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MULTIPLE FUNCTORS
II. THE MONOIDAL CLOSED CATEGORY OF MULTIPLE CATEGORIES

by Andrée and Charles EHRESMANN

This paper is the Part II of our work on multiple functors, which was announced in Part I [5].

In this Part II we define directly (i.e., without reference to sketched structures) and study the category *MCat* of multiple categories. *MCat* is partially monoidal closed, for the «square product» which associates to an m-fold category A and an n-fold category B an $(n+m)$-fold category $B \blacksquare A$, and for a closure functor *Hom* such that $Hom(A, B)$, the $(n-m)$-fold category of «generalized natural transformations», is the set of multiple functors from A to B with compositions deduced «pointwise» from the $(n-m)$ last compositions of B.

One application is a criterium for the existence of colimits in *MCat*, which suggests the introduction of «infinite-fold» categories to embed *MCat* into a complete and cocomplete category. Another one is an existence theorem for generalized limits in n-fold categories, which admits as a particular case a result of Gray [13] and Bourn [3] on representable 2-categories (generalized in Part I to double categories); however the proof given here is more «structural» (and much shorter!).

Other applications are the descriptions of the cartesian closed structure of the category of n-fold categories, and of a monoidal closed structure which «laxifies» it. Part III (to appear in Vol. XIX-4) is devoted to them.

In an Appendix, the constructions of $B \blacksquare A$ and of $Hom(A, B)$ are translated in terms of sketched structures. This leads to similar results on internal multiple sketched structures (in particular internal multiple categories), which will be given in a subsequent paper.

Notations for Hom have been «inversed» relatively to Part I, in order to conform to more usual conventions.

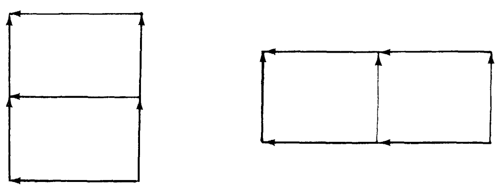
0. Motivating examples.

n-fold categories were introduced in [7] by induction, as categories internal to the category of $(n\text{-}1)$-fold categories. They are also defined as realizations in the category of sets of the sketch of n-fold categories, which is the n-th tensor power of the sketch of categories (see [5]). In this Part, we define and study them directly (i. e., without using the theory of sketched structures).

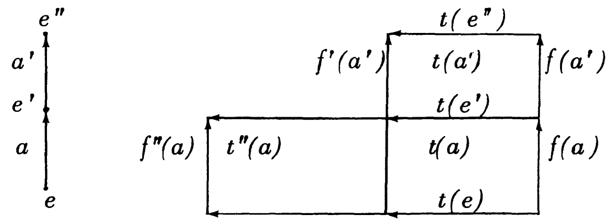
Double categories introduce themselves very naturally as soon as natural transformations are considered. Indeed, if B is a category, its commutative squares

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "\hat{b}'"'] \\ \cdot \arrow[u, "b'"] \arrow[ur, phantom, "="] & \cdot \arrow[l, "\hat{b}"] \arrow[u, "b"'] \end{tikzcd}$$

form a double category $\square B$ for the «vertical and horizontal» compositions :



A natural transformation $t: f \to f': A \to B$ may be seen as a functor from A to the vertical category of squares of B, while the composition of natural transformations is deduced from the horizontal composition :



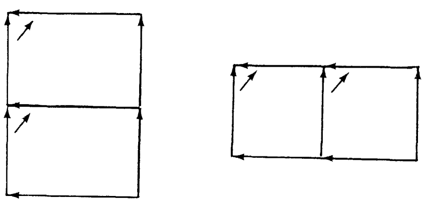
By induction, one defines (see [7], page 398) the multiple category of squares of squares..., which intervene to define transformations between natural transformations and so on... . We will generalize this construction in Part 2.

Other «usual» double categories are the 2-categories (considered by many authors), which are those double categories in which the objects for the second composition are also objects for the first one. For example, natural transformations between small categories form a 2-category, Nat.

$$\begin{tikzcd} C & B \arrow[l, bend right, "g'"'] \arrow[l, bend left, "g"] \arrow[l, phantom, "\Uparrow t'"] & A \arrow[l, bend right, "f'"'] \arrow[l, bend left, "f"] \arrow[l, phantom, "\Uparrow t"] \end{tikzcd}$$

There is also the 2-category of homotopy classes of continuous mappings, very useful in Algebraic Topology.

To a 2-category M is canonically associated the double category Q(M) of its (lax-)squares, with the vertical and horizontal compositions:



(see [8], where Q(Nat) is introduced in 1963 under the name of double category of «quintets», and [11, 2]); such double categories are characterized in [15].

More generally, n-categories are special n-fold categories, in which objects for some of the compositions are also objects for the other ones, and the lax-squares will be generalized in Part III.

A. The category of n-fold categories.

Let n be a positive integer.

An n-fold category A (on the set $\underline{A}$) is a sequence of n categories $(A^0, \dots, A^{n-1})$ with the same set $\underline{A}$ of morphisms, satisfying the *permutability axiom:*

(P) (A^i, A^j) is a double category for each pair (i, j) of integers, such that $i \neq j$, $0 \leqslant i < n$, $0 \leqslant j < n$ (see [5]).

An element of $\underline{A}$ is called a *block of* A , and A^i is the *i-th category of* A. We also say that A is *a multiple category, of multiplicity n*.

The axiom (P) means that, for each i, $0 \leqslant i \leqslant n-1$, the maps source (or domain), target (or codomain) and composition of A^i define functors with respect to the $(n-1)$ other categories A^j. In particular, it follows that the set of objects of A^i defines a subcategory of A^j, for each $j \neq i$. Moreover two of the categories A^i and A^j for $j \neq i$ are identical iff $A^i = A^j$ is a commutative category (i.e., a coproduct of commutative monoids). For example, if C is a commutative monoid, then $(\underbrace{C, \dots, C}_{n \text{ times}})$ is an n-fold category.

In the definition of the n-fold category A, the sequence of categories $(A^0, \dots, A^{n-1})$ is well given. If γ is a permutation of the set

$$n = \{ 0, 1, \dots, n-1 \},$$

then $(A^{\gamma(0)}, \dots, A^{\gamma(n-1)})$ is also an n-fold category on A , but it is different from A as an n-fold category and we denote it A^γ. If $(i_1, \dots, i_m)$ is a sequence of m distinct elements of n , then $(A^{i_1}, \dots, A^{i_m})$ is an m-fold category, denoted more simply by $A^{i_1, \dots, i_m}$. If $\underline{A}^{dis}$ denotes the discrete category on the set $\underline{A}$ (there are only objects), then

$$(A^0, \dots, A^{n-1}, \underbrace{\underline{A}^{dis}, \dots, \underline{A}^{dis}}_{m \text{ times}})$$

is an $(n+m)$-fold category, whatever be the integer m.

If A and B are n-fold categories, an *n-fold functor* $f: A \to B$ from A to B is defined by a map $f: \underline{A} \to \underline{B}$ defining a functor

$$f: A^i \to B^i \quad \text{for each } i < n.$$

Let Cat_n be the category whose objects are the small n-fold categories (i.e., the n-fold categories on small sets, small meaning that they belong to a given universe), and whose morphisms are the n-fold functors between them. By convention, a 0-fold category is a set, a 1-fold category is a category. So Cat_0 is the category *Set* of (small) sets and Cat_1, the category of (small) categories.

For a permutation γ of the set n, we denote by $\tilde{\gamma}: Cat_n \to Cat_n$ the isomorphism «permutation of the compositions»:

$$(f: \mathrm{A} \to \mathrm{B}) \longmapsto (f: \mathrm{A}^{\gamma} \to \mathrm{B}^{\gamma}).$$

These isomorphisms will be useful, since they permit to change the order of compositions when necessary.

PROPOSITION 1. *Cat_n is complete and, for each $i < n$, limits are preserved by the functor $U^i: Cat_n \to Cat$:*

$$(f: \mathrm{A} \to \mathrm{B}) \longmapsto (f: \mathrm{A}^i \to \mathrm{B}^i)$$

forgetting the compositions other than the i-th one.

PROOF. Let $F: K \to Cat_n$ be a functor indexed by a small category K. For each i the composite functor

$$K \xrightarrow{F} Cat_n \xrightarrow{U^i} Cat$$

admits a (projective) limit A^i on the set $\underline{\mathrm{A}}$ of families $(a_e)_e$ indexed by the objects e of K, such that:

$$a_e \in F(e) \text{ and } F(k)(a_e) = a_{e'} \text{ for each } k: e \to e' \text{ in } K.$$

$$\begin{tikzcd} e' & a_{e'} \arrow[r, phantom, "\in"] & F(e') & \\ & & & \mathrm{A} \arrow[ul] \arrow[dl] \\ e \arrow[uu, "k"] & a_e \arrow[uu, mapsto] \arrow[r, phantom, "\in"] & F(e) \arrow[uu, "F(k)"'] & \end{tikzcd}$$

It is easily seen that $(\mathrm{A}^0, \dots, \mathrm{A}^{n-1})$ is an n-fold category A, which is the limit of F. ∇

The following proposition will be used to prove Proposition 3.

PROPOSITION 2. *Let A be an n-fold category and M an infinite subset of $\underline{\mathrm{A}}$. Then the n-fold subcategory M of A generated by M is such that $\underline{\mathrm{M}}$ is equipotent with M.*

PROOF. $\underline{\mathrm{M}}$ is constructed as the union of the increasing sequence of sets M_l, $l \in \mathrm{N}$, defined by induction as follows: $M_0 = M$; if M_l is defined, then M_{l+1} is obtained by adding to M_l, for each $i < n$:

- the source and target in A^i of the blocks m in M_l,

- the composites in A^i of all the couples (m', m) of blocks in M_l admitting a composite in A^i.

Since M is infinite, it is seen by induction that M_{l+1} is equipotent to M_l, hence to M. It follows that $\underline{\mathrm{M}} = \bigcup_{l \in \mathrm{N}} M_l$ is also equipotent to M. ∇

Let m be an integer, $m < n$. There is a faithful functor

$$U_{n,m}: Cat_n \to Cat_m ,$$

which «forgets the $(n\text{-}m)$ first compositions»: it maps A onto $\mathrm{A}^{n-m, \dots, n-1}$ and $f: \mathrm{A} \to \mathrm{B}$ onto $f: U_{n,m}(\mathrm{A}) \to U_{n,m}(\mathrm{B})$.

From Proposition 1, it follows that the functors $U_{n,m}$ preserve limits. We shall prove in Section D that they admit left adjoints.

By composing $U_{n,m}$ with the isomorphism $\tilde{\gamma}: Cat_n \to Cat_n$ corresponding to a permutation γ of the set n (see before Proposition 1), we obtain faithful functors $Cat_n \to Cat_m$ mapping A onto the m-fold category $\mathrm{A}^{i_1, \dots, i_m}$ for every sequence $(i_1, \dots, i_m)$ of m distinct elements of n.

In particular, the functor $U_{n,0}: Cat_n \to Set$ is defined by:

$$(f: \mathrm{A} \to \mathrm{B}) \longmapsto (f: \underline{\mathrm{A}} \to \underline{\mathrm{B}}).$$

PROPOSITION 3. *This faithful functor* $U_{n,0}: Cat_n \to Set$ *admits quasi-quotient objects.*

PROOF. This assertion is deduced from the general existence theorem of quasi-quotient objects of [9], whose hypotheses are satisfied due to Propositions 1 and 2. In fact, we deduce from it the more precise result (used later on):

Let r be a relation on a set $\underline{\mathrm{H}}$ and suppose given a sequence H of n structures of neocategories (i.e., we do not impose unitarity nor associativity) H^i on $\underline{\mathrm{H}}$. Then there exists a universal solution to the problem of finding an n-fold category A and a map $f: \underline{\mathrm{H}} \to \underline{\mathrm{A}}$ compatible with r and defining a neofunctor $f: \mathrm{H}^i \to \mathrm{A}^i$ for each $i < n$. If $\hat{r}: \mathrm{H} \to \mathrm{B}$ is such a universal solution (i.e., every other solution factors through it uniquely), B is an *n-fold category quasi-quotient of* H *by* r. ∇

PROPOSITION 4. *Cat_n is cocomplete. The functor $U_{n,m}: Cat_n \to Cat_m$ preserves coproducts (but not every colimit).*

PROOF. 1° A family $(A_\lambda)_{\lambda \in \Lambda}$ of n-fold categories admits as a coproduct the n-fold category A on the set

$$\{ (a, \lambda) \mid a \in \underline{A}_\lambda,\ \lambda \in \Lambda \}$$

such that A^i is the category coproduct of the categories A^i_λ, $\lambda \in \Lambda$.

2° Let $F: K \to Cat_n$ be a functor indexed by a small category K, and let A be the n-fold category coproduct of the n-fold categories $F(e)$, for all objects e of K. Let r be the relation on $\underline{A}$ defined by:

$$(a, e) \sim (F(k)(a), e') \text{ for each } k: e \to e' \text{ in } K \text{ and } a \in F(e).$$

According to Proposition 3, there exists an n-fold category B quasi-quotient of A by r. From the general construction of colimits from coproducts

$$\begin{tikzcd} e' & F(k)(a) \in F(e') \arrow[dr, "(-,e')"] \arrow[drr, bend left, "t(e')"] & & \\ & \arrow[r, phantom, "\neq"] & A \arrow[r, "\hat{r}"] & B \\ e \arrow[uu, "k"] & a \in F(e) \arrow[uu, mapsto, "F(k)"'] \arrow[ur, "(-,e)"'] \arrow[urr, bend right, "t(e)"'] & & \end{tikzcd}$$

and quasi-quotients [9] it follows that B is a colimit of F, the colimit cone being $t: F \Longrightarrow B$, where

$$t(e) = (F(e) \xrightarrow{(-,e)} A \xrightarrow{\hat{r}} B). \qquad \nabla$$

REMARK. Since the functor $U_{n,m}$ does not preserve all colimits, it does not admit a right adjoint.

B. The monoidal category of multiple categories.

In this section, we consider the category *MCat* of multiple categories, defined as follows:

- Its objects are all the small n-fold categories, for every integer n (hence sets, categories, double categories, ... are objects);

- Let A be an m-fold category and B an n-fold category. If $m \leqslant n$, the morphisms $f: A \to B$, called *multiple functors*, are the m-fold functors f, from A to the m-fold category $B^{0,\ldots,m-1}$ (in which the $(n-m)$ last compositions of B are forgotten). If $m > n$, there is no morphism from A to B.

- The composition is trivially deduced from the composition of maps.

For each integer n, the category Cat_n is a full subcategory of the category $MCat$.

PROPOSITION 5. *1° MCat is complete and the faithful functor*

$$U : MCat \to Set : (f : \mathrm{A} \to \mathrm{B}) \longmapsto (f : \underline{\mathrm{A}} \to \underline{\mathrm{B}})$$

admits quasi-quotient objects.

2° For each integer n, the insertion $Cat_n \hookrightarrow MCat$ preserves limits, colimits and quasi-quotient objects.

PROOF. 1° Let $F : K \to MCat$ be a functor indexed by a small category K. For each object e of K, let n_e be the multiplicity of the multiple category $F(e)$. Let n be the least of the integers n_e, for all objects e of K. By the definition of the multiple functors, we have, for each $m \leqslant n$, a functor $F_m : K \to Cat_m$ such that

$$(k : e \to e') \longmapsto F(k) : F(e)^{0,\dots,m-1} \to F(e')^{0,\dots,m-1}.$$

It follows from Proposition 1 that F_n is the basis of a limit cone in Cat_n, say $l : \mathrm{A} \Longrightarrow F_n$ and that $\mathrm{A}^{0,\dots,m-1}$ is the limit of F_m for each $m < n$.

a) We prove that A is also the limit of F in $MCat$. Indeed, for each object e of K, $l(e) : \mathrm{A} \to F(e)$ is a multiple functor, the multiplicity n of A being lesser than n_e, so that $l : \mathrm{A} \Longrightarrow F$ is also a cone in $MCat$. Let $t : \mathrm{B} \Longrightarrow F$ be a cone in $MCat$. Since $t(e) : \mathrm{B} \to F(e)$ is a multiple functor, the multiplicity m of B is lesser than each n_e; hence $m \leqslant n$ and $t : \mathrm{B} \Longrightarrow F_m$ is a cone in Cat_m. There is a unique $f : \mathrm{B} \to \mathrm{A}^{0,\dots,m-1}$ such that

$$(t : \mathrm{B} \Longrightarrow F_m) = (\mathrm{B} \xrightarrow{f} \mathrm{A}^{0,\dots,m-1} \overset{l}{\Longrightarrow} F_m),$$

$$\begin{tikzcd}
e' & F(e') & & \mathrm{A} \arrow[ll, "l(e')"'] \arrow[dll, "l(e)"] \\
e \arrow[u, "k"] & F(e) \arrow[u, "F(k)"] & & \mathrm{B} \arrow[ll, "t(e)"] \arrow[u, "f"'] \arrow[ull, "t(e')"']
\end{tikzcd}$$

and $f: \mathrm{A} \to \mathrm{B}$ is the unique morphism such that

$$(\, t: \mathrm{B} \Longrightarrow F\,) = (\,\mathrm{B} \xrightarrow{f} \mathrm{A} \overset{l}{\Longrightarrow} F\,).$$

b) Consider now the case where $n_e = n$ for each object e of K, so that F takes its values in Cat_n. According to Proposition 4, there exists a colimit cone $l': F_n \Longrightarrow \mathrm{H}$ in Cat_n. Then $l': F \Longrightarrow \mathrm{H}$ is a colimit cone in $MCat$. Indeed, let $t': F \Longrightarrow \mathrm{B}'$ be an inductive cone, with vertex the p-fold category B'. Then $n \leqslant p$, and $t': F_n \Longrightarrow \mathrm{B}'^{0,\dots,n-1}$ is an inductive cone which factorizes through H :

$$(\, t': F_n \Longrightarrow \mathrm{B}'^{0,\dots,n-1}\,) = (\, F_n \overset{l'}{\Longrightarrow} \mathrm{H} \xrightarrow{f'} \mathrm{B}'^{0,\dots,n-1}\,).$$

$$\begin{tikzcd}
e' & \mathrm{H} \arrow[d, "f'"'] & & F(e') \arrow[ll, "l'(e')"'] \arrow[dll, "t'(e')"'] \\
e \arrow[u, "k"] & \mathrm{B}' & & F(e) \arrow[ll, "t'(e)"] \arrow[ull, "l'(e)"] \arrow[u, "F(k)"']
\end{tikzcd}$$

So $f': \mathrm{H} \to \mathrm{B}'$ is the unique morphism such that

$$(\, t': F \Longrightarrow \mathrm{B}'\,) = (\, F \overset{l'}{\Longrightarrow} \mathrm{H} \xrightarrow{f'} \mathrm{B}'\,).$$

3° Let A be an n-fold category, r a relation on $\underline{\mathrm{A}}$ and $\bar{\mathrm{A}}$ the n-fold category quasi-quotient of A by r (which exists, Proposition 3). Then $\bar{\mathrm{A}}$ is also an object quasi-quotient of A by r with respect to the functor U. Indeed, let $h: \mathrm{A} \to \mathrm{B}$ be a multiple functor compatible with r ; the multiplicity of B must be greater than n, so that there exists in Cat_n a factorization:

$$(\, h: \mathrm{A} \to \mathrm{B}^{0,\dots,n-1}\,) = (\,\mathrm{A} \xrightarrow{\hat{r}} \bar{\mathrm{A}} \xrightarrow{h'} \mathrm{B}^{0,\dots,n-1}\,)$$

$$\begin{tikzcd}
 & \mathrm{A} \arrow[dl, "\hat{r}"'] \arrow[dr, "h"] & \\
\bar{\mathrm{A}} \arrow[rr, "h'"] & & \mathrm{B}
\end{tikzcd}$$

where $\hat{r}: \mathrm{A} \to \bar{\mathrm{A}}$ is the canonical multiple functor. Then $h': \bar{\mathrm{A}} \to \mathrm{B}$ is the unique morphism factorizing h through $\bar{\mathrm{A}}$ in $MCat$. ∇

REMARK. $MCat$ is not cocomplete. In Proposition 10 we shall prove that a functor $F: K \to MCat$ admits a colimit iff the multiplicities of all the $F(e)$ for e object of K are bounded.

There is a partial monoidal structure on $MCat$, whose tensor product extends the square product $B \blacksquare A$ of two categories defined in [5] as being the double category $(\underline{B}^{dis} \times A,\ B \times \underline{A}^{dis})$, where $\underline{B}^{dis}$ denotes the discrete category on $\underline{B}$.

DEFINITION. Let A be an m-fold category and B an n-fold category. We call *square product of* (B, A), denoted by $\mathrm{B} \blacksquare \mathrm{A}$, the $(n+m)$-fold category on the product of sets $\underline{\mathrm{B}} \times \underline{\mathrm{A}}$, defined as follows:

- if $0 \leqslant i < m$, its i-th category is the product $\underline{\mathrm{B}}^{dis} \times \mathrm{A}^{i}$,
- if $0 \leqslant j < n$, its $(m+j)$-th category is the product $\mathrm{B}^{j} \times \underline{\mathrm{A}}^{dis}$.

This defines an $(n+m)$-fold category, which is the product of the $(n+m)$-fold categories:

$$(\underbrace{\underline{\mathrm{B}}^{dis}, \dots, \underline{\mathrm{B}}^{dis}}_{m \text{ times}}, \mathrm{B}^{0}, \dots, \mathrm{B}^{n-1}) \text{ and } (\mathrm{A}^{0}, \dots, \mathrm{A}^{m-1}, \underbrace{\underline{\mathrm{A}}^{dis}, \dots, \underline{\mathrm{A}}^{dis}}_{n \text{ times}}).$$

EXAMPLE. If E is a set, $\mathrm{B} \blacksquare E$ is the n-fold category whose j-th category is $\mathrm{B}^{j} \times E^{dis}$, for $0 \leqslant j < n$.

If H is a p-fold category, a map $g: \underline{\mathrm{B}} \times \underline{\mathrm{A}} \to \underline{\mathrm{H}}$ defines a multiple functor $g: \mathrm{B} \blacksquare \mathrm{A} \to \mathrm{H}$ iff the following conditions are satisfied:

(A1) $m + n \leqslant p$.

(A2) For each block b of B,

$$g(b, -): \mathrm{A} \to \mathrm{H}: a \mapsto g(b, a)$$

is a multiple functor.

(A3) For each block a of A,

$$g(-, a): \mathrm{B} \to \mathrm{H}^{m, \dots, p-1}: b \mapsto g(b, a)$$

is a multiple functor.

In this case we say that $g: (\mathrm{B}, \mathrm{A}) \to \mathrm{H}$ *is an alternative functor.*

In particular, the identity of $\underline{\mathrm{B}} \times \underline{\mathrm{A}}$ defines an alternative functor $id: (\mathrm{B}, \mathrm{A}) \to \mathrm{B} \blacksquare \mathrm{A}$, and any alternative functor $g: (\mathrm{B}, \mathrm{A}) \to \mathrm{H}$ factors through it.

$$\begin{tikzcd} & (B,A) \arrow[dl, "g"'] \arrow[dr, "id"] & \\ H & & B \blacksquare A \arrow[ll, "g"] \end{tikzcd}$$

In other words, $B \blacksquare A$ is the solution of the universal problem «to transform an alternative functor into a multiple functor».

PROPOSITION 6. *There is a functor* $\blacksquare: MCat \times (\coprod_n Cat_n) \to MCat$ *extending the square product, with a restriction giving to* $\coprod_n Cat_n$ *a monoidal structure symmetric «up to an interchange of the compositions». (We say that MCat is partially monoidal.)*

PROOF. 1° We define a functor $\blacksquare: MCat \times (\coprod_n Cat_n) \to MCat$ as follows: If $f: A \to A'$ and $g: B \to B'$ are multiple functors with A and A' of the same multiplicity (this last condition is essential), then

$$g \times f: B \blacksquare A \to B' \blacksquare A' : (b, a) \mapsto (g(b), f(a))$$

is a multiple functor $g \blacksquare f$. The map $(g, f) \mapsto g \blacksquare f$ defines the required functor $\blacksquare$.

2° The square product admits as a unit the set $1 = \{0\}$, the «unitarity isomorphisms» being:

$$A \to A \blacksquare 1 : a \mapsto (a, 0) \quad \text{and} \quad A \to 1 \blacksquare A : a \mapsto (0, a),$$

for each multiple category A. It is associative up to the «associativity isomorphisms»:

$$(B' \blacksquare B) \blacksquare A \to B' \blacksquare (B \blacksquare A) : ((b', b), a) \mapsto (b', (b, a))$$

for any multiple categories A, B, B'.

3° The square product is not symmetric in the usual sense, but there is, if A is an m-fold category and B an n-fold category, the isomorphism:

$$B \blacksquare A \to (A \blacksquare B)^\gamma : (b, a) \mapsto (a, b),$$

where $(A \blacksquare B)^\gamma$ is deduced from $A \blacksquare B$ by the interchange of compositions corresponding to the permutation

$$\gamma : (0, \ldots, m+n-1) \longmapsto (n, \ldots, n+m-1, 0, \ldots, n-1). \qquad \nabla$$

The square product being associative «up to an isomorphism», a sequence $(\mathrm{A}_q, \dots, \mathrm{A}_1)$ of multiple categories admits several composites, depending on the position of the parentheses. Any two of these composites are related by a canonical isomorphism, since $(\coprod_n Cat_n, \blacksquare)$ is monoidal. In particular, all these composites are canonically isomorphic with

$$(\dots((\mathrm{A}_q \blacksquare \mathrm{A}_{q-1}) \blacksquare \dots) \blacksquare \mathrm{A}_2) \blacksquare \mathrm{A}_1 .$$

This composite will be denoted by $\mathrm{A}^{\blacksquare q}$, if $\mathrm{A}_q = \dots = \mathrm{A}_1 = \mathrm{A}$; it is then also defined by induction:

$$\mathrm{A}^{\blacksquare 1} = \mathrm{A}, \quad \mathrm{A}^{\blacksquare q} = \mathrm{A}^{\blacksquare q-1} \blacksquare \mathrm{A} .$$

C. The internal Hom on $MCat$.

Now we define an «internal Hom functor» on the category of multiple categories, so that $MCat$ becomes partially monoidal closed. In particular this Hom associates to a category A and to a double category B the category of B-wise transformations from A (denoted by $T(\mathrm{B}, A)$ in [5]), i.e. the set of functors $f: A \to \mathrm{B}^0$ equipped with the composition deduced «pointwise from B^1»:

$$f' \circ_1 f: A \to \mathrm{B}^0 : a \mapsto f'(a) \circ_1 f(a).$$

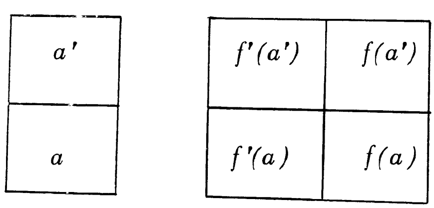
DEFINITION. Let A be an m-fold category and B an n-fold category. We call *multiple category of multiple functors from* A *to* B, and we denote by $Hom(\mathrm{A}, \mathrm{B})$:

- if $m > n$, the void set;

- if $m \leqslant n$, the $(n-m)$-fold category, on the set of the multiple functors $f: \mathrm{A} \to \mathrm{B}$, whose j-th composition, for $0 \leqslant j < n-m$, is

$$(f', f) \mapsto (f' \circ_j f: \mathrm{A} \to \mathrm{B} : a \mapsto f'(a) \circ_{j+m} f(a)),$$

iff the composite $f'(a) \circ_{j+m} f(a)$ exists in B^{j+m} for each block a of A.



So, for each pair

$$(i, j), \quad 0 \leqslant i < m, \quad 0 \leqslant j < n-m,$$

the category $Hom(\mathrm{A}, \mathrm{B})^j$ is a subcategory of the category of $(\mathrm{B}^i, \mathrm{B}^{m+j})$-wise transformations from A^i to $(\mathrm{B}^i, \mathrm{B}^{m+j})$. The permutability axiom is satisfied by $Hom(\mathrm{A}, \mathrm{B})$ since it is satisfied by B and the compositions are defined «pointwise» from that of B.

EXAMPLES. 1° If E is a set, $Hom(E, \mathrm{B})$ is the n-fold category B^E, product of E copies of B (i.e., product in Cat_n of the family $(\mathrm{B}_e)_{e \in E}$, with $\mathrm{B}_e = \mathrm{B}$ for each e in E).

2° If A is a category and B is the double category of squares of a category C, then $Hom(A, \mathrm{B})$ is the category C^A of natural transformations between functors from A to C.

REMARK. In fact, Example 2 motivated the introduction of $Hom(\mathrm{A}, \mathrm{B})$, which was generally defined in 1963 [7], under the name «multiple category of generalized transformations», represented by $\mathfrak{F}(\mathrm{B}, \mathrm{A})$. We interchange here A and B in the notation to adopt a more usual convention.

If $g: \mathrm{A}' \to \mathrm{A}$ is an m-fold functor and $h: \mathrm{B} \to \mathrm{B}'$ a multiple functor,

$$(f: \mathrm{A} \to \mathrm{B}) \longmapsto (\mathrm{A}' \xrightarrow{g} \mathrm{A} \xrightarrow{f} \mathrm{B} \xrightarrow{h} \mathrm{B}')$$

defines a multiple functor

$$Hom(g, h): Hom(\mathrm{A}, \mathrm{B}) \to Hom(\mathrm{A}', \mathrm{B}').$$

This determines the functor

$$Hom: (\coprod_n Cat_n)^{op} \times MCat \to MCat: (g, h) \mapsto Hom(g, h).$$

PROPOSITION 7. *The partial functor* $- \blacksquare \mathrm{A}: MCat \to MCat$, *for each multiple category* A, *admits* $Hom(\mathrm{A}, -): MCat \to MCat$ *as a right adjoint. (We say that* $(MCat, \blacksquare, Hom)$ *is a partial monoidal closed category.) In particular* $\coprod_n Cat_n$, *equipped with restrictions of* $\blacksquare$ *and* Hom, *is a monoidal closed category.*

PROOF. Let H be a p-fold category.

1° The evaluation $ev: (f, a) \mapsto f(a)$ defines an alternative functor

$ev: (Hom(\mathrm{A},\mathrm{H}),\mathrm{A}) \to \mathrm{H}$ since:

- for each block a of A,

$$ev(-,a): Hom(\mathrm{A},\mathrm{H}) \to \mathrm{H}^{m,\dots,p-1} : f \mapsto f(a)$$

is a multiple functor, by the «pointwise» definition of the compositions of $Hom(\mathrm{A},\mathrm{H})$,

- for each f in $Hom(\mathrm{A},\mathrm{H})$,

$$ev(f,-) = f : \mathrm{A} \to \mathrm{H}$$

is a multiple functor.

From the universal property of the square product, it follows that

$$ev: Hom(\mathrm{A},\mathrm{H}) \blacksquare \mathrm{A} \to \mathrm{H}$$

is a multiple functor, which will be the coliberty morphism which defines $Hom(\mathrm{A},\mathrm{H})$ as the cofree object generated by H.

2° Let B be an n-fold category. Then $g: \mathrm{B} \blacksquare \mathrm{A} \to \mathrm{H}$ is a multiple functor iff $g: (\mathrm{B},\mathrm{A}) \to \mathrm{H}$ is an alternative functor, i.e., iff:

- $m+n \leqslant p$ (condition A1),
- there is a map

$$\hat{g}: b \mapsto g(b,-): \mathrm{A} \to \mathrm{H}$$

from $\underline{\mathrm{B}}$ to the set of multiple functors from A to H (condition (A2)),

- for each block a of A, the composite

$$g(-,a) = (\mathrm{B} \xrightarrow{\hat{g}} Hom(\mathrm{A},\mathrm{H}) \xrightarrow{ev(-,a)} \mathrm{H}^{m,\dots p-1})$$

is a multiple functor (condition (A3));

$$\begin{tikzcd} & Hom(\mathrm{A},\mathrm{H}) \\ \mathrm{B} \arrow[ur, "\hat{g}"] & \end{tikzcd} \qquad \begin{tikzcd} Hom(\mathrm{A},\mathrm{H}) \blacksquare \mathrm{A} \arrow[dr, "ev"] & \\ \mathrm{B} \blacksquare \mathrm{A} \arrow[u, "\hat{g} \blacksquare \mathrm{A}"] \arrow[r, "g"'] & \mathrm{H} \end{tikzcd}$$

$$MCat \xrightarrow{\ - \blacksquare \mathrm{A}\ } MCat$$

this is equivalent to say that $\hat{g}: \mathrm{B} \to Hom(\mathrm{A},\mathrm{H})$ is a multiple functor, due to the pointwise definition of the compositions of $Hom(\mathrm{A},\mathrm{H})$. ∇

COROLLARY 1. *Let* A *be an m-fold category; then the «partial» functor*

$- \blacksquare A: Cat_n \to Cat_{m+n}$ admits the functor $Hom(A,-): Cat_{m+n} \to Cat_n$ as a right adjoint. ∇

REMARK. For $m = 1$ and $n = 1$, this Corollary has been proved in [5].

COROLLARY 2. *Let* A, B, H *be multiple categories of multiplicities* m, n *and* p. *There exists a canonical isomorphism*

$$\lambda: Hom(B \blacksquare A, H) \xrightarrow{\sim} Hom(B, Hom(A, H)).$$

If $p \geqslant m+n$, *there is also a canonical isomorphism*

$$Hom(B, Hom(A,H)) \xrightarrow{\sim} Hom(A, Hom(B, H^\pi)),$$

where H^π *is deduced from* H *by the interchange of compositions corresponding to the permutation*

$$\pi: (0, \dots, p-1) \mapsto (m, \dots, m+n-1, 0, \dots, m-1, m+n, \dots, p-1).$$

PROOF. 1° It is well-known for monoidal closed categories [10] that the one-one correspondence

$$(g: B \blacksquare A \to H) \mapsto (\hat{g}: B \to Hom(A,H): b \mapsto g(b,-): A \to H)$$

resulting from the adjunction (see Proof Proposition 7) defines an isomorphism

$$\lambda: Hom(B \blacksquare A, H) \xrightarrow{\sim} Hom(B, Hom(A,H)).$$

(This is also expressed by saying that $Hom(A,-)$ is a right $MCat$-adjoint of $- \blacksquare A$.) This result extends here (with the same proof).

2° Assume $p \geqslant m+n$. We have the «semi-symmetry» isomorphism

$$\sigma: B \blacksquare A \to (A \blacksquare B)^\gamma: (b,a) \mapsto (a,b)$$

(Proposition 6), where γ is the permutation

$$\gamma: (0, \dots, m+n-1) \mapsto (n, \dots, m+n-1, 0, \dots, n-1).$$

For each $(m+n)$-fold category K we have the identification

$$Hom(K^\gamma, H) \approx Hom(K, H^\pi),$$

which comes from the definition of Hom and from the fact that the inverse of γ is a restriction of π and that π is the identity on $(m+n, \dots, p-1)$. So, we get the following string of isomorphisms:

$$\begin{tikzcd}
Hom(B, Hom(A,H)) \arrow[d, "\lambda^{-1}"'] & & & Hom(A, Hom(B, H^{\pi})) \\
Hom(B \blacksquare A, H) \arrow[r, "Hom(\sigma^{-1}, H)"] & Hom((A \blacksquare B)^{\gamma}, H) \approx & Hom(A \blacksquare B, H^{\pi}) \arrow[ur, "\lambda'"'] &
\end{tikzcd} \quad . \nabla$$

The existence of this composite canonical isomorphism can yet be expressed in the following form, if $p = m + n$.

COROLLARY 3. *Let* H *be a p-fold category, with* $p = m+n$, *and* H^{π} *the p-fold category deduced from* H *as in Corollary 2. Then the partial functor* $Hom(-, H): Cat_m^{op} \to Cat_n$ *admits as a left adjoint the opposite of the functor* $Hom(-, H^{\pi}): Cat_n^{op} \to Cat_m$.

PROOF. The liberty morphism corresponding to the n-fold category B is

$$l: B \to Hom(Hom(B, H^{\pi}), H): b \mapsto [\, f \mapsto f(b)].$$

$$\begin{tikzcd}
Hom(B, H^{\pi}) & & Hom(Hom(B, H^{\pi}), H) \arrow[d, dashed, "Hom(g', H)"'] & \\
A \arrow[u, dashed, bend left, "g'"] & & Hom(A, H) & B \arrow[l, "g"] \arrow[ul, "l"']
\end{tikzcd}$$

$$Cat_m^{op} \xrightarrow{\quad Hom(-, H) \quad} Cat_n \qquad . \nabla$$

COROLLARY 4. *Let* B *be an n-fold category,* $p = m+n$ *and* π *the permutation* $(0, \dots, p\text{-}1) \mapsto (m, \dots, p\text{-}1, 0, \dots, m\text{-}1)$. *Then the partial functor* $B \blacksquare -: Cat_m \to Cat_p$ *is a left adjoint of the functor*

$$Hom(B, -^{\pi}) = (Cat_p \xrightarrow{\ \bar{\pi}\ } Cat_p \xrightarrow{Hom(B, -)} Cat_m).$$

PROOF. The liberty morphism corresponding to the m-fold category A is

$$l': A \to Hom(B, (B \blacksquare A)^{\pi}): a \mapsto [\, b \mapsto (b, a)].$$

$$\begin{tikzcd}
B \blacksquare A \arrow[d, dashed, "h'"'] & & Hom(B, (B \blacksquare A)^{\pi}) \arrow[d, "Hom(B, h'^{\pi})"'] & \\
H & & Hom(B, H^{\pi}) & A \arrow[l, "h"] \arrow[ul, "l'"']
\end{tikzcd}$$

$$\begin{tikzcd}
Cat_p \arrow[r, "Hom(B, -^{\pi})"] & Cat_m \arrow[l, bend left, "B \blacksquare -"]
\end{tikzcd} \qquad . \nabla$$

EXAMPLES.

a) Let E be a set and E_n the n-fold category on E whose categories are all discrete. The partial square product functor $- \blacksquare E: Cat_n \to Cat_n$ is identical with the partial product functor $- \times E_n: Cat_n \to Cat_n$. So Corollary 1 implies that the functor $- \times E_n$ admits as a right adjoint the «power functor» $-^E: Cat_n \to Cat_n$: mapping $f: \mathrm{B} \to \mathrm{B}'$ onto

$$f^E: \mathrm{B}^E \to \mathrm{B}'^E: (b_e)_{e \in E} \mapsto (f(b_e))_{e \in E}.$$

More generally, we shall prove in Part III that the partial product functor $- \times \mathrm{B}: Cat_n \to Cat_n$ admits a right adjoint for each n-fold category B, i.e., that Cat_n is cartesian closed.

b) *Functors «forgetting some compositions»:*

We denote by 2 the category

$$1 \xleftarrow{\quad (1,0) \quad} 0,$$

by $2^{\blacksquare m}$ the m-fold category defined by induction (see end of Section B):

$$2^{\blacksquare 1} = 2, \quad 2^{\blacksquare q} = 2^{\blacksquare q-1} \blacksquare 2 \text{ for each integer } q > 1.$$

If B is an n-fold category, a multiple functor $f: 2 \to \mathrm{B}$ is identified with a block $f(1,0)$ of B, and $Hom(2, \mathrm{B})$ is identified with $\mathrm{B}^{1,\dots,n-1}$. So $Hom(2,-): MCat \to MCat$ «is» the functor U_0 «forgetting the 0-th composition» (and mapping a set on the void set). By Proposition 7, this functor U_0 admits as a left adjoint the functor $- \blacksquare 2: MCat \to MCat$.

Let $U_m: MCat \to MCat$ be the composite of U_0 by itself m times: it maps the p-fold category H on:

- the void set if $p < m$,
- $\mathrm{H}^{m,\dots,p-1}$ if $p \geqslant m$.

It admits as a left adjoint the composite of $- \blacksquare 2: MCat \to MCat$ by itself m times, and this functor maps the n-fold category B onto the $(n+m)$-fold category $(\dots(\mathrm{B} \blacksquare 2) \blacksquare \dots 2) \blacksquare 2$, which is canonically isomorphic (end of Section B) with $\mathrm{B} \blacksquare 2^{\blacksquare m}$. Hence U_m also admits as a left adjoint the functor $- \blacksquare 2^{\blacksquare m}: MCat \to MCat$, and U_m may be identified with the functor

$$Hom(2^{\blacksquare m}, -): MCat \to MCat.$$

Taking restrictions of these functors, we get the first assertion of:

PROPOSITION 8. *The functor* $U_{m+n,n}: Cat_{m+n} \to Cat_n$ *forgetting the* m *first compositions admits as a left adjoint the partial functor*

$$- \blacksquare 2^{\blacksquare^m} : Cat_n \to Cat_{m+n}.$$

The functor $U'_{m+n,n}: Cat_{m+n} \to Cat_n$ *forgetting the* m *last compositions admits as a left adjoint the partial functor*

$$2^{\blacksquare^m} \blacksquare - : Cat_n \to Cat_{m+n}.$$

PROOF. We prove the second assertion. From Corollary 4, Proposition 7, it follows that the functor $2^{\blacksquare^m} \blacksquare -: Cat_n \to Cat_{m+n}$ is a left adjoint of

$$Hom(2^{\blacksquare^m}, -^{\pi}) = (Cat_{m+n} \xrightarrow{\tilde{\pi}} Cat_{m+n} \xrightarrow{Hom(2^{\blacksquare^m}, -)} Cat_n)$$

where $\tilde{\pi}$ is the isomorphism associated to the permutation

$$\pi : (0, \dots, m+n-1) \mapsto (n, \dots, m+n-1, 0, \dots, n-1),$$

and this composite functor identifies with

$$U'_{m+n,n} = (Cat_{m+n} \xrightarrow{\tilde{\pi}} Cat_{m+n} \xrightarrow{U_{m+n,n}} Cat_n). \qquad \nabla$$

c) «*Objects - functors*»:

Let 1_m be the «unique» m-fold category on the set $1 = \{0\}$. A multiple functor $f: 1_m \to \mathrm{B}$, where B is an n-fold category, is identified with a block $f(0)$ of B which is moreover an object for the m first categories B^i. Hence the functor $Hom(1_m, -): MCat \to MCat$ maps B onto:

- the void set if $n < m$,

- if $n \geqslant m$, the $(n-m)$-fold subcategory of $\mathrm{B}^{m, \dots, n-1}$ formed by the blocks of B which are objects for each category B^i, for $0 \leqslant i < m$; we will denote it by $|\mathrm{B}|^{m, \dots, n-1}$.

The functor $Hom(1_m, -)$ admits as a left adjoint $- \blacksquare 1_m : MCat \to MCat$ which maps the n-fold category B onto the $(n+m)$-fold category $\mathrm{B} \blacksquare 1_m$, wich is identified with the $(n+m)$-fold category on $\underline{\mathrm{B}}$ whose m first categories are discrete and whose $(m+j)$-th category is B^j, for $0 \leqslant j < n$.

PROPOSITION 9. *The functor* $|U_{n+m,n}| : Cat_{m+n} \to Cat_n$ *restriction of the functor* $Hom(1_m, -)$ *admits both a left and a right adjoint.*

PROOF. The left adjoint is the restriction of the functor $-\blacksquare\, 1_m$, described above. Since $|U_{n+m,n}|$ is equal to the composite

$$Cat_{n+m} \xrightarrow{|U_{n+m,n+m-1}|} Cat_{n+m-1} \to \dots \xrightarrow{|U_{n+1,n}|} Cat_n,$$

it suffices to prove the existence of a right adjoint for

$$|U_{n+1,n}|: Cat_{n+1} \to Cat_n .$$

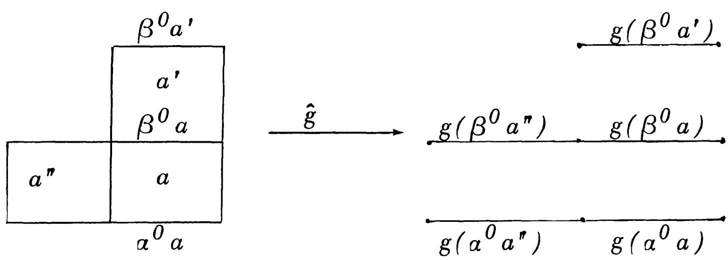
For this, let B be an n-fold category. There is an $(n+1)$-fold category B' on the product $\underline{B}\times\underline{B}$ whose 0-th category is the groupoid of couples of B, and whose $(i+1)$-th category is the product category $B^i \times B^i$, for $0 \leqslant i < n$. The image $|B'|^{1,\dots,n}$ of B' by $|U_{n+1,n}|$ is identified with B by identifying B with the set of objects for the groupoid of its couples. We say that B' is the cofree object generated by B. Indeed, if A is an $(n+1)$-fold category, α^0 and β^0 the source and target of A^0, then a map g defines an n-

$$\begin{tikzcd} & B' & & |B'|^{1,\dots,n} \arrow[rr, Rightarrow] & & B \\ & A \arrow[u, "\hat{g}"] & & |A|^{1,\dots,n} \arrow[u, "g"] \arrow[urr, "g"'] & & \end{tikzcd}$$

$$Cat_{n+1} \xrightarrow{|U_{n+1,n}|} Cat_n$$

fold functor $g: |A|^{1,\dots,n} \to B$ iff the map

$$\hat{g}: a \mapsto (g(\beta^0 a), g(\alpha^0 a))$$



defines an $(n+1)$-fold functor $\hat{g}: A \to B'$. ∇

In particular, the «object-functor» $Cat \to Set$ which maps a category on the set of its objects has a left adjoint mapping the set E onto the discrete category E^{dis}, and a right adjoint mapping E onto the groupoid of its couples.

D. Some applications to the existence of colimits.

1. Construction of colimits in $MCat$.

We have seen (Proposition 5) that $MCat$ is complete. It is not co-complete; however, using Proposition 8, we are going to prove:

PROPOSITION 10. *Let $F: K \to MCat$ be a functor, where K is a small category. Then F admits a colimit iff the multiplicities of the multiple categories $F(e)$, for all objects e of K, are bounded.*

PROOF. The condition is clearly necessary. On the other hand, if there exists a coproduct A of the multiple categories $F(e)$ for all objects e of K, then F will admit as a colimit the multiple category $\bar{A}$ quasi-quotient of A by the relation r:

$$u_e(a) \sim u_{e'}(F(k)(a)) \quad \text{if } k: e \to e' \text{ in } K \text{ and } a \in F(e),$$

where $u_e: F(e) \to A$ is the canonical injection into the coproduct:

$$\begin{tikzcd} e' & F(k)(a) \in F(e') \arrow[dr, "u_{e'}"] & & \\ & \arrow[r, phantom, "\neq"] & A \arrow[r, "\hat{r}"] & \bar{A} \\ e \arrow[uu, "k"] & a \in F(e) \arrow[uu, "F(k)"] \arrow[ur, "u_e"'] & & \end{tikzcd}$$

So it suffices to prove the existence of a coproduct for a family $(A_\lambda)_{\lambda \in \Lambda}$ such that A_λ is an m_λ-fold category and that there exists $n = \sup_{\lambda \in \Lambda} m_\lambda$.

For this, let $B_\lambda = 2^{\blacksquare^{n-m_\lambda}} \blacksquare A_\lambda$ be the free object generated by A_λ with respect to the functor $U'_{n,m_\lambda}: Cat_n \to Cat_{m_\lambda}$ forgetting the $(n-m_\lambda)$ last compositions (see Proposition 8); let $l_\lambda: A_\lambda \to B_\lambda^{0,\dots,m_\lambda-1}$ be the liberty morphism. The family $(B_\lambda)_{\lambda \in \Lambda}$ admits as a coproduct in $MCat$ its coproduct B in Cat_n (by Proposition 5), the canonical injection being

$$v_\lambda : B_\lambda \to B : b \mapsto (b, \lambda).$$

We say that B is also the coproduct of $(B_\lambda)_{\lambda \in \Lambda}$ in $MCat$, the canonical injection being

$$(u_\lambda : A_\lambda \to B) = (A_\lambda \xrightarrow{l_\lambda} B_\lambda \xrightarrow{v_\lambda} B).$$

Indeed, let H be a p-fold category and $f_\lambda : A_\lambda \to H$ a multiple functor for each $\lambda \in \Lambda$. Then $m_\lambda \leqslant p$ for each λ implies $n \leqslant p$, and by definition of a free object generated by A_λ, there exists a unique $g_\lambda : B_\lambda \to H^{0,\dots,n-1}$ with

$$(f_\lambda : A_\lambda \to H) = (A_\lambda \xrightarrow{l_\lambda} B_\lambda \xrightarrow{g_\lambda} H) .$$

The factor $g : B \to H$ of the family $(g_\lambda)_{\lambda \in \Lambda}$ through the coproduct B is the unique morphism rendering commutative the diagram

$$\begin{tikzcd}
B \arrow[dd, "g"'] & & \\
& B_\lambda \arrow[ul, "v_\lambda"'] \arrow[dl, "g_\lambda"'] & A_\lambda \arrow[l, "l_\lambda"] \arrow[ull, "u_\lambda"'] \arrow[dll, "f_\lambda"] \\
H & &
\end{tikzcd}$$

i.e., factorizing $(f_\lambda)_{\lambda \in \Lambda}$ through B. ∇

2. Generalized limits.

Motivated by the example of the category of natural transformations from a category A to a category C, which is identified with the category $Hom(A, \Box C)$, the following terminology was generally introduced in [7], and precised in [5] for double categories.

In this section, B denotes an m-fold category and H an $(m+1)$-fold category such that B is the m-fold subcategory $|B|^{0,\dots,m-1}$ of $H^{0,\dots,m-1}$ formed by those blocks of H which are objects for the last category H^m. Let $|H|^m$ be the subcategory of H^m formed by those blocks of H which are objects for the m first categories H^i. The objects of $|H|^m$ (hence the blocks of H which are objects for all the categories H^j) are called *vertices of* H.

Let A be an m-fold category. The objects of the category $Hom(A, H)$ are the multiple functors $f : A \to H$ taking their values in $|H|^{0,\dots,m-1} = B$; they are identified with the m-fold functors $f : A \to B$. Then, if $g : A \to H$ is a multiple functor, its source in $Hom(A, H)$ is

$$\alpha^m g = (A \xrightarrow{g} H^{0,\dots,m-1} \xrightarrow{\alpha^m} B) ,$$

and its target is

$$\beta^m g = (\mathrm{A} \xrightarrow{g} \mathrm{H}^{0,\ldots,m-1} \xrightarrow{\beta^m} \mathrm{B}),$$

where α^m and β^m are the maps source and target of H^m. We say that g *is a* H*-wise transformation from* $\alpha^m g$ *to* $\beta^m g$, denoted by $g: \alpha^m g \to \beta^m g$.

There is a canonical functor, called the *diagonal functor,*

$$d_{\mathrm{AH}} : |\mathrm{H}|^m \to Hom(\mathrm{A}, \mathrm{H})$$

(which is the functor associated to the alternative functor

$$(|\mathrm{H}|^m, \mathrm{A}) \to \mathrm{H} : (u, a) \mapsto u).$$

This functor maps an object u of $|\mathrm{H}|^m$, i.e., a vertex of H, onto the constant functor

$$u\hat{}: \mathrm{A} \to \mathrm{B} : a \mapsto u,$$

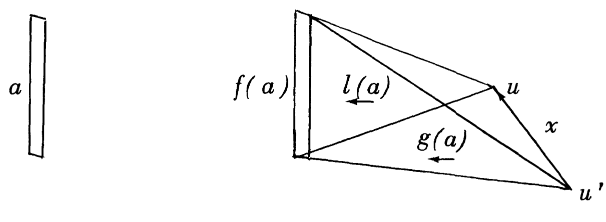
and it maps the morphism $x: u \to u'$ of $|\mathrm{H}|^m$ onto the H-wise transformation «constant on x», denoted by $x\hat{}: u\hat{} \to u'\hat{}$.

DEFINITION. Let $f: \mathrm{A} \to \mathrm{B} = |\mathrm{H}|^{0,\ldots,m-1}$ be an m-fold functor. If u is a free (resp. cofree) object generated by f with respect to the diagonal functor $d_{\mathrm{AH}} : |\mathrm{H}|^m \to Hom(\mathrm{A}, \mathrm{H})$, then u is called an H*-wise colimit* (resp. *limit*) *of* f.

If u is a vertex of H and $g: u\hat{} \to f$ an H-wise transformation, we also say (by reference with the case of natural transformations) that $g: u \Rightarrow f$ is a projective cone. Then u is a limit of $f: \mathrm{A} \to \mathrm{B}$ iff there exists a projective cone $l: u \Rightarrow f$, called a *limit-cone*, such that each projective cone $g: u' \Rightarrow f$ factors in a unique way through l, i.e., there exists a unique morphism $x: u' \to u$ of $|\mathrm{H}|^m$ satisfying:

$$g = (u'\hat{} \xrightarrow{d_{\mathrm{AH}}(x)} u\hat{} \xrightarrow{l} f)$$

(this means $g(a) = l(a) \circ_m x$ for each block a of A).



If the diagonal functor d_{AH} admits a right (resp. left) adjoint, so that each m-fold functor $f: A \to B$ admits a limit (resp. a colimit), we say that B *admits* H-*wise* A-*limits* (resp. A-*colimits*). If B admits H-wise A-limits for each small (resp. finite) m-fold category A, we say that B is H-*wise complete* (resp. *finitely complete*). Similarly is defined the notion: H-wise (finitely) cocomplete.

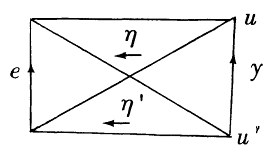
EXAMPLES. 1° If H is a double category (H^0, H^1) and B is the category of 1-morphisms obtained by equipping the set of objects of H^1 with the composition induced by H^0 (denoted by H^1_0 in [5]), these definitions coincide with those given in [5].

2° If $B = |H|^{0,\dots,m-1}$ admits H-wise $2^{\blacksquare^m}$-limits, we also say that H is a *representable* $(m+1)$*-fold category*, by extension of the notion of a representable 2-category introduced by Gray [13] and generalized in [5] to double categories. This means that the insertion functor $|H|^m \hookrightarrow H^m$ admits a right adjoint (since $Hom(2^{\blacksquare^m}, H)$ is identified with H^m). In other words, for each object e of H^m, there exists a vertex u of H, called the *representant of* e, and a block η of H with $\alpha^m \eta = u$, $\beta^m \eta = e$, such that, for each block η' of H with

$$\beta^m \eta' = e \quad \text{and} \quad \alpha^m \eta' = u' = \text{vertex of H},$$

there exists a unique

$$y: u' \to u \ \text{ in } |H|^m \ \text{ with } \eta' = \eta \circ_m y.$$



Dually, H *is corepresentable* if the insertion $|H|^m \hookrightarrow H^m$ admits a left adjoint.

The next proposition gives an existence theorem for H-wise limits. It utilizes the following Lemma, whose proof is given in the Appendix (since it considers multiple categories as sketched structures).

LEMMA. *Cat_m is the inductive closure of* $\{2^{\blacksquare^m}\}$ *(i.e., Cat_m is the smallest subcategory of Cat_m containing $2^{\blacksquare^m}$ and closed by colimits).*

PROPOSITION 11. *Let* H *be a representable $(m+1)$-fold category and let* $B = |H|^{0,\dots,m-1}$. *If $|H|^m$ is complete (resp. finitely complete), then* B *is* H-*wise complete (resp. finitely complete).*

PROOF. Let Ω be the full subcategory of Cat_m whose objects are the m-fold categories P such that B is H-wise P-complete. To say that H is representable means that $2^{\blacksquare^m}$ is an object of Ω. Let A be an m-fold category which is the colimit of a functor $F: K \to \Omega$, where K is small (resp. finite); if we prove that such an A is an object of Ω, it will follow that B is H-wise complete (resp. finitely complete), since Cat_m is the inductive closure of $\{2^{\blacksquare^m}\}$ by the preceding Lemma. For this, let $l': F \Longrightarrow A$ be the colimit cone. Since the functor $Hom(-,H): (Cat_m)^{op} \to Cat$ admits a left adjoint (by Corollary 3, Proposition 7), it transforms the colimit cone l' in Cat_m into a limit cone

$$l: Hom(A,H) \Longrightarrow Hom(F-,H).$$

We have a cone $d: |H|^m \Longrightarrow Hom(F-,H)$ such that

$$d(e) = d_{F(e)H}: |H|^m \to Hom(F(e),H),$$

for each object e of K. The factor of this cone with respect to the limit cone l is the diagonal functor $d_{AH}: |H|^m \to Hom(A,H)$. By hypothesis,

$$\begin{tikzcd}
e' & Hom(F(e'),H) \arrow[d, "Hom(F(k),H)"'] & & Hom(A,H) \arrow[ll, "l(e')"'] \arrow[dll, "l(e)"] \\
e \arrow[u, "k"] & Hom(F(e),H) & & {|H|^m} \arrow[ll, "d_{F(e)H}"] \arrow[ull, "d_{F(e')H}"'] \arrow[u, "d_{AH}"']
\end{tikzcd}$$

$F(e)$ belonging to Ω, each diagonal functor $d(e)$ admits a right adjoint, and $|H|^m$ admits K-limits. Hence a theorem of Appelgate-Tierney [1] asserts that the factor d_{AH} also admits a right adjoint, i.e., B admits H-wise A-limits. Therefore A is also an object of Ω, and a fortiori B is H-wise complete (resp. finitely complete). In fact, if $f: A \to B$ is an m-fold functor, its H-wise limit u is constructed as follows [1]: let u_e be a H-wise limit of the m-fold functor

$$l(e)(f) = (F(e) \xrightarrow{l'(e)} \mathrm{A} \xrightarrow{f} \mathrm{B}).$$

By the universal property of the limit, there exists a unique functor

$$G: K \to |\mathrm{H}|^m \quad \text{such that} \quad G(e) = u_e$$

for each object e of K. This functor G admits a limit u, which is a H-wise limit of $f: \mathrm{A} \to \mathrm{B}$. ∇

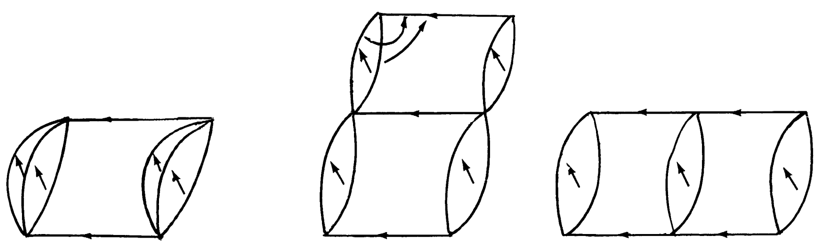
Dually, we prove by a similar method:

PROPOSITION 12. *If* H *is a corepresentable* $(m+1)$*-fold category and if* $|\mathrm{H}|^m$ *is (finitely) cocomplete, then the m-fold category* $\mathrm{B} = |\mathrm{H}|^{0,\dots,m-1}$ *is* H*-wise (finitely) cocomplete.* ∇

EXAMPLES.

a) If H is a double category, we find anew Proposition 3-2 [5] (with a much simpler proof). So if H is the double category $Q(\mathrm{K})$ of up-squares of a 2-category K , it reduces to Gray's Theorem of existence of cartesian quasi-limits [13] , as explained in [5] , page 64.

b) Let K be a 2-category. There is a triple category H , called the *triple category of squares of* $Q(\mathrm{K})$, such that $\mathrm{H}^{0,2}$ is the double category of squares of the vertical category $Q(\mathrm{K})^{\boxminus}$ and that the composition of H^1 is deduced pointwise from that of the horizontal category $Q(\mathrm{K})^{\boxplus}$; its greatest 3-category is the *3-category of cylinders of* K , defined in [2] :



If K is representable, so is $Q(\mathrm{K})$ (by Proposition 6-2 [5]), and also H (this will be proved in Part III, where we construct more generally the multiple category of squares of an n-fold category). If A is a 2-category, an object of $Hom(A, H)$ is identified with a 2-functor $f: A \to K$; a H-wise limit of f is then a catalimit of f in the sense of Bourn [3]. Analimits are obtained by taking down-squares instead of up-squares. So Proposition 11 then reduces to Proposition 7 of Bourn [3], whose proof, of the same type than that of Proposition 3-2 [5], is less «structural» and therefore longer.

E. Infinite-fold categories.

MCat does not admit coproducts for families $(A_\lambda)_{\lambda \in \Lambda}$ such that the multiplicities of the multiple categories A_λ are not bounded; indeed, such a coproduct should have «an infinity» of compositions. This leads to extend as follows *MCat* into a complete and cocomplete category *VMCat* which is partially monoidal closed.

DEFINITION. An N-*fold category* X on the set $\underline{X}$ is an infinite sequence $(X^i)_{i \in N}$ of categories with the same set of morphisms $\underline{X}$, such that, for each pair (i, j) of distinct integers, (X^i, X^j) is a double category. If X' is also an N-fold category, $h: X \to X'$ is an N-*fold functor* if $h: X^i \to X'^i$ is a functor for each integer i.

EXAMPLES.

a) If A is an m-fold category, there is an N-fold category X on $\underline{A}$ with

$$X^i = A^i \quad \text{for } 0 \leqslant i < m,$$
$$X^j = \underline{A}^{dis} \text{ (discrete category on } \underline{A}\text{) for } m \leqslant i \in N.$$

b) Let $(C_n)_{n \in N}$ be a sequence of categories; we define an N-fold category on the set product of the sets $\underline{C}_n$ of morphisms of C_n by taking as i-th category the product category

$$\prod_{n \in N} K_n, \quad \text{where } K_i = C_i \text{ and } K_n = \underline{C}_n^{dis} \text{ if } n \neq i.$$

In particular, if $C_n = 2$ for each integer n, we so obtain the N-fold category, denoted by 2_N, whose i-th category is

$$\underline{2}^{dis} \times \ldots \times \underline{2}^{dis} \times \underset{i\text{-th position}}{2} \times \underline{2}^{dis} \times \ldots \quad (\text{with } \underline{2} = \{0, 1, (1,0)\});$$

its unique non-degenerate «block» is $(u_n)_{n \in N}$, where $u_n = (1,0)$ for each integer n. Hence, an N-fold functor $h: 2_{\mathrm{N}} \to \mathrm{X}$, where X is an N-fold category, may be identified with the block $h((u_n)_{n \in \mathrm{N}})$ of X, image by h of the unique non-degenerate block $(u_n)_{n \in \mathrm{N}}$.

The N-fold functors between small N-fold categories form a category Cat_{N}. For each integer m, there is the faithful functor

$$U'_{\mathrm{N},m}: Cat_{\mathrm{N}} \to Cat_m ,$$

which maps the N-fold category X onto the m-fold category $\mathrm{X}^{0,\ldots,m-1}$ obtained by «keeping only the m first compositions».

REMARK. Cat_{N} may also be defined as the limit of the functor:

$$(n, m) \mapsto U'_{n,m}: Cat_n \to Cat_m$$

(where $U'_{n,m}$ is the functor «forgetting the $(n-m)$ last compositions» defined in Proposition 8), from the category of couples defining the order of N toward the category of categories associated to a universe containing the universe of small sets (if the existence of such a universe is assumed!).

PROPOSITION 13. *Cat_{N} is complete, cocomplete, and the faithful functor $U'_{\mathrm{N},0}: Cat_{\mathrm{N}} \to Set$ «forgetting all the compositions» admits quasi-quotient objects.*

PROOF. 1° From Proposition 2, it follows that, if $F: K \to Cat_{\mathrm{N}}$ is a functor, where K is small, it admits as a limit the N-fold category X such that $\mathrm{X}^{0,\ldots,m-1}$ is the limit of the functor

$$K \xrightarrow{F} Cat_{\mathrm{N}} \xrightarrow{U'_{\mathrm{N},m}} Cat_m ,$$

for each integer m.

2° If $(\mathrm{X}_\lambda)_{\lambda \in \Lambda}$ is a family of N-fold categories, it admits as a coproduct in Cat_{N} the N-fold category X such that X^i is the coproduct of the family of categories $(\mathrm{X}^i_\lambda)_{\lambda \in \Lambda}$.

3° The existence of quasi-quotient objects, and then of colimits, is proved by a method analogous to that used in Propositions 2, 3, 4 to prove similar results with respect to Cat_n, showing first by the same construction the following assertion:

The N-fold subcategory of an N-fold category X generated by an infinite subset M of $\underline{X}$ is equipotent with M. ∇

Let *VMCat* be the category whose objects are the small multiple categories and the small N-fold categories, and of which *MCat* and Cat_N are full subcategories, the only other morphisms being the $g: A \to X$, where A is an m-fold category and $g: A \to X^{0,\dots,m-1}$ an m-fold functor. We shall extend «partially» to *VMCat* the square product and the internal Hom functor of *MCat*.

DEFINITION. If X is an N-fold category and A an m-fold category, the *square product* $X \blacksquare A$ of (X, A) will be the N-fold category on the product set $\underline{X} \times \underline{A}$ whose i-th category is

$$\underline{X}^{dis} \times A^i \text{ if } 0 \leqslant i < m, \quad X^{i-m} \times \underline{A}^{dis} \text{ if } m \leqslant i \in \mathrm{N}.$$

So $X \blacksquare A$ is the N-fold category such that, for each integer $i > m$:

$$(X \blacksquare A)^{0,\dots,i} = X^{0,\dots,i} \blacksquare A.$$

It follows that a map $g: \underline{X} \times \underline{A} \to P$ defines a morphism $g: X \blacksquare A \to P$ iff: P is an N-fold category,

$$g(x, -): A \to P: a \mapsto g(x, a)$$

is a morphism for each block x of X, and for each block a of A,

$$g(-, a): X^i \to P^{m+i}: x \mapsto g(x, a)$$

is a functor for each integer i. Then we say that $g: (X, A) \to P$ *is an alternative morphism.*

In particular, the alternative morphism $id: (X, A) \to X \blacksquare A$ gives the universal solution of the problem of transforming alternative morphisms into N-fold functors.

$$\begin{tikzcd} & (X, A) \arrow[dl, "g"'] \arrow[dr, "id"] & \\ P & & X \blacksquare A \arrow[ll, "g"] \end{tikzcd}$$

DEFINITION. If X is an N-fold category and A an m-fold category, we denote by $Hom(A, X)$ the N-fold category on the set of morphisms from A to

X , whose i-th composition is deduced «pointwise» from that of X^{m+i} , so that, for each integer i :

$$Hom(A, X)^{0,\dots,i-1} = Hom(A, X^{0,\dots,m+i-1}) .$$

PROPOSITION 14. *The square product functor and the internal Hom functor of MCat extend into functors, still denoted:*

$$\blacksquare : VMCat \times \coprod_n Cat_n \to VMCat \text{ and } Hom : (\coprod_n Cat_n)^{op} \times VMCat \to VMCat.$$

For each multiple category A *the partial functor*

$$Hom(A, -) : VMCat \to VMCat$$

is right adjoint to $- \blacksquare A : VMCat \to VMCat$.

PROOF. The proof is similar to that of Proposition 7. The extended functor $\blacksquare$ maps $(h : X \to X', f : A \to A')$ onto the N-fold functor

$$h \times f : X \blacksquare A \to X' \blacksquare A' : (x, a) \mapsto (h(x), f(a)).$$

The extended functor Hom maps $(f' : B \to A,\ h : X \to X')$ onto the morphism $Hom(f', h) : Hom(A, X) \to Hom(B, X')$ defined by

$$(g : A \to X) \mapsto (B \xrightarrow{f'} A \xrightarrow{g} X \xrightarrow{h} X').$$

If A is an m-fold category and X an N-fold category, $Hom(A, X)$ is the cofree object generated by X with respect to the partial functor

$$- \blacksquare A : VMCat \to VMCat ,$$

the coliberty morphism being

$$ev : Hom(A, X) \blacksquare A \to X : (f, a) \mapsto f(a).$$

$$\begin{tikzcd}
Hom(A,X) & Hom(A,X) \blacksquare A \arrow[dr, "ev"] & \\
X' \arrow[u, "\hat{g}"] & X' \blacksquare A \arrow[u, "\hat{g} \blacksquare A"] \arrow[r, "g"'] & X \\
VMCat \arrow[r, "- \blacksquare A"] & VMCat . &
\end{tikzcd}$$

∇

COROLLARY. *The functor* $U'_{N,m} : Cat_N \to Cat_m$ *«keeping only the* m *first compositions» admits as a left adjoint the functor* $2_N \blacksquare - : Cat_m \to Cat_N$.

PROOF. Let A be an m-fold category ; then $2_N \blacksquare A$ (where 2_N is defined in Example b) is a free object generated by A with respect to $U'_{N,m}$, the liberty morphism being:

$$l: A \to (2_N \blacksquare A)^{0,\dots,m-1} : a \mapsto ((u_n)_{n \in N}, a),$$

where $u_n = (1,0): 0 \to 1$ for each integer n. Indeed, let X be an N-fold category. By the proposition, there is a canonical 1-1 correspondence between N-fold functors $h: 2_N \blacksquare A \to X$ and N-fold functors $2_N \to Hom(A, X)$, which are identified with blocks of $Hom(A,X)$, i.e., with m-fold functors $f: A \to X^{0,\dots,m-1}$. The morphism associated to $h: 2_N \blacksquare A \to X$ is

$$f: A \to X^{0,\dots,m-1} : a \mapsto h((u_n)_{n \in N}, a),$$

and h if the unique factor of f through l.

$$\begin{tikzcd} 2_N \blacksquare A \arrow[d, "h"'] & & (2_N \blacksquare A)^{0,\dots,m-1} \arrow[d, "h"'] & \\ X & & X^{0,\dots,m-1} & & A \arrow[ull, "l"'] \arrow[ll, "f"] \\ Cat_N \arrow[rr, "U'_{N,m}"'] & & Cat_m & \end{tikzcd}$$

∇

This Corollary, similar to Proposition 8, is used to prove:

PROPOSITION 15. *VMCat is complete, cocomplete, and the functor «forgetting all the compositions» $U: VMCat \to Set$ admits quasi-quotient objects.*

PROOF. The proof is analogous to that of Propositions 5 and 10, using the fact that Cat_N and Cat_m, for each integer m, are complete and cocomplete. More precisely:

1° The functor $F: K \to VMCat$, where M is small, admits as a limit in $VMCat$:

- if F takes its values in Cat_N, the limit in Cat_N of the restriction $F: K \to Cat_N$,

- otherwise, let n be the least of the multiplicities (finite or infinite) of the objects $F(e)$, for all objects e of K; then the limit of F in $VMCat$ is the limit of the composite functor:

$$K \xrightarrow{F} VMCat \xrightarrow{U'_{N,n}} Cat_n.$$

2° F admits as a colimit the quasi-quotient of the coproduct P of the objects $F(e)$, e object of K, in $VMCat$, this quasi-quotient being computed in Cat_{N} if P is an N-fold category, in $MCat$ otherwise.

3° A family $(\mathrm{P}_\lambda)_{\lambda \in \Lambda}$ of objects of $VMCat$ admits as a coproduct:

- its coproduct in $MCat$ if the multiplicities of the objects P_λ are all finite and bounded;

- and otherwise the coproduct of $(\mathrm{X}_\lambda)_{\lambda \in \Lambda}$ in Cat_{N}, where $\mathrm{X}_\lambda = \mathrm{P}_\lambda$ if P_λ is an N-fold category, and $\mathrm{X}_\lambda = 2_{\mathrm{N}} \blacksquare \mathrm{P}_\lambda$ if P_λ is an n_λ-fold category for some integer n_λ. ∇

REMARK.

The functor $\mathrm{X} \blacksquare - : \coprod_n Cat_n \to VMCat$, where X is an N-fold category, cannot be extended into a functor from $VMCat$, since to define $\mathrm{X} \blacksquare \mathrm{A}$ we have first considered «all the compositions of A ». In the same way, the functor $Hom(-, \mathrm{X}) : (\coprod_n Cat_n)^{op} \to VMCat$ cannot be extended trivially into a functor from $(VMCat)^{op}$. However, we may define as follows an internal Hom functor

$$Hom_{\mathrm{N}} : (Cat_{\mathrm{N}})^{op} \times Cat_{\mathrm{N}}) \to Cat_{\mathrm{N}}$$

and a functor $\blacklozenge : Cat_{\mathrm{N}} \times Cat_{\mathrm{N}} \to Cat_{\mathrm{N}}$ such that the partial functors

$$- \blacklozenge \mathrm{X}, \; Hom_{\mathrm{N}}(\mathrm{X}, -) : Cat_{\mathrm{N}} \to Cat_{\mathrm{N}}$$

are adjoint, for each N-fold category X . If X' is also an N-fold category:

$\mathrm{X}' \blacklozenge \mathrm{X}$ is the N-fold category whose $2i$-th category is $\underline{\mathrm{X}}'^{dis} \times \mathrm{X}^i$ and whose $(2i+1)$-th category is $\mathrm{X}'^i \times \underline{\mathrm{X}}^{dis}$;

denoting by X'^{even} and X'^{odd} respectively the N-fold categories

$$\mathrm{X}'^{even} = (\mathrm{X}'^{2i})_{i \in \mathrm{N}} \quad \text{and} \quad \mathrm{X}'^{odd} = (\mathrm{X}'^{2i+1})_{i \in \mathrm{N}},$$

we take for $Hom_{\mathrm{N}}(\mathrm{X}, \mathrm{X}')$ the N-fold category on the set of N-fold functors $h : \mathrm{X} \to \mathrm{X}'^{even}$ whose compositions are deduced «pointwise» from that of X'^{odd}, so that

$$h' \circ_i h : \mathrm{X} \to \mathrm{X}'^{even} : x \mapsto h'(x) \circ_{2i+1} h(x) \text{ iff this is defined.}$$

But this does not give a monoidal closed structure on Cat_{N}. It is not associative nor unitary (up to isomorphisms or interchange of compositions).

APPENDIX

In this paper, we have defined multiple categories directly, but they can also be considered (in several ways) as sketched structures. Here we interpret the constructions of the square product and of *Hom* in terms of «multiple internal categories».

A. Multiple categories as sketched structures.

For the notations and results on sketched structures and internal categories, we refer to Section 0 [5]. We only recall that the category underlying the sketch $\sigma\mathcal{J}$ of categories (denoted more simply $\sigma = (\Sigma, \Gamma)$) is the full subcategory Σ of the opposite of the simplicial category whose obtects are the integers $1, 2, 3, 4$. The «idea» of this sketch is

$$\begin{tikzcd} 1 & 2 \arrow[l, bend right, "\alpha"'] \arrow[l, bend left, "\beta"] & 3 \arrow[l, "\kappa"'] \end{tikzcd} .$$

This means that a realization $F: \sigma \to \mathrm{K}$, or «category in(ternal to) K» is uniquely determined by $F(\alpha)$, $F(\beta)$, $F(\kappa)$, whatever be the category K.

If C is a category, the realization $\sigma \to Set$ canonically associated to C maps α, β, κ respectively on the maps source, target and composition of C.

Multiple categories appear as sketched structures in three different ways:

1° *The category Cat_n of n-fold categories is equivalent to the category Cat^{σ}_{n-1} of categories in Cat_{n-1}.*

Indeed, if B is an n-fold category, the realization $\sigma \to Cat_{n-1}$ canonically associated to B maps α, β, κ on the maps source α^{n-1}, target β^{n-1} and composition κ^{n-1} of B^{n-1}, considered as $(n-1)$-fold functors with respect to the $(n-1)$ first compositions of B, so that:

$$\begin{tikzcd} 1 & 2 \arrow[l, bend right, "\alpha"'] \arrow[l, bend left, "\beta"] & 3 \arrow[l, "\kappa"'] \end{tikzcd} \mapsto \begin{tikzcd} {|\mathrm{B}|^{0,\dots,n-2}} & \mathrm{B}^{0,\dots,n-2} \arrow[l, bend right, "\alpha^{n-1}"'] \arrow[l, bend left, "\beta^{n-1}"] & (\mathrm{B} *_{n-1} \mathrm{B})^{0,\dots,n-2} \arrow[l, "\kappa^{n-1}"'] \end{tikzcd}$$

where $(B *_{n-1} B)^{0,\dots,n-2}$ is the $(n-1)$-fold subcategory of the product $(n-1)$-fold category $B^{0,\dots,n-2} \times B^{0,\dots,n-2}$ formed by the couples (b', b) having a composite $b' \circ_{n-1} b$ in B^{n-1}.

2° *Cat_n is equivalent to the category Set^{σ_n} of realizations in Set of the «sketch of n-fold categories» σ_n.*

Indeed, σ_n is the n-th tensor power $\overset{n}{\otimes}\sigma$ of σ (see [5]) defined inductively by:

$$\sigma_1 = \sigma \quad \text{and} \quad \sigma_n = \sigma_{n-1} \otimes \sigma .$$

Its underlying category Σ_n is:

$$\Sigma_n = \Sigma_{n-1} \times \Sigma = (\dots(\Sigma \times \Sigma) \times \dots \times \Sigma) \times \Sigma \;;$$

a morphism of Σ_n will be more simply written as a sequence $(x_0, \dots, x_{n-1})$ of morphisms of Σ (i.e., we omit the parentheses).

For $0 \leqslant i < n$, there is a one-one functor $\delta_n^i : \Sigma \to \Sigma_n$, which maps x onto the sequence $(2, \dots, 2, x, 2, \dots, 2)$ in which all the factors are 2 except the i-th one, which is x. This functor defines a morphism of sketches $\delta_n^i : \sigma \to \sigma_n$. If $F : \sigma_n \to K$ is a realization in a category K, also called an *n-fold category in* K, then F is uniquely determined by the n categories F^i in K such that

$$F^i = (\sigma \xrightarrow{\delta_n^i} \sigma_n \xrightarrow{F} K), \quad \text{for } 0 \leqslant i < n .$$

If B is an n-fold category, the realization $B : \sigma_n \to Set$ (canonically) associated to B is such that

$$B^i = (\sigma \xrightarrow{\delta_n^i} \sigma_n \xrightarrow{B} Set)$$

is the realization in Set associated to the category B^i, for each $i < n$. This determines an equivalence $\eta_n : Cat_n \to Set^{\sigma_n}$.

3° *For each integer $m < n$, the category Cat_n is equivalent to the category $(Set^{\sigma_m})^{\sigma_{n-m}}$*, and to the category $(Cat_m)^{\sigma_{n-m}}$ *of $(n-m)$-fold categories in Cat_m.*

Indeed, from the universal property of the tensor product of sketches (which equips the category of sketches of a monoidal closed structure, see

[4,14]), we deduce the canonical isomorphisms $\sigma_{n+m} \xrightarrow[\sim]{associativity} \sigma_n \otimes \sigma_m$,

$$Set^{\sigma_n} \overset{\sim}{\to} Set^{\sigma_m \otimes \sigma_{n-m}} \overset{\sim}{\to} (Set^{\sigma_m})^{\sigma_{n-m}}.$$

More precisely, let B be an n-fold category; then the realization $\hat{B}: \sigma_{n-m} \to Cat_m$ (canonically) associated to B maps $(2,\dots,2)$ onto the m-fold category $B^{0,\dots,m-1}$ and it is determined by the fact that for $j < n-m$, the composite

$$\sigma \xrightarrow{\delta^j_{n-m}} \sigma_{n-m} \xrightarrow{\hat{B}} Cat_m$$

is the category in Cat_m associated (as in 1 above) to the $(m+1)$-fold category $B^{0,\dots,m-1,m+j}$, so that it is defined by:

$$\left(\cdot \underset{\beta}{\overset{\alpha}{\leftleftarrows}} \cdot \xleftarrow{\ \kappa\ } \cdot\right) \longmapsto \left(|B|^{0,\dots,m-1} \underset{\beta^{m+j}}{\overset{\alpha^{m+j}}{\leftleftarrows}} B^{0,\dots,m-1} \xleftarrow[(B *_{m+j} B)^{0,\dots,m-1}]{\ \kappa^{m+j}\ }\right)$$

The realization $\bar{B}: \sigma_{n-m} \to Set^{\sigma_m}$ associated to B is the composite of $\hat{B}$ with the equivalence $\eta_m: Cat_m \to Set^{\sigma_m}$ (defined in 2), so that

$$\sigma \xrightarrow{\delta^j_{n-m}} \sigma_{n-m} \xrightarrow{\bar{B}} Set^{\sigma_m},$$

for $0 \leqslant j < n-m$, is the category in Set^{σ_m} associated to $B^{0,\dots,m-1,m+j}$.

B. Realizations associated to $B \blacksquare A$ **and to** $Hom(A, B)$.

In this section, we denote by A an m-fold category, by B an n-fold category, by

$$A: \sigma_m \to Set \quad \text{and} \quad B: \sigma_n \to Set$$

the associated realizations in Set.

PROPOSITION 1. *The realization in Set associated to* $B \blacksquare A$ *is*

$$P = (\sigma_{n+m} \xrightarrow{\lambda} \sigma_n \otimes \sigma_m \xrightarrow{B \times A} Set \times Set \xrightarrow{-\times-} Set),$$

where λ *is the isomorphism*

$$(x_0, \dots, x_{n+m-1}) \mapsto ((x_m, \dots, x_{n+m-1}), (x_0, \dots, x_{m-1}))$$

and where the last functor is the (cartesian) product functor.

PROOF. We will use the following facts:

- If K and K' are categories with associated realizations K, K' from σ in Set, then the realization associated to the product category $K \times K'$ is

$$(K, K'): \sigma \to Set: x \mapsto K(x) \times K'(x).$$

- If E is a set, the discrete category E^{dis} admits as its associated realization $E\hat{}: \sigma \to Set$, where $E\hat{}$ is the functor «constant on E ».

Now, we have the functor $P: \Sigma_{n+m} \to Set$ defined by:

$$(x_0, \dots, x_{n+m-1}) \mapsto B(x_m, \dots, x_{n+m-1}) \times A(x_0, \dots, x_{m-1}).$$

The composite P^i:

$$\Sigma \xrightarrow{\delta^i_{n+m}} \Sigma_{n+m} \xrightarrow{P} Set$$

is defined by:

1° $x \mapsto B(2, \dots, 2) \times A(\underbrace{2, \dots, x, \dots, 2}_{i\text{-th position}}) = \underline{B} \times A(\delta^i_m(x))$,

if $0 \leqslant i < m$, so that P^i is then the realization from σ associated to the product category $\underline{B}^{dis} \times A^i$;

2° $x \mapsto B(\underbrace{2, \dots, x, \dots, 2}_{j\text{-th position}}) \times A(2, \dots, 2) = B(\delta^j_n(x)) \times \underline{A}$,

if $m \leqslant i = j + m < n + m$, so that P^{j+m} is the realization associated to the product category $B^j \times \underline{A}^{dis}$.

Hence, the realization associated to $B \blacksquare A$ is $P: \sigma_{n+m} \to Set$. ∇

COROLLARY. *The $(n+m)$-fold category* K *whose associated realization is*

$$\sigma_{n+m} \xrightarrow[\sim]{ass.} \sigma_n \otimes \sigma_m \xrightarrow{B \times A} Set \times Set \xrightarrow{- \times -} Set$$

is deduced from $A \blacksquare B$ *by the «symmetry isomorphism»* $(a, b) \mapsto (b, a)$.

If K is a category, for each object e of K we denote the partial Hom functor by $K(e, -): K \to Set$.

PROPOSITION 2. *If $m < n$, the realization in Set associated to the $(n$-$m)$-fold category $Hom(A, B)$ is*

$$H = (\sigma_{n-m} \xrightarrow{\hat{B}} Cat_m \xrightarrow{Cat_m(A, -)} Set),$$

where $\hat{B}$ is the $(n$-$m)$-fold category in Cat_m associated to B *(by* A-3*); it is equivalent to the realization*

$$H' = (\sigma_{n-m} \xrightarrow{\hat{B}} Cat_m \xrightarrow{\eta_m} Set^{\sigma_m} \xrightarrow{Set^{\sigma_m}(A, -)} Set)$$

(where η_m is the equivalence defined in A-2).

PROOF. $\hat{B}$ is a realization and a partial Hom functor preserves limits, so that H and H' are realizations.

1° Since $\hat{B}(2,\dots,2) = \mathrm{B}^{0,\dots,m-1}$, the functor H maps $(2,\dots,2)$ onto $Cat_m(\mathrm{A},\mathrm{B}^{0,\dots,m-1})$, which is the set of multiple functors from A to B. For $0 \leqslant j < n-m$, let us consider the category H^j whose associated realization is:

$$\sigma \xrightarrow{\delta^j_{n-m}} \sigma_{n-m} \xrightarrow{\hat{B}} Cat_m \xrightarrow{Cat_m(\mathrm{A},-)} Set.$$

The composite of the two first functors is defined by:

$$\overset{\alpha}{\underset{\beta}{\leftleftarrows}} \xleftarrow{\kappa} \quad\longmapsto\quad |\mathrm{B}|^{0,\dots,m-1} \overset{\alpha^{m+j}}{\underset{\beta^{m+j}}{\leftleftarrows}} \mathrm{B}^{0,\dots,m-1} \xleftarrow{\kappa^{m+j}} (\mathrm{B}*_{m+j}\mathrm{B})^{0,\dots,m-1}$$

It follows that the composition map of H^j is

$$Cat_m(\mathrm{A},\kappa^{m+j}) : Cat_m(\mathrm{A},(\mathrm{B}*_{m+j}\mathrm{B})^{0,\dots,m-1}) \to Cat_m(\mathrm{A},\mathrm{B}^{0,\dots,m-1}) \;;$$

an element of $Cat_m(\mathrm{A},(\mathrm{B}*_{m+j}\mathrm{B})^{0,\dots,m-1})$ is identified with a couple (f',f) of multiple functors from A to B such that $\alpha^{m+j}f' = \beta^{m+j}f$; by $Cat_m(\mathrm{A},\kappa^{m+j})$, it is mapped onto

$$\kappa^{m+j}\circ(f',f) : \mathrm{A} \to \mathrm{B} : a \mapsto f'(a) \circ_{m+j} f(a),$$

which is equal to the composite $f' \circ_j f$ in $Hom(\mathrm{A},\mathrm{B})^j$. Therefore, we have $\mathrm{H}^j = Hom(\mathrm{A},\mathrm{B})^j$ for each j, and H is the realization associated to the $(n-m)$-fold category $Hom(\mathrm{A},\mathrm{B})$.

2° H' is equivalent to H. Indeed, let A' be an m-fold category and $A' : \sigma_m \to Set$ the associated realization. The composite

$$Cat_m \xrightarrow{\eta_m} Set^{\sigma_m} \xrightarrow{Set^{\sigma_m}(A,-)} Set$$

maps A' onto the set $Set^{\sigma_m}(A,A')$ of natural transformations from A to A', which is in 1-1 correspondence with the set $Cat_m(\mathrm{A},\mathrm{A}')$ of m-fold functors from A to A'. Hence the above composite is equivalent to

$$Cat_m(\mathrm{A},-) : Cat_m \to Set.$$

It follows that H' is equivalent to H. ∇

REMARK. The reason for introducing H' in the above proposition is that it is constructed by using only realizations associated to A and B (while A itself remains in H). Propositions 1 and 2 suggest definitions of the square product and of the functor Hom for general internal multiple sketched structures; in this way all the results of the present paper may be extended, as will be shown in a subsequent paper.

C. An application.

This Section is devoted to prove that Cat_n is «generated from $2^{\blacksquare n}$ by colimits».

We denote by $Y_n : \Sigma_n^{op} \to Set^{\Sigma_n}$ (= category of natural transformations from Σ_n into Set) the Yoneda embedding, which maps an object u of Σ_n onto the partial Hom functor $\Sigma_n(u,-): \Sigma_n \to Set$. It is known [6, 5] that Y_n defines a σ_n-costructure in Set^{σ_n} (i. e., a realization

$$Y_n : \sigma_n \to (Set^{\sigma_n})^{op}),$$

called the *Yoneda* σ_n-costructure, denoted by $Y_n : \sigma_n^{op} \to Set^{\sigma_n}$. Since Cat_n is equivalent to Set^{σ_n}, there is also a canonical σ_n-costructure in Cat_n, defined by:

$$Y'_n = (\sigma_n^{op} \xrightarrow{Y_n} Set^{\sigma_n} \xrightarrow{\zeta_n} Cat_n),$$

where ζ_n is the canonical equivalence (see A-2).

In particular, if $n = 1$, the σ-costructure Y'_1 in Cat maps the integer q, for $q = 1, 2, 3, 4$, onto the category q defining the usual order on $q = \{ 0, \dots, q-1 \}$ (see Proposition 9-0 [5]).

More generally, we have the following result, used in Proposition 4.

PROPOSITION 3. *The canonical σ_n-costructure Y'_n in Cat_n maps an object $(q_0, \dots, q_{n-1})$ of Σ_n onto an n-fold category isomorphic with*

$$q_{n-1} \blacksquare (\dots \blacksquare q_0) .$$

PROOF. The proof is by induction. As said above, the assertion is true for $n = 1$. Let us assume it is true for $(n-1)$-fold categories. Let u be an object $(q_0, \dots, q_{n-1})$ of Σ_n ; by Y'_n, it is mapped onto the n-fold category whose associated realization is $\Sigma_n(u,-): \sigma_n \to Set$. As $\Sigma_n = \Sigma_{n-1} \times \Sigma$, the partial Hom functor $\Sigma_n(u,-)$ is equal to the composite

$$\Sigma_n = \Sigma_{n-1} \times \Sigma \xrightarrow{\Sigma_{n-1}((q_0,\dots,q_{n-2}),-)\times\Sigma(q_{n-1},-)} Set \times Set \xrightarrow{-\times-} Set.$$

The induction hypothesis indicates that

$$\Sigma_{n-1}((q_0,\dots,q_{n-2}),-): \sigma_{n-1} \to Set$$

is the realization associated to an $(n-1)$-fold category isomorphic with

$$q_{n-2} \blacksquare (\dots \blacksquare q_0),$$

and $\Sigma(q_{n-1},-): \sigma \to Set$ is associated to q_{n-1}. Then Corollary, Proposition 1 asserts that the n-fold category whose associated realization is the above composite (equal to $\Sigma_n(u,-)$) is isomorphic with

$$q_{n-1} \blacksquare (q_{n-2} \blacksquare (\dots \blacksquare q_0)).$$

This achieves the proof by induction. ∇

PROPOSITION 4. *Cat_n is the inductive closure of* $\{2^{\blacksquare^n}\}$.

PROOF. In C-0 [5], it is proved that Σ is the Γ-closure of $\{2\}$ (where Γ is the set of distinguished cones of σ), so that by Proposition 7-0 [5] it follows that Set^{σ_n} is the inductive closure of $\{Y_n(2,\dots,2)\}$. Since

$$Y'_n = (\sigma_n^{op} \xrightarrow{Y_n} Set^{\sigma_n} \xrightarrow{\zeta_n} Cat_n),$$

where ζ_n is an equivalence, Cat_n is the inductive closure of

$$\{\zeta_n(Y_n(2,\dots,2)) = Y'_n(2,\dots,2)\}.$$

By Proposition 3, $Y'_n(2,\dots,2)$ is isomorphic with $\underbrace{2 \blacksquare (\dots \blacksquare 2)}_{n \text{ times}}$, which is isomorphic with

$$2^{\blacksquare^n} = \underbrace{(2 \blacksquare \dots) \blacksquare 2}_{n \text{ times}}.$$

More precisely, it is shown that the subcategory image of Y'_n is the pushout closure of $\{2^{\blacksquare^n}\}$, because q_j, for $q_j = 1, 2, 3, 4$, is deduced from 2 by pushouts [5], $q_j \blacksquare -$ preserve pushouts and $Y'_n(q_0,\dots,q_{n-1})$ is isomorphic to $q_{n-1} \blacksquare (\dots \blacksquare q_0)$. Then an n-fold category B is the colimit of the composite of Y'_n with the opposite of the discrete fibration $K_B \to \Sigma_n$ corresponding to the realization $B: \sigma_n \to Set$ associated to B. ∇

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