H^{2n-1} = B^{n-2} \times \underline{A}^{dis} \to (SqK)^{2n-1} = \boxbar K^0$$



($b': y' \to y''$ in B^{n-1}).

b) To $g: H \to SqK$ is canonically associated (by the adjunction between the *Link* and *Square* functors) a $(2n-1)$-fold functor $\hat{g}: LkH \to K$ such that $\hat{g}[b, a] = (b, a)$ for any (b, a) in $\underline{B} \times \underline{A}$, since (b, a) is the diagonal of the square $g(b, a)$ of $K^0 = B^{n-1} \times A^{n-1}$ (Proof, Proposition 1).

c) There exists a $(2n-1)$-fold functor

$$\begin{tikzcd} \mathrm{K} \\ Lk\mathrm{H} \arrow[u, "\hat{g}"] \end{tikzcd} \qquad \begin{tikzcd} Sq\mathrm{K} & \\ Sq(Lk\mathrm{H}) \arrow[u, "Sq\hat{g}"] & \mathrm{H} \arrow[ul, "g"'] \arrow[l, "l"] \end{tikzcd}$$

$$\hat{g}': \mathrm{K} \to Lk\mathrm{H} : (b, a) \mapsto [b, a].$$

(i) For $0 \leqslant j < 2n-2$, since $\mathrm{K}^{j+1} = \mathrm{H}^j$ and the composition of $(Lk\mathrm{H})^{j+1}$ is deduced pointwise from that of H^j, it follows that $\hat{g}': \mathrm{K}^{j+1} \to (Lk\mathrm{H})^{j+1}$ is a functor.

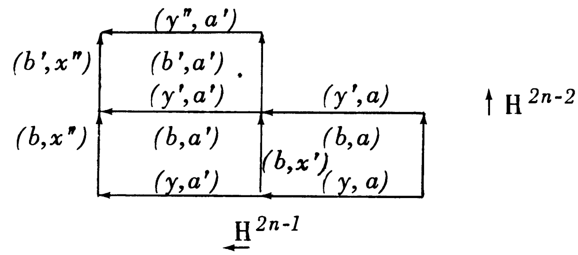
(ii) It remains to prove that $\hat{g}': \mathrm{K}^0 \to (Lk\mathrm{H})^0$ is a functor. For this, let the composite

$$(b', a') \circ_0 (b, a) = (b' \circ_{n-1} b,\ a' \circ_{n-1} a)$$

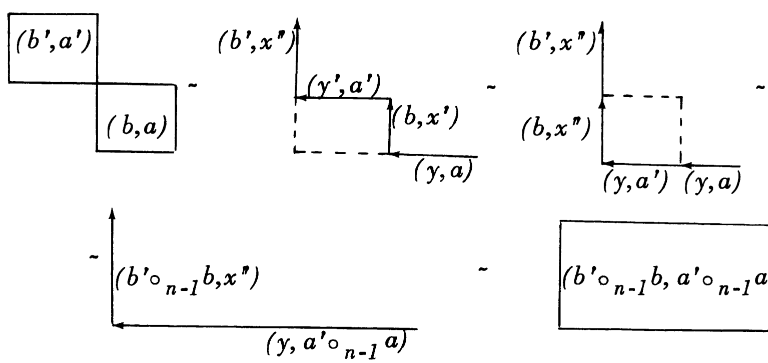
be defined in $\mathrm{K}^0 = \mathrm{B}^{n-1} \times \mathrm{A}^{n-1}$, so that

$a: x \to x'$ and $a': x' \to x''$ in A^{n-1}, $b: y \to y'$ and $b': y' \to y''$ in B^{n-1}.

Since $\mathrm{H}^{2n-2} = \underline{\mathrm{B}}^{dis} \times \mathrm{A}^{n-1}$ and $\mathrm{H}^{2n-1} = \mathrm{B}^{n-1} \times \underline{\mathrm{A}}^{dis}$, in the relation on paths



used to define $Lk\mathrm{H}$ (Proof, Proposition 1), we have successively



This implies

$$\hat{g}'(b', a') \circ_0 \hat{g}'(b, a) = [b', a'] \circ_0 [b, a] = [(b', a'), (b, a)] =$$

$$= [\, b' \circ_{n-1} b,\ a' \circ_{n-1} a] = \hat{g}'((b', a') \circ_0 (b, a)).$$

Hence $\hat{g}' : \mathrm{K}^0 \to (Lk\mathrm{H})^0$ is also a functor.

d) $\hat{g}'$ is the inverse of $\hat{g} : Lk\mathrm{H} \to \mathrm{K}$. Indeed,

$$\hat{g}\hat{g}'(b, a) = \hat{g}\ [\, b, a] = (b, a)$$

for each block (b, a) of K, so that $\hat{g}\hat{g}'$ is an identity. On the other hand, the equalities

$$\hat{g}'\hat{g}\,[\, b, a] = \hat{g}'(b, a) = [\, b, a]$$

imply that $\hat{g}'\hat{g}$ is also an identity, the blocks $[\, b, a]$ generating (by definition) the $(2n-1)$-fold category $Lk\mathrm{H}$. So $\hat{g} : Lk\mathrm{H} \to \mathrm{K}$ is an isomorphism.

2° Let us suppose proven that $Lk_{2n,2n-m}\mathrm{H}$, for $1 \leqslant m \leqslant n-1$, is isomorphic to the $(2n-m)$-fold category K_m such that

$(\mathrm{B}^{n-m} \times \mathrm{A}^{n-m}, \dots, \mathrm{B}^{n-1} \times \mathrm{A}^{n-1}, \underline{\mathrm{B}}^{dis} \times \mathrm{A}^0, \dots, \underline{\mathrm{B}}^{dis} \times \mathrm{A}^{n-m-1}, \mathrm{B}^{n-m-1} \times \underline{\mathrm{A}}^{dis})$.

Then a proof similar to the preceding one proves that $Lk\mathrm{K}_m$, and a fortiori

$$Lk(Lk_{2n,\,2n-m}\mathrm{H}) = Lk_{2n,\,2n-m-1}\mathrm{H}$$

is isomorphic to the $(2n-m-1)$-fold category K_{m+1}. By induction, it follows that $Lk_{2n,n}\mathrm{H}$ is isomorphic to

$$\mathrm{B} \times \mathrm{A} = (\mathrm{B}^0 \times \mathrm{A}^0, \dots, \mathrm{B}^{n-1} \times \mathrm{A}^{n-1}). \qquad \nabla$$

COROLLARY. *For each n-fold category* A, *the «partial» product functor* $- \times \mathrm{A} : Cat_n \to Cat_n$ *is equivalent to the composite functor*

$$Cat_n \xrightarrow{- \,\blacksquare\, \mathrm{A}} Cat_{2n} \xrightarrow{\tilde{\gamma}^{-1}} Cat_{2n} \xrightarrow{Lk_{2n,n}} Cat_n. \qquad \nabla$$

DEFINITION. The composite functor

$$Cat_n \xrightarrow{Sq_{n,2n}} Cat_{2n} \xrightarrow{\tilde{\gamma}} Cat_{2n}$$

will be called the *n-square functor*, denoted by $\square_n : Cat_n \to Cat_{2n}$.

PROPOSITION 7. Cat_n *is a cartesian closed category whose internal Hom functor* $Hom_n : (Cat_n)^{op} \times Cat_n \to Cat_n$ *is such that, for any n-fold category* A, *the partial functor* $Hom_n(\mathrm{A}, -)$ *is equal to the composite:*

$$Cat_n \xrightarrow{\square_n} Cat_{2n} \xrightarrow{Hom(\mathrm{A},\, -)} Cat_n.$$

PROOF. 1° Since Cat_n admits (finite) products, to prove that it is cartesian closed it suffices to show that the partial product functor $- \times \mathbb{A} : Cat_n \to Cat_n$ admits a right adjoint [13]. By the Corollary of Proposition 6, this functor is equivalent to the composite of three functors:

$-\blacksquare \mathbb{A} : Cat_n \to Cat_{2n}$ who has a right adjoint $Hom(\mathbb{A}, -)$ (due to the partial monoidal closed structure of $M\,Cat$, Proposition 7 [5]),

$\tilde{\gamma}^{-1} : Cat_{2n} \to Cat_{2n}$ whose inverse $\tilde{\gamma}$ is a right adjoint,

$Lk_{2n,n} : Cat_{2n} \to Cat_n$ who admits $Sq_{n,2n}$ as a right adjoint.

By transitivity of adjunctions, this implies that $- \times \mathbb{A}$ admits as a right adjoint the composite $Hom_n(\mathbb{A}, -) =$

$$\begin{tikzcd} Cat_n \arrow[r, "Sq_{n,2n}"] \arrow[rr, bend right, "\square_n"'] & Cat_{2n} \arrow[r, "\tilde{\gamma}"] & Cat_{2n} \arrow[r, "Hom(\mathbb{A}, -)"] & Cat_n . \end{tikzcd}$$

2° The corresponding internal Hom functor (or closure functor)

$$Hom_n : (Cat_n)^{op} \times Cat_n \to Cat_n$$

maps the couple of n-fold functors $(f: \mathbb{A}' \to \mathbb{A},\ g: \mathbb{B} \to \mathbb{B}')$ onto the n-fold functor

$$Hom_n(f, g) : Hom_n(\mathbb{A}, \mathbb{B}) = Hom(\mathbb{A}, \square_n \mathbb{B}) \to Hom_n(\mathbb{A}', \mathbb{B}')$$

mapping $h : \mathbb{A} \to \square_n \mathbb{B}$ onto

$$\mathbb{A}' \xrightarrow{f} \mathbb{A} \xrightarrow{h} \square_n \mathbb{B} \xrightarrow{\square_n g} \square_n \mathbb{B}'.$$

3° Let us describe more explicitely the adjunction between $- \times \mathbb{A}$ and $Hom_n(\mathbb{A}, -) : Cat_n \to Cat_n$. Let $\mathbb{B}$ be an n-fold category.

a) There is a map $\partial : \underline{\square_n \mathbb{B}} \to \underline{\mathbb{B}}$ (it is not a multiple functor, but a map between the sets of blocks) which maps an n-square of $\mathbb{B}$ onto « its diagonal » defined as follows: For each $i < n$, there is the diagonal map

$$d_i : \underline{Sq_{n,n+i+1} \mathbb{B}} = \underline{Sq(Sq_{n,n+i} \mathbb{B})} \to \underline{Sq_{n,n+i} \mathbb{B}}$$

which maps the square

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "\hat{s}'"'] \\ \cdot \arrow[u, "s'"] & \cdot \arrow[l, "\hat{s}"] \arrow[u, "s"'] \end{tikzcd} \quad \text{of} \quad (Sq_{n,n+i} \mathbb{B})^0$$

onto its diagonal $\hat{s}' \circ_0 s = s' \circ_0 \hat{s}$. Then ∂ is the composite map $d_0 \ldots d_{n-1}$:

$$\underline{\Box_n \mathrm{B}} = \underline{Sq_{n,2n}\mathrm{B}} \xrightarrow[d_{n-1}]{} \underline{Sq_{n,2n-1}\mathrm{B}} \to \dots \to \underline{Sq\mathrm{B}} \xrightarrow[d_0]{} \underline{\mathrm{B}}\,.$$

b) The 1-1 correspondence due to the adjunction between $-\times \mathrm{A}$ and $Hom_n(\mathrm{A},-)$ maps the n-fold functor $h: \mathrm{A}' \to Hom_n(\mathrm{A},\mathrm{B})$ onto the n-fold functor

$$\hat{h}: \mathrm{A}'\times \mathrm{A} \to \mathrm{B}: (a', a) \mapsto \partial(h(a')(a)).$$

Indeed, the adjunction between $Hom(\mathrm{A},-)$ and $-\blacksquare\, \mathrm{A}$ associates to h the n-fold functor

$$h_0^{\gamma}: \mathrm{A}' \blacksquare\, \mathrm{A} \to \Box_n \mathrm{B}: (a', a) \mapsto h(a')(a),$$

and therefore the n-fold functor

$$h_0: (\mathrm{A}' \blacksquare\, \mathrm{A})^{\gamma^{-1}} \to (\Box_n \mathrm{B})^{\gamma^{-1}} = Sq_{n,2n}\mathrm{B}\ ;$$

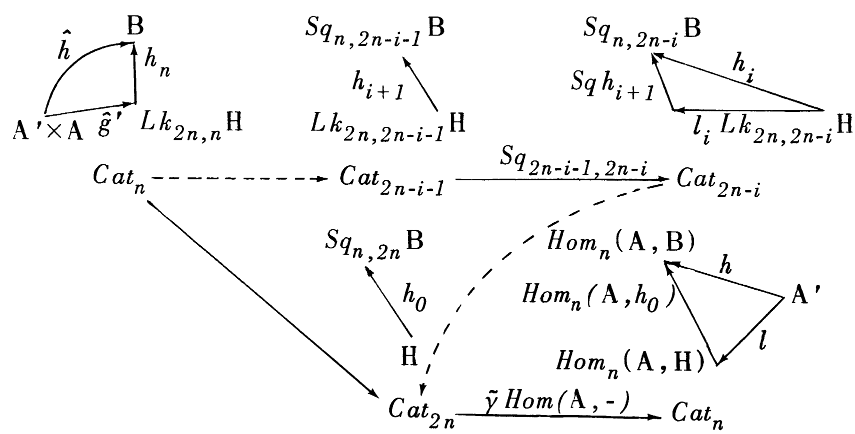
we write H instead of $(\mathrm{A}' \blacksquare\, \mathrm{A})^{\gamma^{-1}}$. By induction, we define

$$h_{i+1}: Lk_{2n,2n-i-1}\mathrm{H} = Lk(Lk_{2n,2n-i}\mathrm{H}) \to Sq_{n,2n-i-1}\mathrm{B}\,,$$

for each $i < n$, as the $(2n-i-1)$-fold functor associated (by the adjunction between $Lk_{2n-i,2n-i-1}$ and $Sq_{2n-i-1,2n-i}: Cat_{2n-i-1} \to Cat_{2n-i}$) to

$$h_i: Lk_{2n,2n-i}\mathrm{H} \to Sq_{n,2n-i}\mathrm{B} = Sq(Sq_{n,2n-i-1}\mathrm{B})\ ;$$

by construction, h_{i+1} maps a block of $Lk_{2n,2n-i-1}\mathrm{H}$ of the form $[a', a]$ (see Proof, Proposition 6) onto the diagonal $d_{n-i-1}h_i[a', a]$ of the square



$h_i[a', a]$ of $(Sq_{n,2n-i-1}\mathrm{B})^0$. It follows that to h is associated

$$\hat{h} = (\mathrm{A}'\times\mathrm{A} \xrightarrow[\hat{g}']{} Lk_{2n,n}\mathrm{H} \xrightarrow[h_n]{} \mathrm{B}),$$

where $\hat{g}'$ is the canonical isomorphism $(a', a) \mapsto [a', a]$ (see Proof, Proposition 6); $\hat{h}$ maps (a', a) onto

$$d_0 \dots d_{n-1} h(a')(a) = \partial h(a')(a).$$

c) The coliberty morphism defining $Hom_n(\mathrm{A}, \mathrm{B})$ as a cofree object generated by B is the «evaluation»:

$$ev: Hom_n(\mathrm{A}, \mathrm{B}) \times \mathrm{A} \to \mathrm{B}: (f, a) \mapsto \partial f(a),$$

since it corresponds to the identity of $Hom_n(\mathrm{A}, \mathrm{B})$. In particular, if A is

$$\begin{tikzcd} & Hom_n(\mathrm{A},\mathrm{B}) \\ \mathrm{A}' \arrow[ur, "h"] \end{tikzcd} \qquad \begin{tikzcd} Hom_n(\mathrm{A},\mathrm{B}) \times \mathrm{A} \arrow[drr, "ev"] & & \\ \mathrm{A}'\times\mathrm{A} \arrow[u, "h \times id"] \arrow[rr, "\hat{h}"'] & & \mathrm{B} \end{tikzcd}$$

the n-fold category $2^{\blacksquare^n}$ (see [5]), with only one non-degenerate block z, then

$$Hom_n(2^{\blacksquare^n}, \mathrm{B}) = Hom(2^{\blacksquare^n}, \square_n \mathrm{B})$$

is identified with $(\square_n \mathrm{B})^{n,\dots,2n-1}$, and the evaluation becomes the n-fold functor $ev: (\square_n \mathrm{B})^{n,\dots,2n-1} \times 2^{\blacksquare^n} \to \mathrm{B}$ such that the map

$$ev(-, z): \underline{\square_n \mathrm{B}} \to \underline{\mathrm{B}}: s \mapsto \partial s$$

is the diagonal map ∂ defined in a. ∇

COROLLARY 1. *The vertices of* $Hom_n(\mathrm{A}, \mathrm{B})$ *are identified with the n-fold functors from* A *to* B.

PROOF. The final object 1_n of Cat_n is the unique n-fold category on the set $1 = \{0\}$. The vertices of $Hom_n(\mathrm{A}, \mathrm{B})$ are identified [5] with the n-fold functors $1_n \to Hom_n(\mathrm{A}, \mathrm{B})$, which are in 1-1 correspondence (by adjunction) with the n-fold functors from $1_n \times \mathrm{A} \approx \mathrm{A}$ to B. To $f: \mathrm{A} \to \mathrm{B}$ corresponds the vertex of $Hom_n(\mathrm{A}, \mathrm{B})$ mapping a onto the degenerate n-square (vertex of $\square_n \mathrm{B}$) determined by $f(a)$. ∇

COROLLARY 2. *There is a canonical isomorphism*

$$\lambda: Hom_n(\mathrm{A}', Hom_n(\mathrm{A}, \mathrm{B})) \xrightarrow{\sim} Hom_n(\mathrm{A}'\times\mathrm{A}, \mathrm{B})$$

extending the 1-1 correspondence (Proof above):

$$(h: A' \to Hom_n(A,B)) \mapsto (\hat{h}: A' \times A \to B: (a',a) \mapsto \partial h(a')(a)).$$

PROOF. It is a general result on cartesian (as well as monoidal) closed categories [13] ; it means that $Hom_n(A,-): Cat_n \to Cat_n$ is a Cat_n-right adjoint of $- \times A$. ∇

COROLLARY 3. *There is a canonical n-fold « composition » functor*

$$\kappa_{A,B,B'}: Hom_n(A,B) \times Hom_n(B,B') \to Hom_n(A,B'):$$
$$(f,f') \mapsto (f'': A \to \square_n B') \text{ with } \partial_{B'} f'' = \partial_{B'} f' \partial_B f: \underline{A} \to \underline{B}.$$

PROOF. This is also a general result on cartesian closed categories ; in fact, $\kappa_{A,B,B'}$ corresponds to the composite n-fold functor:

$$\begin{tikzcd}
(Hom_n(A,B) \times Hom_n(B,B')) \times A \arrow[r, "\cong"] & Hom_n(B,B') \times (Hom_n(A,B) \times A) \arrow[d, "id \times ev_{A,B}"] \\
& Hom_n(B,B') \times B \arrow[r, "ev_{B,B'}"'] & B'
\end{tikzcd}$$

mapping (f,f',a) onto $\partial_{B'} f'(\partial f(a))$. ∇

This Corollary 3 means that Cat_n is a Cat_n-category (i.e., a category enriched in the cartesian closed category Cat_n) and it will be used in Proposition 8.

REMARK. The *existence* of a cartesian closed structure on Cat_n may also be deduced, by induction, from Corollary 3, Proposition 23 [7] , as follows: since Cat is cartesian closed, the sketch σ of categories is cartesian [7] ; so, if Cat_i is cartesian closed, the category Cat_i^σ of categories in Cat_i is cartesian closed by this Corollary, as well as the equivalent category Cat_{i+1} (see Appendix [5]). However the *explicit construction of* Hom_n cannot be deduced from this (or from another) existence result.

EXAMPLE. *The cartesian closed category* Cat_2 :

Let A and B be double categories. Then $\square_2 B$ is the 4-fold category deduced from $SqSqB$ (described in the Example above) by permutation of the 1-st and 2-nd compositions. Hence, $Hom_2(A,B)$ is constructed as follows:

- Its blocks are the double functors from A to the double category

$(SqSq\mathrm{B})^{0,2}$ of «frames» whose compositions are «one frame behind the the other» and «one frame above the other».

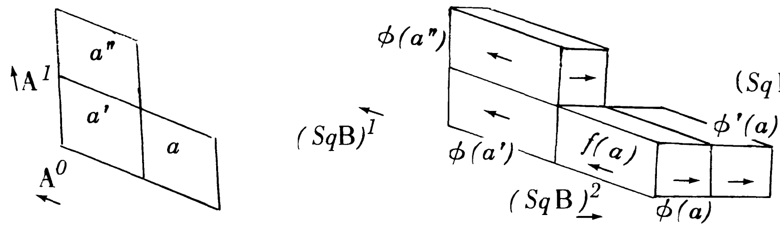
- Its compositions are deduced pointwise from the compositions «one frame inside the other» and «one frame beside the other».

- Its vertices «are» the double functors $f: \mathrm{A} \to \mathrm{B}$.

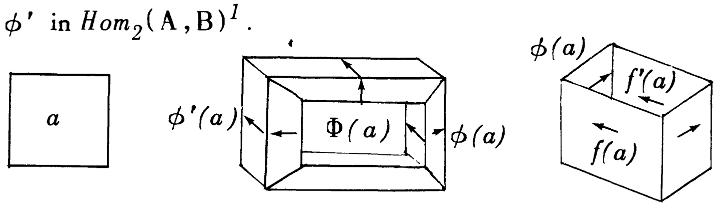
- The 3-fold subcategory $|(SqSq\mathrm{B})^3|^{0,2,1}$ of $(SqSq\mathrm{B})^{0,2,1}$ formed by the objects of $(SqSq\mathrm{B})^3$ is identified with $(Sq\mathrm{B})^{1,0,2}$ by the isomorphism

$$-^{\boxplus}: (Sq\mathrm{B})^{1,0,2} \xrightarrow{\sim} |(SqSq\mathrm{B})^3| : s \mapsto s^{\boxplus}.$$

Then an object of $Hom_2(\mathrm{A},\mathrm{B})^1$ (which is a double functor $\mathrm{A} \to (SqSq\mathrm{B})^{0,2}$ taking its values in $|(SqSq\mathrm{B})^3|$) will be identified with a double functor $\phi: \mathrm{A} \to (Sq\mathrm{B})^{1,0}$, and the subcategory of $Hom_2(\mathrm{A},\mathrm{B})^0$ formed by these objects «is» $Hom(\mathrm{A},(Sq\mathrm{B})^{1,0,2})$. The objects of this last category are



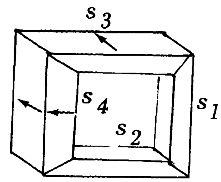
themselves identified with the double functors $f: \mathrm{A} \to \mathrm{B}$. With the terminology of [7], a double functor $\phi: \mathrm{A} \to (Sq\mathrm{B})^{1,0}$ is called a *double natural transformation* (i.e., a natural transformation internal to *Cat*) from f to f', if $\phi: f \to f'$ in $Hom(\mathrm{A},(Sq\mathrm{B})^{1,0,2})$. This may suggest to call the block $\Phi: \mathrm{A} \to \square_2\mathrm{B}$ of $Hom_2(\mathrm{A},\mathrm{B})$ *a hypertransformation from* ϕ *to* ϕ' where $\Phi: \phi \to \phi'$ in $Hom_2(\mathrm{A},\mathrm{B})^1$.



- If $h: \mathrm{A}' \to Hom_2(\mathrm{A},\mathrm{B})$ is a double functor, the double functor canonically associated (by adjunction) $\hat{h}: \mathrm{A}' \times \mathrm{A} \to \mathrm{B}$ maps (a', a) onto the diagonal of the frame $h(a')(a)$, which is equal to

$$(b'_4 \circ_0 \hat{b}_4) \circ_1 (b'_2 \circ_0 \hat{b}_2)$$

if

$h(a')(a) =$  where $s_i =$

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "\hat{b}'_i"'] \\ \cdot \arrow[u, "b'_i"] & \cdot \arrow[u, "b_i"'] \arrow[l, "\hat{b}_i"] \end{tikzcd}$$

APPLICATION. *The* $(n+1)$*-category* Nat_n *of hypertransformations.*

The following Proposition 8 shows that Cat_n is the category of 1-morphisms of an $(n+1)$-category Nat_n which, for $n = 1$, is the 2-category of natural transformations. It is based on the Lemma, whose proof is given in the Appendix:

LEMMA. *Let* V *denote a cartesian category with commuting coproducts (in the sense of Penon* [21]*) and* A *be a* V*-category. If* V *admits coproducts indexed by the class of objects of* A, *then there is a category in* V *whose object of morphisms is the coproduct of* $A(e, e')$, *for any objects* e *and* e' *of* A, *and whose composition «glues together» the composition morphisms*

$$\kappa_{e,e',e''}: A(e, e')\times A(e', e'') \to A(e, e'').$$

PROPOSITION 8. *There is an* $(n+1)$*-fold category* Nat_n *satisfying the following conditions:*

1^o $(\mathrm{Nat}_n)^{0,\dots,n-1}$ *is the n-fold category coproduct of the n-fold categories* $Hom_n(\mathrm{A}, \mathrm{B})$, *for any (small) n-fold categories* A, B.

2^o *Its n-th composition* κ_n *is (notations Corollary 3, Proposition 7):*

$(f, f') \mapsto \kappa_{\mathrm{A},\mathrm{B},\mathrm{B}'}(f, f')$ *iff* f *in* $Hom_n(\mathrm{A}, \mathrm{B})$ *and* f' *in* $Hom_n(\mathrm{B}, \mathrm{B}')$.

3^o Cat_n *is the category of 1-morphisms of* Nat_n.

PROOF. 1^o Let $\hat{Cat}_n$ be the category of n-fold categories associated to a universe $\hat{\mathrm{U}}$ to which belongs the universe U of small sets, and a fortiori the class of objects of Cat_n. Then $\hat{Cat}_n$ is also cartesian closed. The faithful functor «forgetting all the compositions» from $\hat{Cat}_n$ toward the category $\hat{Set}$ (of sets associated to the universe $\hat{\mathrm{U}}$) preserves coproducts and limits, and it reflects isomorphisms (an n-fold functor defined by a 1-1 and onto map is an isomorphism); hence Corollary 1, Proposition 1-6 of Penon

[21] asserts that $\hat{Cat}_n$ has commuting coproducts (in [21] «small» is now to be replaced by: belonging to $\hat{\mathrm{U}}$).

2° As Cat_n is cartesian closed, it «is» a Cat_n-category [3], and it determines also a $\hat{Cat}_n$-category, the insertion functor $Cat_n \hookrightarrow \hat{Cat}_n$ preserving the cartesian closed structure. More precisely, we have the $\hat{Cat}_n$-category H_n defined as follows:

- its objects are the small n-fold categories A, B, ..., and

$$\mathrm{H}_n(\mathrm{A},\mathrm{B}) = Hom_n(\mathrm{A},\mathrm{B}) \ ;$$

- the «unitarity» morphisms are of the form $j_{\mathrm{A}}: 1_n \to \mathrm{H}_n(\mathrm{A},\mathrm{A})$, where $j_{\mathrm{A}}(0)$ is the vertex of $Hom_n(\mathrm{A},\mathrm{A})$ identified with $id: \mathrm{A} \to \mathrm{A}$;

- the «composition» morphisms $\kappa_{\mathrm{A},\mathrm{B},\mathrm{B}'}$ are those defined in Corollary 3, Proposition 7.

3° The Lemma associates to H_n a category H_n in $\hat{Cat}_n$ defined as

$$\begin{tikzcd} 1 & 2 \arrow[l, bend right, "\alpha"'] \arrow[l, bend left, "\beta"] & 3 \arrow[l, "\kappa"'] \end{tikzcd} \qquad \begin{tikzcd} \underline{Cat}_n^{dis} & H_n(2) \arrow[l, bend right, "\alpha^n"'] \arrow[l, bend left, "\beta^n"] & H_n(3) \arrow[l, "\kappa^n"'] \end{tikzcd}$$

follows:

- its object of morphisms $H_n(2)$ is the n-fold category $\coprod_{\mathrm{A},\mathrm{B}} Hom_n(\mathrm{A},\mathrm{B})$ coproduct of the n-fold categories $Hom_n(\mathrm{A},\mathrm{B})$, for any (small) n-fold categories A, B (as the sets $\underline{Hom_n(\mathrm{A},\mathrm{B})}$ are disjoint, this coproduct is on their union);

- its object of objects $H_n(1)$ is the «discrete» n-fold category on Cat_n (since it is the coproduct of $\underline{Cat_n}$ copies of the final object 1_n);

- the morphisms source α^n and target β^n send a block $f: \mathrm{A} \to \square_n \mathrm{B}$ of $Hom_n(\mathrm{A},\mathrm{B})$ onto A and B respectively;

- the composition morphism κ^n is the union of the n-fold «composition» functors $\kappa_{\mathrm{A},\mathrm{B},\mathrm{B}'}$ (Corollary 3, Proposition 7).

4° By the equivalence between categories in $\hat{Cat}_n$ and $(n+1)$-fold cat-categories (see Appendix [5]), $H_n: \sigma \to \hat{Cat}_n$ is the realization associated to the $(n+1)$-fold category Nat_n such that:

$$(\mathrm{Nat}_n)^{0,\dots,n} = H_n(2) = \coprod_{\mathrm{A},\mathrm{B}} Hom_n(\mathrm{A},\mathrm{B}) \ ,$$

$(\mathrm{Nat}_n)^n$ is the category whose composition is κ^n.

Nat_n is, in fact, an *(n+1)*-category, its vertices being the objects of the n-th category $(\mathrm{Nat}_n)^n$. A block of Nat_n will be called an *hypertransformation* (as suggested by the Example above). The hypertransformations being objects of the n first categories $(\mathrm{Nat}_n)^i$ are the vertices of the n-fold categories $Hom_n(\mathrm{A}, \mathrm{B})$, hence (Corollary 1, Proposition 7) they are the n-fold functors; the subcategory of $(\mathrm{Nat}_n)^n$ that they form is so identified with Cat_n, by definition of κ^n. ∇

DEFINITION. The *(n+1)*-fold category Nat_n defined in Proposition 8 is called *the (n+1)-category of hypertransformations between n-fold (small) categories.*

For $n = 1$, the 2-category Nat_1 is the (usual) 2-category Nat of natural transformations (introduced in 1963 in [11]).

REMARK. The functors

$$Sq_{n,n+1}: Cat_n \to Cat_{n+1} \quad \text{for all integers } n,$$

do not extend into an endofunctor of the category $MCat$ of multiple categories (considered in [5]). This comes from the fact that in $Sq\mathrm{B}$, we have put the vertical and horizontal compositions of squares at the two last places, the compositions deduced pointwise from that of B being first indicated. However, it exists a functor $MCat \to MCat$ which maps an n-fold category B onto the *(n+1)*-fold category $(Sq\mathrm{B})^{n-1,n,0,\dots,n-2}$, deduced from $Sq\mathrm{B}$ by permutation of compositions, for $n \geqslant 1$, and a set E onto the discrete category E^{dis}.

This functor admits a «partial» left adjoint (it is not defined on $Set \hookrightarrow MCat$) which associates to an *(n+1)*-fold category A the n-fold category $Lk(\mathrm{A}^{2,\dots,n,0,1})$ of *(0,1)*-links of A.

We have not considered these functors, because their iterates are not interesting, while the iterates of the *Square* and *Link* functors have played in important role in this Section, since they consider successively all the compositions of a multiple category.

APPENDIX

Enriched categories as internal categories

The aim of this Appendix is to prove that, under mild enough conditions on a cartesian category V, the V-categories (categories enriched in V, in the sense of [13,3]) «are» those categories internal to V whose object of objects is discrete, i.e., is a coproduct of copies of the final object I.

The main condition is that V is a *category with commuting coproducts*; this notion, due to Penon [21], means that:

- V admits finite limits and (small) coproducts preserved by pullbacks (in fact, Penon requires the existence of all small limits, but only finite limits are used);

- if the coproduct of two morphisms of V is an isomorphism, both are isomorphisms.

It implies (Proposition 2-3 [21]) that the partial product functors

$$- \times V : \mathrm{V} \to \mathrm{V} \quad \text{et} \quad V \times - : \mathrm{V} \to \mathrm{V}$$

preserve coproducts, for each object V of V.

Let V be a category with commuting coproducts. A fortiori, it is a cartesian category (i.e., it admits finite products), and we may consider the V-categories (for this cartesian structure). We denote by:

- V-*Cat* the category of V-categories A whose class A_0 of objects is small;

- *Cat* V the category of categories in(ternal to) V.

PROPOSITION A. *Let* V *be a category with commuting coproducts. Then there exists a functor* $\Gamma : \mathrm{V}\text{-}Cat \to Cat\,\mathrm{V}$ *associating to the* V-*category* A *a category in* V *whose object of morphisms is the coproduct of* $A(e, e')$ *for every couple* (e, e') *of objects of* A.

PROOF. 1° We will use the following assertion: If $(V_\lambda)_{\lambda \in \Lambda}$ and $(V'_\mu)_{\mu \in \mathrm{M}}$ are families of objects of V, if $\phi : \Lambda \to \mathrm{M}$ is a map and if $v_\lambda : V_\lambda \to V'_{\phi\lambda}$ is a morphism of V, for each $\lambda \in \Lambda$, then there exists a unique morphism v

between coproducts such that, for each $\lambda \in \Lambda$, the diagram

$$\begin{tikzcd}
\coprod_{\mu \in M} V'_\mu & & \coprod_{\lambda \in \Lambda} V_\lambda \arrow[ll, "v"'] \\
V'_{\phi\lambda} \arrow[u, "j_{\phi\lambda}"] & & V_\lambda \arrow[ll, "v_\lambda"'] \arrow[u, "j_\lambda"']
\end{tikzcd}$$

commutes, where j_λ and $j_{\phi\lambda}$ always denote the injections into the coproducts. Indeed, v, called the *factor of* $(v_\lambda)_\lambda$ *with respect to* ϕ, is defined as follows:

$$v = \Big(\coprod_\lambda V_\lambda \xrightarrow{\;\sim\;} \coprod_\mu \big(\coprod_{\lambda \in \phi^{-1}(\mu)} V_\lambda \big) \xrightarrow{\;\coprod_\mu v^\mu\;} \coprod_\mu V'_\mu \Big),$$

where $v^\mu : \coprod_{\lambda \in \phi^{-1}(\mu)} V_\lambda \to V'_\mu$ is the factor of $(v_\lambda)_\lambda$ through the coproduct $\coprod_{\lambda \in \phi^{-1}(\mu)} V_\lambda$.

2° *Construction of the category* ΓA in V, for a V-category A such that there exist in V coproducts indexed by the class A_0 of objects of A.

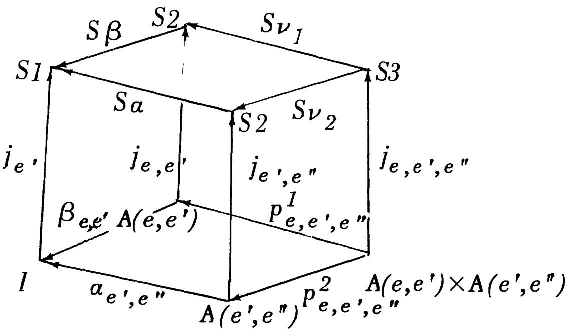
a) Since A_0 is finite or equipotent with $A_0 \times A_0$ and $A_0 \times A_0 \times A_0$, there exist in V coproducts:

S1 of the family $(I_e)_e$ indexed by A_0, where I_e is equal to the final object I of V for each object e of A,

S2 of the family $(A(e,e'))_{e,e'}$, indexed by $A_0 \times A_0$, where $A(e,e')$ is the «object of morphisms from e to e' in A»,

S3 of the family $(A(e,e') \times A(e',e''))_{e,e',e''}$ indexed by $A_0 \times A_0 \times A_0$.

b) (i) There exist unique morphisms $S\alpha$, $S\beta$, $S\nu_i$ rendering commutative the «cube»:



where $p^i_{e,e',e''}$ are the projections of the product, $\alpha_{e',e''}$ and $\beta_{e,e'}$ are the unique morphisms toward the final object I (the name of such a morphism will often be omitted). Indeed, $S\alpha$, $S\beta$, $S\nu_i$ are respectively the factors of:

$(\alpha_{e',e''})$ with respect to the projection $A_0 \times A_0 \to A_0 : (e', e'') \mapsto e'$,
$(\beta_{e,e'})$ with respect to the map $A_0^2 \to A_0 : (e, e') \mapsto e'$,
$(p^i_{e,e',e''})$ with respect to the maps $q_i : A_0 \times A_0 \times A_0 \to A_0 \times A_0$ with

$$q_1(e, e', e'') = (e, e') \text{ and } q_2(e, e', e'') = (e', e'').$$

Since the down face of the cube commutes (there is only one morphism $I_{e,e',e''}: A(e,e') \times A(e', e'') \to I$), by unicity of the factor of $(I_{e,e',e''})$ with respect to the projection $A_0 \times A_0 \times A_0 \to A_0 : (e, e', e'') \mapsto e'$, the up face of the cube also commutes.

(ii) The square

$$\text{(D)} \qquad \begin{tikzcd} & S2 \arrow[dl, "S\beta"'] & \\ S1 & & S3 \arrow[ul, "S\nu_1"'] \arrow[dl, "S\nu_2"] \\ & S2 \arrow[ul, "S\alpha"] & \end{tikzcd}$$

is a pullback. Indeed, for each object e' of A we have the pullback

$$(\mathrm{D}_{e'}) \qquad \begin{tikzcd} & \coprod_e A(e, e') \arrow[dl, "\beta_{e'}"'] & \\ I & & (\coprod_e A(e,e')) \times (\coprod_{e''} A(e', e'')) \arrow[ul, "p_1^{e'}"'] \arrow[dl, "p_2^{e'}"] \\ & \coprod_{e''} A(e', e'') \arrow[ul, "\alpha_{e'}"] & \end{tikzcd}$$

where $p_i^{e'}$ are projections of the product, since I is a final object. V having commuting coproducts, the theorem of commutation of Penon (Corollary 3, Proposition 1-8 [21]) asserts that the square (D') coproduct of the squares $(\mathrm{D}_{e'})$ is also a pullback. Now (D') is the down face of the cube

$$\begin{tikzcd} & S2 \arrow[dl, "S\beta"'] & & S3 \arrow[ll, "S\nu_1"'] \arrow[dl, "S\nu_2"] \\ S1 & & S2 \arrow[ll, "S\alpha"'] & \\ & {} \arrow[uu, "\wr"] \arrow[dl, "\coprod_{e'} \beta_{e'}"'] & & \coprod_{e'}((\coprod_e A(e,e')) \times (\coprod_{e''} A(e', e''))) \arrow[ll, "\coprod_{e'} p_1^{e'}"'] \arrow[dl, "\coprod_{e'} p_2^{e'}"] \arrow[uu, "\delta"'] \\ S1 \arrow[uu, equal] & & \coprod_{e'}(\coprod_{e''} A(e', e'')) \arrow[ll, "\coprod_{e'} \alpha_{e'}"] \arrow[uu, "\wr"] & \end{tikzcd}$$

The vertical edges of this cube are canonical isomorphisms between coproducts (the existence of δ follows from the preservation of coproducts by the partial product functors in V). By construction of the factors $S\alpha$, $S\beta$, $S\nu_i$, this cube commutes, so that its up face (D) is also a pullback.

(iii) There exist unique morphisms $S\iota$ and $S\kappa$ rendering commutative the squares

$$\begin{tikzcd} S1 \arrow[r, "S\iota"] & S2 \\ 1 \arrow[u, "j_e"] \arrow[r, "u_e"'] & A(e,e) \arrow[u, "j_{e,e}"'] \end{tikzcd} \qquad \begin{tikzcd} S2 & S3 \arrow[l, "S\kappa"'] \\ A(e,e'') \arrow[u, "j_{e,e''}"] & A(e,e')\times A(e',e'') \arrow[l, "\kappa_{e,e',e''}"] \arrow[u, "j_{e,e',e''}"'] \end{tikzcd}$$

where u_e and $\kappa_{e,e',e''}$ are the «identity» morphisms and the «composition» morphisms of the V-category A . Indeed, $S\iota$ and $S\kappa$ are respectively the factors of

$(u_e)_e$ with respect to the map $A_0 \to A_0 \times A_0 : e \mapsto (e,e)$,

$(\kappa_{e,e',e''})$ with respect to $A_0 \times A_0 \times A_0 \to A_0 \times A_0 : (e,e',e'') \mapsto (e,e'')$.

c) This defines a category S in V , i.e., a realization $S: \sigma \to$ V of the sketch σ of categories (see [4] and [5] Appendix):

$$\begin{tikzcd} 1 \arrow[r, "\iota"] & 2 \arrow[l, bend right, "\beta"'] \arrow[l, bend left, "\alpha"] & 3 \arrow[l, "\kappa"'] \arrow[l, bend right, "\nu_1"'] \arrow[l, bend left, "\nu_2"] \end{tikzcd} \qquad \begin{tikzcd} S1 \arrow[r, "S\iota"] & S2 \arrow[l, bend right, "S\beta"'] \arrow[l, bend left, "S\alpha"] & S3 \arrow[l, "S\kappa"'] \arrow[l, bend right, "S\nu_1"'] \arrow[l, bend left, "S\nu_2"] \end{tikzcd}$$

(i) For a couple (e,e') of objects of A , let $u_{e,e'}$ be equal to

$$A(e,e') \xrightarrow{\sim} 1 \times A(e,e') \xrightarrow{u_e \times id} A(e,e) \times A(e,e')$$

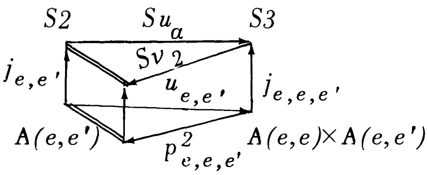
and Su_α be the factor of $(u_{e,e'})_{e,e'}$ with respect to the map

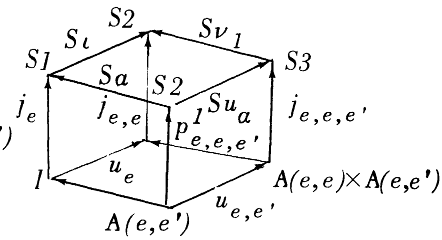
$$A_0 \times A_0 \to A_0 \times A_0 \times A_0 : (e,e') \mapsto (e,e,e').$$

Then

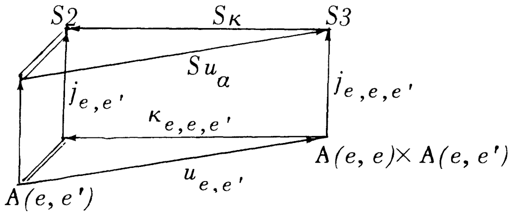
$$S\nu_1 . Su_\alpha = S\iota . S\alpha, \quad S\nu_2 . Su_\alpha = id_{S2} = S\kappa . Su_\alpha$$

(«source» unitarity axiom of an internal category). Indeed, by unicity of the factors and by definition of $u_{e,e'}$, for every objects e and e' of A the two following diagrams commute, so that the two first equalities are valid.





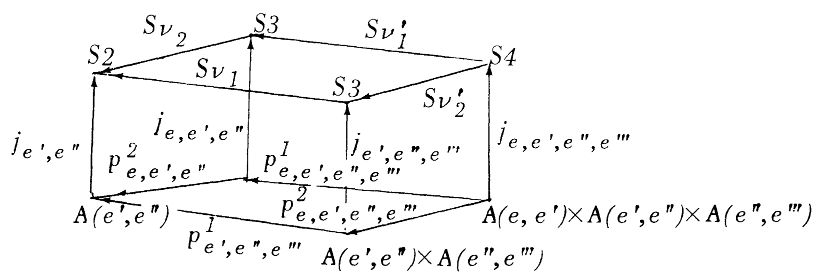
The validity of the third equation is deduced from the commutativity of the diagram



(whose down triangle commutes due to the unitarity axiom satisfied by A).

(ii) A similar proof shows that S satisfies the «target» unitarity axiom of an internal category.

(iii) S also satisfies the associativity axiom of an internal category. Indeed, for objects e, e', e'', e''' of A, there exists a commutative cube



where $j_{e,e',e'',e'''}$ is the injection toward the coproduct $S4$ of the family $(A(e,e')\times A(e',e'')\times A(e'',e'''))_{e,e',e'',e'''}$ indexed by A_0^4, and where $S\nu'_i$ is the factor of the family $(p^i_{e,e',e'',e'''})$ of projections with respect to the map $q'_i : A_0^4 \to A_0^3$ defined by

$$q'_1(e,e',e'',e''') = (e,e',e''), \quad q'_2(e,e',e'',e''') = (e',e'',e''').$$

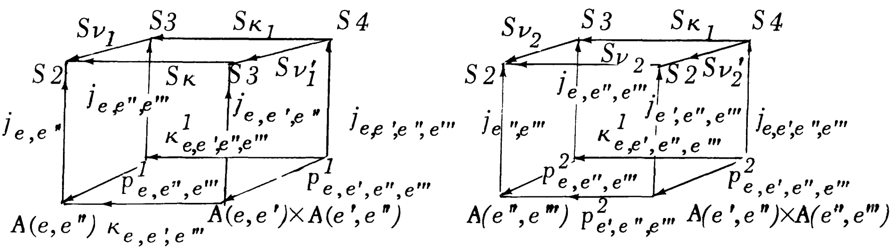
As the down face of this cube is a pullback, a proof analogous to that of Part b proves that the up face of this cube is a pullback. Now, let us denote by $\kappa^1_{e,e',e'',e'''}$ the composite

$$A(e,e')\times A(e',e'')\times A(e'',e''') \xrightarrow{\sim} (A(e,e')\times A(e',e''))\times A(e'',e''') \xrightarrow{\kappa_{e,e',e''}\times id} A(e,e'')\times A(e'',e''')$$

$S\kappa_1$ factor of the family $(\kappa^1_{e,e',e'',e'''})$ with respect to the projection

$$A_0^4 \to A_0^3 : (e,e',e'',e''') \mapsto (e,e'',e''')$$

renders commutative the cubes



(by definition of $\kappa^1_{e,e',e'',e'''}$ and of the different factors), so that

$$S\nu_1 . S\kappa_1 = S\kappa . S\nu'_1 \quad\text{and}\quad S\nu_2 . S\kappa_1 = S\nu_2 . S\nu'_2 .$$

In the same way, there is a factor $S\kappa_2 : S4 \to S3$ of the family of composites $\kappa^2_{e,e',e'',e'''} =$

$$A(e,e')\times A(e',e'')\times A(e'',e''') \xrightarrow{\sim} A(e,e')\times(A(e',e'')\times A(e'',e''')) \xrightarrow{id\times\kappa_{e',e'',e'''}} A(e,e')\times A(e',e''')$$

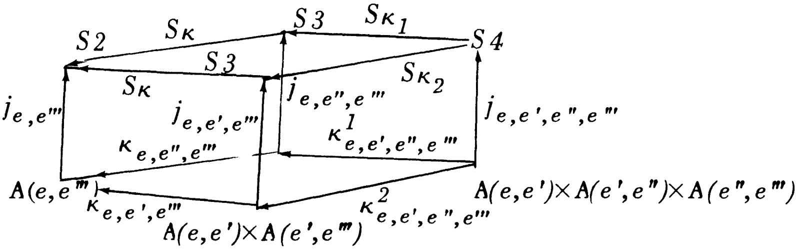
with respect to the projection

$$A_0^4 \to A_0^3 : (e,e',e'',e''') \mapsto (e,e',e'''),$$

and $S\kappa_2$ satisfies the equalities

$$S\nu_1 . S\kappa_2 = S\nu_1 . S\nu'_1 \quad\text{and}\quad S\nu_2 . S\kappa_2 = S\kappa . S\nu'_2 .$$

The associativity axiom $S\kappa . S\kappa_1 = S\kappa . S\kappa_2$ then follows from the unicity of factors and from the following cube, whose down face commutes due to the associativity axiom satisfied by the V-category A and whose lateral faces are commutative, by definition of the different factors. Hence, S defines a realization $S : \sigma \to \mathrm{V}$ of σ in V, i.e., a category internal to V, which will be denoted by ΓA.



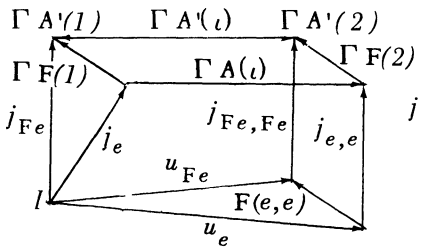
2° a) Let $F : A \to A'$ be a V-functor, $F_0 : A_0 \to A'_0 : e \mapsto Fe$ the map between objects and $F(e,e') : A(e,e') \to A(Fe, Fe')$ the canonical morphism, for every couple (e,e') of objects of A. There exist factors

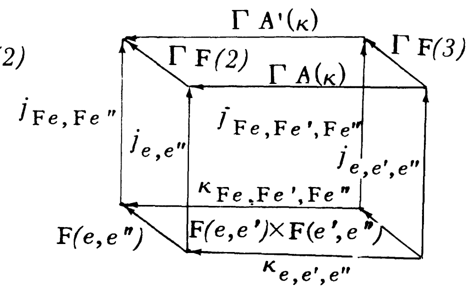
$\Gamma F(2) : \Gamma A(2) \to \Gamma A'(2)$ of $(F(e,e'))_{e,e'}$, with respect to $F_0 \times F_0$,

$\Gamma F(1) : \Gamma A(1) \to \Gamma A'(1)$ of $(1_e = 1)_e$ with respect to F_0,

$\Gamma F(3) : \Gamma A(3) \to \Gamma A'(3)$ of $(F(e,e') \times F(e',e''))_{e,e',e''}$ with respect to $F_0 \times F_0 \times F_0$.

These factors render commutative the diagrams





whose down faces commute by definition of a V-functor. This proves that $\Gamma F : \Gamma A \to \Gamma A'$ is a functor in V.

$$\begin{tikzcd}
\Gamma A'(1) & \Gamma A'(2) \arrow[l, "\Gamma A'(\iota)"'] & \Gamma A'(3) \arrow[l, "\Gamma A'(\kappa)"'] \\
\Gamma A(1) \arrow[u, "\Gamma F(1)"] & \Gamma A(2) \arrow[l, "\Gamma A(\iota)"] \arrow[u, "\Gamma F(2)"] & \Gamma A(3) \arrow[l, "\Gamma A(\kappa)"] \arrow[u, "\Gamma F(3)"']
\end{tikzcd}$$

b) This defines a functor $\Gamma : \text{V-}\mathcal{Cat} \to \mathcal{Cat}\,\text{V} : F \mapsto \Gamma F$, due to the unicity of the factors defining $\Gamma F(i)$, $i = 1, 2, 3$. ∇

PROPOSITION B. *The functor $\Gamma : \text{V-}\mathcal{Cat} \to \mathcal{Cat}\,\text{V}$ constructed above admits a right adjoint.*

PROOF. Let B be a category in V.

1° We define a V-category $\mathrm{B} = \Gamma' B$. The class B_0 of its objects is the set of morphisms $e : I \to B1$. If $e : I \to B1$ and $e' : I \to B1$ are such objects, $\mathrm{B}(e,e')$ is defined by the pullback

$$(\mathrm{D}_{e,e'}) \qquad \begin{tikzcd} & B2 \arrow[dl, "{[B\alpha, B\beta]}"'] & \\ B1\times B1 & & \mathrm{B}(e,e') \arrow[ul, "t_{e,e'}"'] \arrow[dl] \\ & I \arrow[ul, "{[e,e']}"] & \end{tikzcd}$$

where $[e,e']$ and $[B\alpha, B\beta]$ are the factors of (e,e') and $(B\alpha, B\beta)$ through the product $B1 \times B1$. There exists a factor $u_e : I \to \mathrm{B}(e,e)$, through the pullback $(\mathrm{D}_{e,e})$, of the diagram

$$\begin{tikzcd} & B2 \arrow[dl, "{[B\alpha, B\beta]}"'] & B1 \arrow[l, "B\iota"'] & \\ B1\times B1 & & & I \arrow[ul, "e"'] \arrow[dll, equal] \\ & I \arrow[ul, "{[e,e]}"] & & \end{tikzcd}$$

(which commutes, since $B\alpha . B\iota$ and $B\beta . B\iota$ are identities). Let e'' be another «object» $e'' : I \to B1$. The commutative diagram

$$\begin{tikzcd} & B2 \arrow[dl, "B\beta"'] & \mathrm{B}(e,e') \arrow[l, "t_{e,e'}"'] & \\ B1 & & I \arrow[ll, "e'"'] & \mathrm{B}(e,e') \times \mathrm{B}(e',e'') \arrow[ul, "proj"'] \arrow[l] \arrow[dl, "proj"] \\ & B2 \arrow[ul, "B\alpha"] & \mathrm{B}(e',e'') \arrow[l, "t_{e',e''}"] & \end{tikzcd}$$

factors uniquely through the pullback

$$\begin{tikzcd} & B2 \arrow[dl, "B\beta"'] & \\ B1 & & B3 \arrow[ul, "B\nu_1"'] \arrow[dl, "B\nu_2"] \\ & B2 \arrow[ul, "B\alpha"] & \end{tikzcd}$$

into $t_{e,e',e''} : \mathrm{B}(e,e') \times \mathrm{B}(e',e'') \to B3$, and the diagram

$$\begin{tikzcd} & B2 \arrow[dl, "{[B\alpha, B\beta]}"'] & B3 \arrow[l, "B\kappa"'] & \\ B1\times B1 & & & \mathrm{B}(e,e') \times \mathrm{B}(e',e') \arrow[ul, "t_{e,e',e''}"'] \arrow[dll] \\ & I \arrow[ul, "{[e,e'']}"] & & \end{tikzcd}$$

commutes (this uses the equalities

$$B\alpha . B\kappa = B\alpha . B\nu_1 \quad \text{and} \quad B\beta . B\kappa = B\beta . B\nu_2$$

of an internal category, and the commutativity of $(\mathrm{D}_{e,e'})$ and $(\mathrm{D}_{e',e''})$). Hence this diagram factors uniquely through the pullback $(\mathrm{D}_{e,e''})$ into

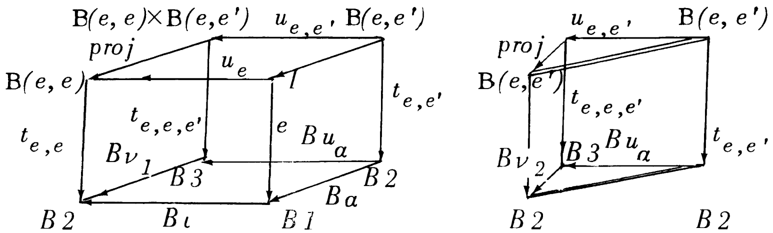
$$\kappa_{e,e',e''} : \mathrm{B}(e,e')\times \mathrm{B}(e',e'') \to \mathrm{B}(e,e'').$$

b) This defines a V-category B.

(i) Let us denote by $u_{e,e'}$ the composite

$$\mathrm{B}(e,e') \xrightarrow{\sim} I\times \mathrm{B}(e,e') \xrightarrow{u_e \times id} \mathrm{B}(e,e)\times \mathrm{B}(e,e').$$

In the diagrams

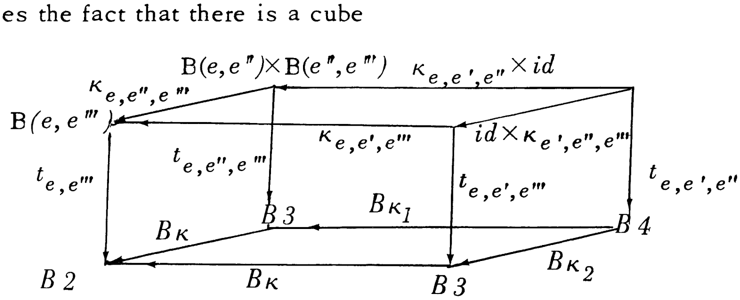


all the faces commute, except perhaps the back one; as $B\nu_i$ are projections of a pullback, it follows that this last face also commutes. So, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{B}(e,e') \arrow[d, "t_{e,e'}"'] & \arrow[l, "\kappa_{e,e,e'}"'] \arrow[d, "t_{e,e,e'}"] & \mathrm{B}(e,e') \arrow[l, "u_{e,e'}"'] \arrow[d, "t_{e,e'}"] \\
B2 & B3 \arrow[l, "B\kappa"] & B2 \arrow[l, "Bu_\alpha"] \arrow[ll, bend left, "id"]
\end{tikzcd}$$

and the unicity of the factor through the pullback $\mathrm{B}(e,e')$ implies that $\kappa_{e,e,e'}.\, u_{e,e'}$ is an identity. Therefore, B satisfies the unitarity axiom.

(ii) A similar method proves that B satisfies the associativity axiom. It uses the fact that there is a cube



in which all the vertical edges are projections of pullbacks and all faces, except perhaps the up face commute; so this up face also commutes.

2° There is an internal functor $t: \Gamma\mathrm{B} \to B$. Indeed, let $t(2)$ be the factor through the coproduct $\Gamma\mathrm{B}(2)$ (constructed in Proposition A) of the family $(t_{e,e'}: \mathrm{B}(e,e') \to B2)_{e,e'}$ indexed by $\mathrm{B}_0 \times \mathrm{B}_0$, so that:

$$\begin{tikzcd}
\Gamma\mathrm{B}(2) \arrow[d, "t(2)"'] & & \mathrm{B}(e,e') \arrow[ll, "j_{e,e'}"'] \arrow[dll, "t_{e,e'}"] \\
B2 & &
\end{tikzcd}$$

commutes. Let $t(1): \Gamma\mathrm{B}(1) \to B1$ be the factor through the coproduct $\Gamma\mathrm{B}(1)$ of the family $(e)_e$ indexed by B_0, so that

$$\begin{tikzcd}
\Gamma\mathrm{B}(1) \arrow[d, "t(1)"'] & & I \arrow[ll, "j_e"'] \arrow[dll, "e"] \\
B1 & &
\end{tikzcd}$$

commutes. Then the back face of the diagram

$$\begin{tikzcd}
\Gamma\mathrm{B}(1) \arrow[rr, "\Gamma\mathrm{B}(\iota)"] \arrow[dd, "t(1)"'] & & \Gamma\mathrm{B}(2) \arrow[dd, "t(2)"] & \\
& I \arrow[ul, "j_e"] \arrow[rr, "u_e"] \arrow[dl, "e"] & & \mathrm{B}(e,e) \arrow[ul, "j_{e,e}"'] \arrow[dl, "t_{e,e}"] \\
B1 \arrow[rr, "B\iota"'] & & B2 &
\end{tikzcd}$$

commutes, because all the other faces commute and $\Gamma\mathrm{B}(1)$ is a coproduct. Similarly, the back face of the diagram

$$\begin{tikzcd}
\Gamma\mathrm{B}(2) \arrow[dd, "t(2)"'] & & \Gamma\mathrm{B}(3) \arrow[ll, "\Gamma\mathrm{B}(\kappa)"'] \arrow[dd, "t(3)"] & \\
& \mathrm{B}(e,e'') \arrow[ul, "j_{e,e''}"'] \arrow[dl, "t_{e,e''}"] & & \mathrm{B}(e,e')\times\mathrm{B}(e',e'') \arrow[ul, "j_{e,e',e''}"'] \arrow[ll, "\kappa_{e,e',e''}"'] \arrow[dl, "t_{e,e',e''}"] \\
B2 & & B3 \arrow[ll, "B\kappa"] &
\end{tikzcd}$$

commutes, where $t(3)$ is the factor of $(t_{e,e',e''})$ through the coproduct $\Gamma\mathrm{B}(3)$. We have so defined an internal to V functor $t: \Gamma\mathrm{B} \to B$.

3° $t: \Gamma\mathrm{B} \to B$ is the coliberty morphism defining B as a cofree object generated by B. Indeed, let A be a V-category and $t': \Gamma A \to B$ be a functor in V. We are going to construct a V-functor $\mathrm{T}: A \to \mathrm{B}$.

b) For each object a of A, let $\mathrm{T}a$ be the object of B:

$$\mathrm{T}a = (I \xrightarrow{j_a} \Gamma A(1) \xrightarrow{t'(1)} B1)$$

where j_a is always the injection into the coproduct; this defines a map $\mathrm{T}_0 : A_0 \to \mathrm{B}_0$. If a and a' are objects of A, the two small squares of the diagram

$$\begin{tikzcd}
I \arrow[dr, "{[j_a, j_{a'}]}"] \arrow[dd, "{[\mathrm{T}a, \mathrm{T}a']}"'] & & \mathrm{A}(a, a') \arrow[ll] \arrow[d, "j_{a,a'}"] \\
& \Gamma A(1)^2 \arrow[dl, "t'(1)\times t'(1)"'] & \Gamma A(2) \arrow[l, "{[\Gamma A\alpha, \Gamma A\beta]}"'] \arrow[d, "t'(2)"] \\
B1\times B1 & & B2 \arrow[ll, "{[B\alpha, B\beta]}"]
\end{tikzcd}$$

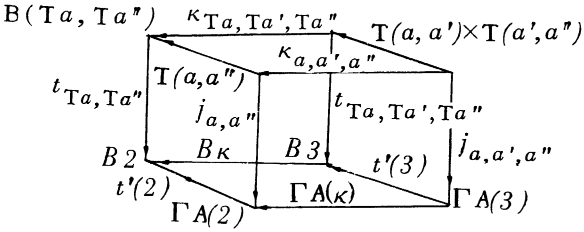
are commutative (by definition of ΓA and t' being an internal functor). Hence the exterior square is commutative, and it factors through the pull-back ($\mathrm{D}_{\mathrm{T}a, \mathrm{T}a'}$) into a unique $\mathrm{T}(a, a') : A(a, a') \to \mathrm{B}(\mathrm{T}a, \mathrm{T}a')$.

$$\begin{tikzcd}
\mathrm{B}(\mathrm{T}a, \mathrm{T}a') \arrow[d, "t_{\mathrm{T}a, \mathrm{T}a'}"'] & \mathrm{A}(a, a') \arrow[l, "{\mathrm{T}(a, a')}"'] \arrow[d, "j_{a,a'}"] \\
B2 & \Gamma A(2) \arrow[l, "t'(2)"]
\end{tikzcd}$$

b) This defines a V-functor $\mathrm{T} : A \to \mathrm{B}$. Indeed, for each object a of A, the up face of the diagram

$$\begin{tikzcd}
\mathrm{B}(\mathrm{T}a, \mathrm{T}a) \arrow[dd, "t_{\mathrm{T}a, \mathrm{T}a}"'] & & & & A(a, a) \arrow[llll, "{\mathrm{T}(a, a)}"'] \arrow[dd, "j_{a,a}"] \\
& & I \arrow[ull, "u_{\mathrm{T}a}"'] \arrow[urr, "u_a"] \arrow[ddl, "\mathrm{T}a"'] \arrow[ddr, "j_a"] & & \\
B2 & & & & \Gamma A(2) \arrow[llll, "t'(2)"'] \\
& B1 \arrow[ul, "B\iota"] & & \Gamma A(1) \arrow[ll, "t'(1)"] \arrow[ur, "\Gamma A(\iota)"'] &
\end{tikzcd}$$

commutes, since all the other faces commute and $\mathrm{B}(\mathrm{T}a, \mathrm{T}a')$ is a pull-back. Similarly, the up face of the following cube



commutes, all the other faces commuting and $B(Ta, Ta'')$ being a pullback. Hence, $T: A \to B$ is a V-functor.

c) The down face of the diagram

$$\begin{tikzcd}
B(Ta, Ta') \arrow[d, "j_{Ta,Ta'}"'] \arrow[drr, "t_{Ta,Ta'}"] & & & A(a,a') \arrow[lll, "T(a,a')"'] \arrow[d, "j_{a,a'}"] \\
\Gamma B(2) \arrow[dr, "t(2)"'] & & & \Gamma A(2) \arrow[lll, "\Gamma T(2)"'] \arrow[dll, "t'(2)"] \\
& B_2 & &
\end{tikzcd}$$

commutes, whatever be the objects a', a of A since the other faces commute and $\Gamma A(2)$ is the coproduct of $(A(a,a'))_{a,a'}$. It follows that

$$(t': \Gamma A \to B) = (\Gamma A \xrightarrow{\Gamma T} \Gamma B \xrightarrow{t} B).$$

$$\begin{tikzcd}
B & & \Gamma B \arrow[dr, "t"] & \\
A \arrow[u, "T"] & & \Gamma A \arrow[u, "\Gamma T"] \arrow[r, "t'"'] & B
\end{tikzcd}$$

Finally, the unicity of the V-functor T satisfying this equality results from the unicity of the morphisms $T(a,a')$. So B is a cofree object generated by B with respect to $\Gamma: V\text{-}Cat \to Cat\,V$. ∇

DEFINITION. A category in V is called *pseudo-discrete* if its object of objects is a coproduct of copies of the final object I.

By the construction of the functor Γ (Proposition A), it takes its values into the full subcategory $PsCat\,V$ of $Cat\,V$ whose objects are the pseudo-discrete categories in V. Hence it admits as a restriction a functor, also denoted by $\Gamma: V\text{-}Cat \to PsCat\,V$. Remark that the existence of this functor is conjectured (without precise hypotheses) in the Appendix III of the book [8].

PROPOSITION C. *Let V be a category with commuting coproducts, I its final object. If the functor $Hom(I,-): V \to V$ preserves coproducts, then the functor $\Gamma: V\text{-}Cat \to PsCat\,V$ is an equivalence.*

PROOF. Let $\Gamma': PsCat\,V \to V\text{-}Cat$ be the right adjoint of Γ constructed in Proposition B.

1° *The composite*

$$PsCat\,\mathrm{V} \xrightarrow{\Gamma'} \mathrm{V}\text{-}Cat \xrightarrow{\Gamma} PsCat\,\mathrm{V}$$

is equivalent to the identity:

Indeed, it suffices to prove that, for each pseudo-discrete category B in V, the coliberty morphism $t': \Gamma\Gamma' B \to B$ is an isomorphism. By hypothesis, $B1$ is the coproduct of a family $(I_\lambda = I)_{\lambda \in \Lambda}$ and

$$Hom(I, B1) \approx \coprod_\lambda Hom(I, I_\lambda) \approx \Lambda$$

since $Hom(I,I)$ is reduced to the identity of I; hence $B1$ is also a coproduct of the family $(I_e = I)_e$ indexed by the set $\mathrm{B}_0 = Hom(I, B1)$ of morphisms $e: I \to B1$, the e-th injection being e itself. As the partial product functors preserve coproducts, $B1 \times B1$ is the coproduct of the family $(I_{e,e'} = I)_{e,e'}$ indexed by $\mathrm{B}_0 \times \mathrm{B}_0$, the injections being the factor $[e, e']: I \to B1 \times B1$ into the product. We take the pullback

$$\begin{tikzcd} I & \mathrm{B}(e,e') \arrow[l] \arrow[d, "t_{e,e'}"] \\ B1 \times B1 \arrow[u, leftarrow, "{[e,e']}"] & B2 \arrow[l, "{[B\alpha, B\beta]}"] \end{tikzcd}$$

used to define $\mathrm{B} = \Gamma B$. The category V admitting commuting coproducts, by pulling back along $[B\alpha, B\beta]$ the coproduct $B1 \times B1$, we get $B2$ as a coproduct of $(\mathrm{B}(e,e'))_{e,e'}$, the injections being the morphisms $t_{e,e'}: \mathrm{B}(e,e') \to B2$. So the factor $t(2): \Gamma\,\mathrm{B}(2) \to B2$ of $(t_{e,e'})_{e,e'}$ is an isomorphism. This implies that $t: \Gamma\,\mathrm{B} \to B$ is an isomorphism.

2° *The composite*

$$\mathrm{V}\text{-}Cat \xrightarrow{\Gamma} PsCat\,\mathrm{V} \xrightarrow{\Gamma'} \mathrm{V}\text{-}Cat$$

is also equivalent to the identity, so that $PsCat\,\mathrm{V}$ and V-Cat are equivalent. Indeed, let A be a V-category; by adjunction, there is a V-functor $\mathrm{T}: A \to \Gamma'\Gamma A$ such that $\mathrm{T}a$ is the injection $j_a: I \to \Gamma A(1)$ for each object a of A and that the following diagram commutes, for each couple (a, a') of objects of A (we take up the notations of Proposition B, in which we choose $B = \Gamma A$).

$$\begin{tikzcd} \Gamma'\Gamma A(\mathrm{T}a,\mathrm{T}a') \arrow[dr, "t_{\mathrm{T}a,\mathrm{T}a'}"'] & & A(a,a') \arrow[ll, "\mathrm{T}(a,a')"'] \arrow[dl, "j_{a,a'}"] \\ & \Gamma A(2) & \end{tikzcd}$$

We are going to prove that T is an isomorphism.

a) $\mathrm{T}_0 : A_0 \to (\Gamma'\Gamma A)_0$ is 1-1 and onto: $\Gamma A(1)$ is the coproduct of the family $(I = I_a)_a$ indexed by the set A_0 of objects of A ; since $Hom(I,-): \mathrm{V} \to \mathrm{V}$ preserves coproducts, we have

$$Hom(I, \Gamma A(1)) \approx \coprod_a Hom(I, I_a) \approx A_0 ,$$

so that T_0 is an isomorphism.

b) For every objects a, a' of A, there is a pullback

$$\begin{tikzcd} I \arrow[d, "{[j_a, j_{a'}]}"'] & & \Gamma'\Gamma A(\mathrm{T}a,\mathrm{T}a') \arrow[ll] \arrow[d, "t_{\mathrm{T}a,\mathrm{T}a'}"] \\ \Gamma A(1) \times \Gamma A(1) & & \Gamma A(2) \arrow[ll, "{[\Gamma A\alpha, \Gamma A\beta]}"] \end{tikzcd}$$

defining $\Gamma'\Gamma A(\mathrm{T}a,\mathrm{T}a')$. We deduce as in Part 1 that $\Gamma A(2)$ is the coproduct of $(\Gamma'\Gamma A(\mathrm{T}a,\mathrm{T}a'))_{a,a'}$ with injections $t_{\mathrm{T}a,\mathrm{T}a'}$. But (by definition) $\Gamma A(2)$ is also the coproduct of $(A(a,a'))_{a,a'}$, and the commutativity of the diagrams defining $\mathrm{T}(a,a')$ implies that the identity of $\Gamma A(2)$ is the coproduct of $(\mathrm{T}(a,a'))_{a,a'}$. So, by definition of a category with commuting coproducts, each $\mathrm{T}(a,a')$ is an isomorphism. Hence $\mathrm{T} : A \to \Gamma'\Gamma A$ is an isomorphism. ∇

COROLLARY. *If* V *is a category with commuting coproducts, the functor* Γ*:* V*-Cat* $\to$ *PsCat*V *is an equivalence iff the endofunctor* $Hom(I,-)$ *preserves coproducts of copies of the final object* I.

PROOF. The preceding proof shows that the condition is sufficient. On the other hand, let us suppose that Γ : V-*Cat* $\to$ *PsCat*V is an equivalence and let S be the coproduct of a family $(I_\lambda = I)_{\lambda \in \Lambda}$. There exists a V-category A (the «V-groupoid of pairs of Λ») such that Λ is the set of its objects and $A(\lambda, \lambda') = I$ for each couple (λ, λ') of objects. The canonical V-functor

$$\mathrm{T} : A(\lambda, \lambda') \to \Gamma'\Gamma A(\mathrm{T}\lambda, \mathrm{T}\lambda')$$

being an isomorphism by hypothesis, its «restriction to the objects»:

$$T_0 : (A_0 = \Lambda) \to (\Gamma'\Gamma A)_0 = Hom(I, S)$$

is an isomorphism, and $Hom(I,S) \approx \Lambda \approx \prod_{\lambda \in \Lambda} Hom(I, I_\lambda)$. ∇

EXAMPLES.

1° There are many examples of categories V with commuting coproducts (see Penon [21]):

- the elementary topoi admitting coproducts,

- the categories admitting finite limits and coproducts and equipped with a faithful functor toward *Set* preserving pullbacks and coproducts and reflecting isomorphisms; in particular, the initialstructure categories (Wischnewsky [22], or topological categories in the sense of Herrlich [18]), the categories Cat_n for any integer n.

The condition that $Hom(I, -): V \to V$ preserves coproducts means that I is connected (in the sense of Hoffmann [19], see also Proposition 3-12 of Penon [21]). It is satisfied in the categories of a «topological nature», as well as in Cat_n. Remark that an $(n+1)$-fold category H (considered as a category in Cat_n, see Appendix [5]) is pseudo-discrete, and therefore «is» a Cat_n-category, by Proposition C, iff the objects of the last category H^n are also objects for the n first categories H^i (in an $(n+1)$-category, the objects of H^n are only supposed to be objects for H^{n-1}). The $(n+1)$-category Nat_n constructed in Proposition 8 «is» pseudo-discrete.

2° Proposition C is also valid if V is the category of r-differentiable manifolds (modelled on Banach spaces), though only some pullbacks exist in it (the pullbacks used in the proof will exist). Hence categories whose Hom are equipped with «compatible» r-differentiable structures «are» those r-differentiable categories (in the sense of [12]) in which the topology induced on the class of objects is discrete.

REFERENCES

1. J. BENABOU, Introduction to bicategories, *Lecture Notes in Math.* 47, Springer (1967).
2. J. BENABOU, Fibrations petites et localement petites, *C.R.A.S. Paris* 281 (1975), 897-900.
3. E. DUBUC, Kan extensions in enriched categories, *Lecture Notes in Math.* 145, Springer (1970).
4. A. & C. EHRESMANN, Multiple functors I, *Cahiers Topo. et Géo. Diff.* XV-3 (1974), 215-292.
5. A. & C. EHRESMANN, Multiple functors II, *id.* XIX-3 (1978), 295-333.
6. A. & C. EHRESMANN, Multiple functors IV, *id.* XX-1 (1979), to appear.
7. A. & C. EHRESMANN, Categories of sketched structures, *id.* XIII-2 (1972).
8. C. EHRESMANN, *Catégories et Structures*, Dunod, Paris, 1965.
9. C. EHRESMANN, Catégories structurées d'opérateurs, *C. R. A. S., Paris*, 256 (1963), 2080.
10. C. EHRESMANN, Structures quasi-quotients, *Math. Ann.* 171 (1967), 293-363.
11. C. EHRESMANN, Catégories structurées III, *Cahiers Topo. et Géo, Diff.* V (1963).
12. C. EHRESMANN, Catégories topologiques et catégories différentiables, *Col. Géo. Diff. globale* Bruxelles (1958), 137-152.
13. EILENBERG & KELLY, Closed categories, *Proc. Conf. on categ. Algebra* La Jolla 1965, Springer, 1966.
14. FOLTZ & LAIR, Fermeture standard des catégories algébriques, *Cahiers Topo. et Géo. Diff.* XIII-3 (1972).
15. J.W. GRAY, Fibred and cofibred categories, *Proc. Conf. on categ. Algebra* La Jolla 1965, Springer, 1966.
16. A. GROTHENDIECK, Catégories fibrées et descente, *Séminaire I. H. E. S.* Paris (1961).
17. GUITART & VAN DEN BRIL, Décompositions et lax-complétions, *Cahiers Topo. et Géo. Diff.* XVIII-4 (1977), 333-408.
18. H. HERRLICH, Topological functors, *Gen. Topo. and Appl.* 4 (1974).
19. R.E. HOFFMANN, A categorical concept of connectedness, Résumés Coll. Amiens, *Cahiers Topo. et Géo. Diff.* XIV-2 (1973).
20. PARE & SCHUMACHER, Abstract families and the adjoint functor theorems, *Lecture Notes in Math.* 661, Springer (1978), 1-125.
21. J. PENON, Catégories à sommes commutables, *Cahiers Topo.* XIV-3 (1973).
22. M. WISCHNEWSKY, *Initialkategorien*, Thesis, Univ. of München, 1972.