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COMPACT AND SMALL RANK PERTURBATIONS OF CONFORMALLY SYMPLECTIC STRUCTURES

by Heikki HAAHTI

I. INTRODUCTION AND SUMMARY.

1. In what follows almost symplectic C^3-manifolds $(\mathfrak{M}, \Omega)$ modeled on Banach spaces and with a C^2-fundamental form Ω are considered. The tensorfield Ω defines, by definition of the almost symplecticity, for each $x \in \mathfrak{M}$, a skew-symmetric bounded bilinear form $\Omega(x): \mathfrak{M}_x \times \mathfrak{M}_x \to R$ of the Banach metrisable tangent space $\mathfrak{M}_x$, such that the following conditions of «strong regularity» holds [1]:

The linear and bounded map of the Banach space $\mathfrak{M}_x$ into its dual space [2] $\mathfrak{M}_x^* = \mathfrak{L}(\mathfrak{M}_x; R)$ given by

$$\mathfrak{M} \ni h \to i(h)\Omega(x) = \{ k \to \Omega(x) h k \}$$

is bijective.

1) Instead of «strong regularity» the notions «regularity» and «non-singularity» also are used in the litterature. If instead of bijectivity merely injectivity of the map in question is postulated, the term «weak regularity» is used [1]. Strong and weak regularity are equivalent conditions in the finite-dimensional case, implying that the dimension of $\mathfrak{M}$ must be even.

2) If $A, B, \dots, C$ and D are $m+1$ given Banach spaces, the symbol

$$\mathfrak{L}(A \times B \times \dots \times C; D)$$

denotes the Banach space of all bounded m-linear functions $M: A \times B \times \dots \times C \to D$; the values of M are denoted sometimes without parenthesis, writing

$$M(h, k, \dots, l) = M h k \dots l$$

if no misunderstanding can arise. Of the topologies in $\mathfrak{L}(A \times B \times C. \times C; D)$ only that one is used which is given by the supremum norm $|M| = \sup |M h k \dots l|$ (for $|h| = |k| = \dots = |l| = 1$). In the case $A = B = \dots = C = E$ we write

$$\mathfrak{L}^m(E; D) = \mathfrak{L}(A \times B \times \dots \times C; D)$$

and the closed linear subspace in $\mathfrak{L}^m(E; D)$ consisting of all skew-symmetric m-linear functions with values in D is denoted by $\mathfrak{L}^m_a(E; D)$.

By Banach's theorem this linear map is then a linear homeomorphism. The postulate of the strong regularity implies in particular that all tangent spaces $\mathfrak{M}_x$ and hence also all parameter spaces of the manifold $\mathfrak{M}$ are *reflexive* Banach-metrisable vector spaces (see [22], 5).

2. Two almost symplectic manifolds $(\mathfrak{M}, \Omega)$ and $(\bar{\mathfrak{M}}, \bar{\Omega})$ are *conformally diffeomorphic* (resp. *isometric*) if there exists a C^3-diffeomorphism

$$y: (\mathfrak{M}, \Omega) \to (\bar{\mathfrak{M}}, \bar{\Omega})$$

satisfying

$$(1) \qquad e^{a(x)}\Omega(x) = y_*(x)\bar{\Omega}\,, \quad \text{for all } x \in \mathfrak{M}\,,$$

where a is a real function (resp. where $a(x) = 0$) and where y_* denotes the «pull-back» from y.

An almost symplectic manifold $(\mathfrak{M}, \Omega)$ is *locally conformally symplectic* (and shortly «conformally symplectic» or «conformally flat»), if a neighborhood $\mathfrak{U}$ of every point $x_0 \in \mathfrak{M}$ is conformally diffeomorphic to a neighborhood in some Banach space (E, A), where

$$A = <.,.>: E \times E \to \mathrm{R}$$

is a strongly regular, skew-symmetric bilinear form. Denoting by

$$dy = y'(x): \mathfrak{M}_x \to E_{y(x)} = E$$

the derivative of the map y, the conformality reads explicitly:

$$(1') \qquad <y'(x)h,\ y'(x)k> = e^{a(x)}\Omega(x)hk\,,$$

for all $h, k \in \mathfrak{M}_x$ and $x \in \mathfrak{U}$.

In the case $a(x) = 0$ of isometry, $(\mathfrak{M}, \Omega)$ is *symplectic*. Evidently the Banach space (E, A) above is symplectic. In the case where the dimension $dim\,\mathfrak{M} = 2n$ is finite, $(\mathfrak{M}, \Omega)$ is conformally symplectic (resp. symplectic) iff near every point local coordinates can be introduced, such that Ω transforms to

$$e^{-a(x)}(dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \dots + dx_n \wedge dy_n),$$

where a is a real function (resp. where $a(x) = 0$).

3. Since $e^{a(x)}\Omega(x)$ can be transformed on a conformally symplectic ma-

nifold $(\mathfrak{M},\Omega)$ to a form A not depending on the point $y(x)$, we have in particular for the outer derivative:

$$d(e^{a(x)}\Omega(x))=0.$$

Hence, by *Poincaré Lemma*, the conformal flatness of $(\mathfrak{M},\Omega)$ implies the existence of «integration factors» $\lambda(x)=e^{a(x)}$ such that the differential equation $dX=\lambda\Omega$ has locally defined 1-forms X as solutions.

Also the conformal flatness implies that the «distribution» of isotropic linear subspaces is «integrable» [3] on $(\mathfrak{M},\Omega)$. Indeed, given a conformal map $y:(\mathfrak{M},\Omega)\to(E,A)$ defined on a neighborhood $\mathfrak{U}$ of $x_0\in\mathfrak{M}$, an isotropic linear subspace $\mathfrak{I}_{x_0}\subset(\mathfrak{M}_{x_0},\Omega(x_0))$ of the tangent space at x_0 is mapped by the value $y'(x_0)$ of the derivative to an isotropic linear subspace $N\subset(E,A)$. Furthermore, the intersection of $V=y(\mathfrak{U})\subset E$ with the plane $y(x_0)+N=J$ is mapped by y^{-1} to a submanifold

$$\mathfrak{N}=y^{-1}(V\cap J)\subset\mathfrak{M}\quad\text{of}\quad(\mathfrak{M},\Omega)$$

for which every tangent space $\mathfrak{N}_x$ is an isotropic subspace of $(\mathfrak{M}_x,\Omega(x))$, since $y'(x)\mathfrak{N}_x=N$, and $\mathfrak{I}_{x_0}$ is the tangent space of $\mathfrak{N}$ at x_0.

4. The well-known tensorial condition - vanishing of the Riemannian curvature tensor - for Riemannian and pseudo-Riemannian manifolds to be locally euclidean has the *Theorem of Darboux* as an analogue in the almost symplectic geometry. This latter reads:

An almost symplectic manifold $(\mathfrak{M},\Omega)$ is locally flat, that is, symplectic, iff the outer derivative $d\Omega$ vanishes.

This for the finite-dimensional case classical theorem has been proved by *A. Weinstein* [21, 22] for infinite-dimensional manifolds, using a method due to *J. Moser* [17].

The vanishing of the conformal curvature tensor of Hermann Weyl again, which characterizes the conformal flatness of Riemannian and pseudo-

[3] By this we mean the following: For each pair $(x_0,\mathfrak{I}_{x_0})$, where $x_0\in\mathfrak{M}$ and where $\mathfrak{I}_{x_0}$ is a $\Omega(x_0)$-isotropic linear subspace of the tangent space $\mathfrak{M}_{x_0}$, there exists a submanifold $\mathfrak{N}\subset\mathfrak{M}$ having isotropic tangent spaces $\mathfrak{N}_x$ and with $x_0\in\mathfrak{N}$, $\mathfrak{N}_{x_0}=\mathfrak{I}_{x_0}$. In such a case one also calls the 2-form Ω «integrable» [4].

Riemannian manifolds [23, 6] corresponds to theorems due to *Lee, Ehresmann* and *Libermann* in the finite-dimensional almost symplectic geometry [4,13,14,16]. One of the classical results can be stated as follows:

An almost symplectic $2n$-dimensional manifold $(\mathfrak{M}, \Omega)$ with $dim\,\mathfrak{M} > 4$ is conformally symplectic iff the outer derivative $d\Omega$ is «divisible» by Ω, that is, iff there exists a one-form R on $\mathfrak{M}$ such that in the equation

$$d\Omega = R \wedge \Omega + C$$

we have

$$(1'') \qquad C = 0\ .$$

Furthermore, if such a factor R exists, it is (in all cases $dim\,\mathfrak{M} \geqslant 4$) uniquely determined by the expression $R(x) = W(x)$, where

$$(2) \qquad 2(n-1)W(x) = Trace\ d\Omega(x)$$

is defined by means of tensor-contraction: For all points $x \in \mathfrak{M}$ and tangent vectors $h \in \mathfrak{M}_x$,

$$2(n-1)W(x)h = Trace\ \{(k,l) \to d\Omega(x)(h,k,l)\}\ .$$

Here the «trace» of the bilinear form

$$B = \{(k,l) \to d\Omega(x)(h,k,l)\} \in \mathfrak{L}^2(\mathfrak{M}_x; \mathrm{R})$$

is defined in a familiar way as that of the linear transformation

$$T = \Omega^{-1} B \in \mathfrak{L}(\mathfrak{M}_x; \mathfrak{M}_x)$$

coming from B by «raising the index» with Ω:

$$B(k,l) = \Omega(x)(Tk, l) \quad \text{for all} \quad k,\ l \in \mathfrak{M}_x \ ^{4)}.$$

Thus, the conformally symplectic case can be equivalently characterized

4) Denoting in a local chart representation by $\Omega_{ij} = \Omega(x)(e_i, e_j)$ the values of the bilinear form $\Omega(x)$ for basis vectors e_i $(i = 1, 2, \ldots, 2n)$ and writing

$$(d\Omega)_{ijk} = \frac{\partial \Omega_{ij}}{\partial x^k} + \frac{\partial \Omega_{jk}}{\partial x^i} + \frac{\partial \Omega_{ki}}{\partial x^j}\ ,$$

the formula (2) is written in the index notation

$$(2') \qquad 2(n-1)W_i = (d\Omega)_{ij}{}^{j}\ ,$$

where $d\Omega_{ij}{}^{k} = (d\Omega)_{ijs}\Omega^{sk}$, $\Omega_{ij}\Omega^{jk} = \delta_i^k$ = Kronecker symbol.

by the vanishing of the tensor of C.H. Lee:

$$(2') \qquad L = d\Omega - W \wedge \Omega ,$$

an analogon of the Weyl conformal curvature tensor. Furthermore, this condition is equivalent both to the existence of an «integration factor» near every point and to the «integrability of the distribution» given by the isotropic linear subspaces $\mathcal{N}_x \subset \mathcal{M}_x$ (indeed, even to the integrability of the distribution given merely by the *maximal* isotropic linear subspaces). If the dimension is $2n = 4$, then $L = 0$ and $W = R$ for *all* almost symplectic manifolds and $(\mathcal{M}, \Omega)$ is conformally symplectic iff:

$$(3) \qquad dR = 0.$$

5. In the present generalizations of finite-dimensional geometry to the infinite-dimensional one, the key point of the solution goes back to the way in which one avoids the use of the usual trace-operator occuring in (2). The solution is based on a simple lemma according to which a bounded linear operator T of an infinite-dimensional real Banach space has at most one representation $T = \lambda I + K$, where I is a given non-compact and bounded operator (in particular the identity operator occurs in the sequel) and where K is compact, λ being a real number. It comes out that the operator

$$\frac{1}{2(n-1)} \, Trace$$

can be replaced by the operator

$$tr: T = \lambda I + K \to \lambda ,$$

the «reduced trace», to get the generalization in question. Furthermore, if in the decomposition $T = \lambda I + K$ the compact operator K has in particular a finite rank (or if it is, more generally, nuclear), then the usual trace of K exists and one has besides of the above operator «*tr*» the linear operator

$$\delta: T = \lambda I + K \to Trace K.$$

The operators tr and δ induce corresponding «contraction operators» on multilinear forms in an analogous way as does the usual trace-operator in finite-dimensional tensor calculus. In particular there exists on an infinite-dimensional almost symplectic manifold $(\mathfrak{M}, \Omega)$ at most one representation

$$(4) \qquad d\Omega(x) = R(x) \wedge \Omega(x) + C(x)$$

of $d\Omega$ as a «compact perturbation» of a multiple of Ω, whereby

$$R(x) = tr\, d\Omega(x),$$

and we prove that $(\mathfrak{M}, \Omega)$ is conformally symplectic iff $C = 0$ (see Theorem 1, number 16). Also this condition is equivalent to the existence of an integration factor (see Corollary 3, number 18) and to the integrability of the isotropic distribution, the latter fact being proved in [11].

6. The unique representation $T = \lambda I + C$ of a linear operator T as a compact perturbation of λI (which was the basis for the above mentioned solution in the infinite-dimensional geometry) has the following simple lemma as «an analogon» in the finite-dimensional linear algebra: There exists in a m-dimensional vector space E at most one representation $T = \lambda I + C$ of a linear transformation $T \in \mathfrak{L}(E;E)$, where I is the identity-operator and where $rank\, C < \frac{m}{2}$. For such a T we have the two operators

$$tr\colon T = \lambda I + C \to \lambda \quad \text{and} \quad \delta\colon T \to Trace\, C$$

related to the usual trace of T by

$$Trace\, T = m\, tr T + \delta T,$$

the domain of these two operators being, however, not a linear subspace of $L(E;E)$ anymore. These operators induce corresponding operators for multilinear forms. In particular there exists on a $2n$-dimensional almost symplectic manifold $(\mathfrak{M}, \Omega)$ at most one representation (4) of $d\Omega = R \wedge \Omega + C$ such that «the perturbation C has small-rank», that is, for all $x \in \mathfrak{M}$,

$$(4') \qquad rank_{(2,3)} C(x) < n - 1,$$

where by definition

$$rank_{(2,3)} C(x) = \max_{x \in \mathfrak{M}} rank\, i(h) C(x)$$

(see number 30). If an almost symplectic $2n$-dimensional manifold $(\mathfrak{M}, \Omega)$

admits such a small-rank perturbation $C = d\Omega - R\wedge\Omega$ then the perturbation tensor is related to the classical «conformal curvature tensor»

$$L = d\Omega - W\wedge\Omega$$

given in (2'), by

$$L = C - \frac{1}{2(n-1)}(\mathit{Trace}\, C)\wedge\Omega\,. \tag{5}$$

Here the expression $\mathit{Trace}\, C$ is defined analogously to the right-hand side of (2). In the case of conformal symplecticity, $C = L = 0$, $W = R$.

7. To get a uniform interpretation of the above mentioned classical results and their generalizations, a family, say ρ, of almost symplectic C^3-manifolds $(\mathfrak{M}, \Omega)$, the manifolds with «perturbed conformally symplectic structure», is introduced as follows: an infinite-dimensional (resp. finite-dimensional) space $(\mathfrak{M}, \Omega)$ belongs to ρ iff the outer derivative of the C^2 fundamental form Ω admits a representation $d\Omega = R\wedge\Omega + C$ as a compact (resp. small-rank) perturbation of a multiple of Ω. It comes out (Proposition 4, number 42) that every almost symplectic C^3-manifold $(\bar{\mathfrak{M}}, \bar{\Omega})$ which is conformally diffeomorphic to a $(\mathfrak{M}, \Omega)\in\rho$ also belongs to ρ, whereby the characteristic tensor fields R and C obey simple laws of transformation. The family ρ divides into conformally invariant equivalence classes. In particular all conformally symplectic spaces $(\mathfrak{M}, \Omega)$ belong to ρ. They are included in the bigger conformally invariant class given by the tensorial condition

$$0 = D = dC - R\wedge C\;;$$

these are just those spaces $(\mathfrak{M}, \Omega)$ of ρ where $dR = 0$ or, equivalently, which are (locally) conformally diffeomorphic with a «perturbed symplectic space» $(\bar{\mathfrak{M}}, \bar{\Omega})$, that is, with an almost symplectic space for which $d\bar{\Omega} = \bar{C}$ is compact (resp. has a small-rank; see Proposition 5, number 44).

In Section 3 and in Appendix we present some lemmas on multilinear forms in Banach spaces which may have an interest of their own.

For the sake of self-containedness, we have preferred an elementary and detailed tratment rather than shorter considerations with more references. That part of the material presented here which concerns the infinite-dimensional case of the conformal flatness theorem alone is based on [9], the geometry of the compact perturbations of infinite-dimensional conformally flat structures is - with some improvements - from [5], and the material was completed with the finite-dimensional case of the small-rank perturbations later.

8. ACKNOWLEDGEMENT. By means of financial aid of the *Finnish Academy* I was able to visit the Warwick University in May 1974, and working then with [5], I had opportunity to inspiring discussions with *J. Eells* and *D. Elworthy*. This remark I add with gratitude.

2. THE THEOREM ON CONFORMAL SYMPLECTICITY.

9. Suppose $(\mathfrak{M},\Omega)$ is a conformally symplectic (or as we also called it, a conformally flat) C^3-manifold with the C^2-fundamental form Ω, and with

$$2 < dimension\,\mathfrak{M} = 2n \leqslant \infty .$$

Given $x_0 \in \mathfrak{M}$ there thus exists a Banach space E with a bilinear, strongly regular skew-symmetric form

$$A = <.,.>: E \times E \to \mathrm{R},$$

a local C^3-diffeomorphism $y: \mathfrak{M} \to E$ and a real function α, such that in a neighborhood $\mathfrak{U}$ of x_0 the derivative $y' = dy$ of y satisfies

$$(6') \qquad <y'(x)h,\, y'(x)k> = e^{\alpha(x)}\Omega(x)hk$$

for all tangent vectors $h, k \in \mathfrak{M}_x$ and $x \in \mathfrak{U}$. Since for any given $\bar{x} \in \mathfrak{U}$, two C^2-vector fields $\bar{h}, \bar{k}$ can be chosen so that $\Omega(x)[\bar{h}(x), \bar{k}(x)] \neq 0$ in a neighborhood of x, the function

$$x \to \alpha(x) = ln\frac{<y'(x)\bar{h}(x),\, y'(x)\bar{k}(x)>}{\Omega(x)[\bar{h}(x),\bar{k}(x)]}$$

is necessarily C^2. We may apply the usual rules of differentiation by taking on both sides of (6') the outer derivative d. The symmetry of the second (usual) derivative then implies

$$0 = d(\overset{\bullet}{e^{a}}\Omega) = e^{a}(da\wedge\Omega + d\Omega),$$

so for all $x \in \mathfrak{U}$,

$$(6) \qquad 0 = da(x)\wedge\Omega(x) + d\Omega(x).$$

10. We get from here conditions for the fundamental form Ω alone by means of the following

LEMMA 1. *There exists at every point x of an almost symplectic C^3-manifold $(\mathfrak{M},\Omega)$ with the C^2-fundamental form Ω at most one linear function $R(x): \mathfrak{M}_x \to \mathrm{R}$ in the Banach-metrisable tangent space $\mathfrak{M}_x$, satisfying*

$$(6'') \qquad d\Omega(x) = R(x)\wedge\Omega(x).$$

If such a function $R(x)$ exists for every $x\in\mathfrak{M}$ it is necessarily bounded, the mapping $\mathfrak{M}\ni x\to R(x)$ defines a C^1 one form R on $\mathfrak{M}$ and when the dimension $2n$ is finite, $R = W$ is given by (2) number 4.

In the finite-dimensional case, Lemma 1 follows by means of tensor contraction on both sides of (6"). In the case $dim\,\mathfrak{M} = \infty$, it follows as a corollary of Proposition 3, number 33.

11. The space $(\mathfrak{M},\Omega)$ being conformally symplectic it follows from (6) that the one form R exists and satisfies the equation

$$(7) \qquad -R = da\,,$$

so we have the representation

$$(7') \qquad d\Omega = R\wedge\Omega + C\,,$$

with

$$(7'') \qquad C = 0$$

for $d\Omega$ and the condition

$$(8) \qquad dR = 0\,.$$

The necessary conditions (7") and (8) are conditions for Ω alone.

12. We go on proving that in the case $4 < dim\,\mathfrak{M} \leqslant \infty$ the existence of the linear function $R(x): \mathfrak{M}_x \to \mathrm{R}$ with

$$C(x) = d\Omega(x) - R(x)\wedge\Omega(x) = 0 \quad \text{for all} \quad x \in \mathfrak{M}$$

and in the case $dim\,\mathfrak{M} = 4$ the condition $dR \overset{\bullet}{=} 0$ is sufficient for the conformal symplecticity of $(\mathfrak{M}, \Omega)$. By Lemma 1 the one form R is C^1, so exterior derivatives can be taken on both sides of the equation $C = d\Omega - R\wedge\Omega$. Since $dd\Omega = 0$, we get

$$\begin{aligned} dC &= - dR\wedge\Omega + R\wedge d\Omega \\ &= -dR\wedge\Omega + R\wedge(R\wedge\Omega + C) \\ &= -dR\wedge\Omega + (R\wedge R)\wedge\Omega + R\wedge C, \end{aligned}$$

so by $R\wedge R = 0$,

$$dC = - dR\wedge\Omega + R\wedge C. \tag{9}$$

13. Now in the case $dim\,\mathfrak{M} > 4$ our condition $C = 0$ implies $dR = 0$, the equation which was postulated if $dim\,\mathfrak{M} = 4$. We namely have from (9) for all $x \in \mathfrak{M}$:

$$dR(x)\wedge\Omega(x) = 0\,, \tag{9'}$$

and since the dimension of the tangent space is $dim\,\mathfrak{M}_x = dim\,\mathfrak{M} > 4$, and since $\Omega(x)$ is regular, it follows from (9') that $dR = 0$ (see Appendix, Lemma 9, number 46).

14. It follows that the initial value problem

$$d\alpha(x) = -R(x), \quad \alpha(x_0) = 0$$

for an unknown real function $\alpha : \mathfrak{M} \to \mathrm{R}$ has a unique solution in a neighborhood $\mathfrak{U} \subset \mathfrak{M}$ of $x_0 \in \mathfrak{M}$ (where $\mathfrak{U} = \mathfrak{M}$ can be chosen, if $\mathfrak{M}$ is simply connected). Define for all x: $\bar{\Omega}(x) = e^{\alpha(x)}\Omega(x)$. Then in $\mathfrak{U}$,

$$d\bar{\Omega} = e^{\alpha}(d\alpha\wedge\Omega + d\Omega) = e^{\alpha}(-R\wedge\Omega + d\Omega),$$

so

$$d\bar{\Omega} = e^{\alpha} C. \tag{10}$$

15. In the case $dim\,\mathfrak{M} > 4$ we have here by hypothesis $C = 0$. If, however, $dim\,\mathfrak{M} = 4$, this equation *holds identically* for all almost symplectic manifolds, whether conformally symplectic or not [5]. Consequently, for all cases in question we get from (10):

$$(11) \qquad d\bar{\Omega} = 0.$$

16. Now the theorem of *Darboux-Moser-Weinstein* comes to use. Choose a chart (V, ψ) around $x_0 \in \mathfrak{M}$ with the Banach space E as a parameter space. The representant

$$(12) \qquad A = \psi_*^{-1} \bar{\Omega}(x_0)$$

of Ω at $u_0 = \psi(x_0)$ defines in E a skew-symmetric and strongly regular bilinear form. By the theorem the conditions (11) and (12) imply that there exists a local C^3-diffeomorphism y of $(\mathfrak{M}, \Omega)$ in (E, A) with $y(x_0) = 0$, say, such that the isometry condition $\bar{\Omega} = y_* A$ holds. Since

$$\bar{\Omega}(x) = e^{\alpha(x)} \Omega(x),$$

the diffeomorphism y is, together with the real function α, a solution of our conformal mapping problem. Summing up we get

THEOREM 1. *Let $(\mathfrak{M}, \Omega)$ be an almost symplectic C^3-manifold modeled on Banach spaces, with the fundamental form Ω of class C^2 and with*

$$4 < \dim \mathfrak{M} = 2n \leqslant \infty.$$

Then $(\mathfrak{M}, \Omega)$ is locally conformally symplectic if and only if the following condition (C) is satisfied:

(C) For every point $x \in \mathfrak{M}$ there exists a linear function $R(x): \mathfrak{M}_x \to \mathrm{R}$ defined in the Banach-metrisable tangent space $\mathfrak{M}_x$ and satisfying

$$(13) \qquad d\Omega(x) = R(x) \wedge \Omega(x) + C(x)$$

with

$$(14) \qquad C(x) = 0.$$

Furthermore, if such an $R(x)$ exists, it is uniquely given and bounded, de-

5) Indeed, denoting $\mathfrak{M}_x = E, \Omega(x) = A$, we have in the 4-dimensional symplectic vector space (E, A) the nonvanishing determinant function $D = A \wedge A$ and the skew-symmetric trilinear functions $M \in \mathfrak{L}^3_a(E; \mathrm{R})$ are in (1-1)-correspondence with the vectors $v_M \in E$ by $M = i(v_M)D$, as is directly verified taking for instance an «orthonormal» basis $(e_i)_{i=1}^4$ with $A(e_1, e_2) = 1 = A(e_3, e_4)$ and $A(e_i, e_j) = 0$, for $|i - j| \geqslant 2$. Developing $M = i(v_M)D = i(v_M)(A \wedge A)$ one gets $M = R \wedge A$, with $R \in E^*$, so in particular for $d\Omega = M$, $d\Omega = R \wedge \Omega$.

fining on $\mathfrak{M}$ *a closed one-form* $\mathfrak{M} \ni x \to R(x)$, *which in the finite-dimensional case has for all* $x \in \mathfrak{M}$ *and* $h \in \mathfrak{M}_x$ *the expression* [6])

$$(15) \quad R(x)h = \frac{1}{2(n-1)} \, Trace \, \{ (k,l) \to d\Omega(x)(h,k,l) \}.$$

In the case $\dim \mathfrak{M} = 4$ *the condition* (C) *is true for almost symplectic manifolds, and* $(\mathfrak{M}, \Omega)$ *is locally conformally symplectic iff* R *is closed:*

$$(C') \qquad\qquad dR = 0 .$$

17. In the formulation of Theorem 1 the «conformal flatness» condition was given in the weak form, where the flat Banach space (E, A) was not a priori there. On the other hand, one may ask whether an almost symplectic manifold $(\mathfrak{M}, \Omega)$ is locally conformally diffeomorphic to a *given* symplectic Banach space (E, A).

By Darboux Theorem the question is equivalent to the following mapping problem: Can $(\mathfrak{M}, \Omega)$ be mapped locally and conformally on a given symplectic manifold $(\bar{\mathfrak{M}}, \bar{\Omega})$? From Theorem 1 it follows:

COROLLARY 1. *Suppose that of the two given almost symplectic* C^3*-manifolds* $(\mathfrak{M}, \Omega)$ *and* $(\bar{\mathfrak{M}}, \bar{\Omega})$ *the one, say* $(\bar{\mathfrak{M}}, \bar{\Omega})$, *is symplectic:* $d\bar{\Omega} = 0$. *Suppose furthermore that* $4 < \dim \mathfrak{M} \leqslant \infty$ *(resp. that* $4 = \dim \mathfrak{M}$ *). Given two points* $x_0 \in \mathfrak{M}$ *and* $\bar{x}_0 \in \bar{\mathfrak{M}}$, *a neighborhood of* x_0 *can be mapped by a conformal* C^3*-diffeomorphism*

$$y : (\mathfrak{M}, \Omega) \to (\bar{\mathfrak{M}}, \bar{\Omega}) \quad with \quad y(x_0) = \bar{x}_0$$

in $\bar{\mathfrak{M}}$ *iff the condition* (C) *(resp.* (C')*) of Theorem 1 holds and if furthermore the linear symplectic spaces* $(\mathfrak{M}_{x_0}, \Omega(x_0))$ *and* $(\bar{\mathfrak{M}}_{\bar{x}_0}, \bar{\Omega}(\bar{x}_0))$ *are isometrically isomorph.*

The last mentioned condition means that there exists a linear homeomorphism

$$T : \mathfrak{M}_{x_0} \to \bar{\mathfrak{M}}_{\bar{x}_0} \quad \text{such that} \quad T_* \bar{\Omega}(\bar{x}_0) = \Omega(x_0).$$

The proof of Corollary 1 is immediate (see for instance [5], page 27-28).

6) Here the trace of the bilinear function is defined as indicated in number 4.

Now suppose $\mathfrak{M}$ and $\bar{\mathfrak{M}}$ are modeled on Hilbert spaces. Then the tangent spaces $\mathfrak{M}_{x_0}$ and $\bar{\mathfrak{M}}_{\bar{x}_0}$ are Hilbert-metrisable and if they have same dimension (see [2], IV.4.15), they are isometrically isomorphic as Hilbert spaces. In this case $(\mathfrak{M}_{x_0}, \Omega(x_0))$ is symplectically isomorphic (see Appendix, number 48) with $(\bar{\mathfrak{M}}_{\bar{x}_0}, \bar{\Omega}(\bar{x}_0))$. The dimension of a manifold being equal to the common dimension of its tangent spaces, it follows from Corollary 1:

COROLLARY 2. *Suppose the manifolds* $(\mathfrak{M}, \Omega)$ *and* $(\bar{\mathfrak{M}}, \bar{\Omega})$ *are modeled on Hilbert spaces,* $d\bar{\Omega} = 0$ *and*

$$4 < \dim \mathfrak{M} = \dim \bar{\mathfrak{M}} \leqslant \infty \quad (\text{resp. } 4 = \dim \mathfrak{M} = \dim \bar{\mathfrak{M}}).$$

Then $(\mathfrak{M}, \Omega)$ *is locally conformally diffeomorphic with* $(\bar{\mathfrak{M}}, \bar{\Omega})$ *iff the condition* (C) *(resp. the condition* (C')*) of Theorem 1 holds.*

18. Recall that by Poincaré Lemma there are locally defined p-forms X satisfying $dX = Y$ iff the given $(p+1)$-form Y is closed: $dY = 0$. The solution of the conformal mapping problem at hand can be equivalently reformulated in the context of integration of forms, as follows.

COROLLARY 3. *Let* $(\mathfrak{M}, \Omega)$ *be an almost symplectic* C^3*-manifold with the* C^2*-fundamental form* Ω *and with* $4 < \dim \mathfrak{M} \leqslant \infty$ *(resp.* $4 = \dim \mathfrak{M}$*). Then, there exists in a neighborhood of every point* $x_0 \in \mathfrak{M}$ *a real function*

$$x \to \lambda(x) = e^{a(x)},$$

an «integration factor» such that the equation

$$dX(x) = \lambda(x)\,\Omega(x) \tag{16}$$

is locally integrable iff the condition (C) *(resp. condition* (C')*) of Theorem 1 holds.*

Indeed, if X is a solution of (16), then

$$0 = ddX = e^{a}\{ da \wedge \Omega + d\Omega \},$$

so

$$d\Omega = R \wedge \Omega + C \quad \text{with} \quad R = -da \quad \text{and} \quad C = 0.$$

On the other hand, in the numbers 12-15, we have seen that the conditions

$$d\Omega = R \wedge \Omega + C, \quad C = 0$$

imply the existence of an integration factor $\lambda(x) = e^{\alpha(x)}$ such that the form $\bar{\Omega}(x) = \lambda(x)\Omega(x)$ is closed: $d\bar{\Omega} = 0$. By Poincaré Lemma hence Corollary 3 is true.

19. As mentioned in the Introduction, the conformal flatness of $(\mathfrak{M}, \Omega)$ also is equivalent to the integrability of the distribution consisting of all maximal isotropic linear subspaces of the tangent spaces when $dim\,\mathfrak{M} > 4$. To show that the integrability implies the condition (C) of conformal flatness one needs a lemma on multilinear algebra, a generalization of a result due to *Lepage* and *Papy* [15, 19]. The proofs are given in [11].

3. LINEAR SPACE THEORY OF COMPACT AND SMALL-RANK PERTURBATIONS.

20. The trilinear form $C(x)$ in (13), being zero, is in particular «compact». To prove Lemma 1 in number 10 on which Theorem 1 was based, we first consider independently skew-symmetric trilinear forms in an infinite-dimensional Banach space, which are «compactly perturbed multiples» of a fundamental bilinear form. In fact, all the paragraph is not needed for the proof of Lemma 1 alone; also the generalizations of conformally symplectic spaces given in 5 as well as the proofs of [11] are based on Section 3.

21. A vector $h \in E$ of a given Banach space E defines a linear map

$$i(h): \mathfrak{L}^{n+1}(E; F) \to \mathfrak{L}^{n}(E; F)$$

between spaces of multilinear functions given by

$$(17) \qquad i(h)M = Mh = \{(k_1, \ldots, k_n) \to M h k_1 \ldots k_n\}$$

for all $M \in \mathfrak{L}^{n+1}(E; F)$ (F is a Banach space). For $h_1, \ldots, h_m \in E$, we write

$$(17') \qquad i(h_1, \ldots, h_m) = i(h_m) \circ i(h_{m-1}) \circ \ldots \circ i(h_1),$$

getting a linear map

$$i(h_1, \ldots, h_m): \mathfrak{L}^{n+m}(E; F) \to \mathfrak{L}^{n}(E; F)$$

with $|i(h_1, h_2, \dots, h_m)| \leqslant |h_1|\,|h_2| \dots |h_m|$.

Suppose there is given in the Banach space E a bounded bilinear form $A = \langle .\,,. \rangle$ satisfying the condition of strong regularity: The linear and bounded map

$$A_b : E \to E^* \quad \text{given by} \quad A_b = \{ x \to i(x).A \}$$

is a bijection. Then A_b^{-1} is bounded and we get for every $m = 1, 2, \dots$ a linear homeomorphism

$$L_m : \mathfrak{L}^m(E;\, E) \to \mathfrak{L}^{m+1}(E;\, \mathrm{R}),$$

which sends the m-linear form $N \in \mathfrak{L}^m(E;\, E)$ to the real-valued $(m+1)$-linear form $M = L_m N$ given by

$$(L_m N)(h_1, \dots, h_{m+1}) = \langle N h_1 \dots h_m,\, h_{m+1} \rangle = A_b(N h_1 \dots h_m) h_{m+1}$$

$(h_1, h_2, \dots, h_{m+1} \in E)$. The inverse map L_m^{-1} is given by

$$(L_m^{-1} M)(h_1, h_2, \dots, h_m) = A_b^{-1}\, i(h_1, h_2, \dots, h_m) M$$

for all $h_1, h_2, \dots, h_m \in E$, and we have

$$(18) \qquad |L_m| = |A_b| = |A| \qquad \text{and} \qquad (18') \qquad |L_m^{-1}| = |A_b^{-1}|$$

for all $m = 1, 2, \dots$. To be shorter we write henceforth, when possible, for all $m = 1, 2, \dots$:

$$(19) \qquad L_m = A_b = A, \quad L_m^{-1} = A_b^{-1} = A^{-1}.$$

This notation is applied in particular for the case $E = \mathfrak{M}_x$ = a tangent space and $A = \Omega(x)$ = the value of a fundamental tensor field.

22. We call a bilinear function $C : E \times E \to \mathrm{R}$ *compact* iff the corresponding linear map $C_b = \{ x \to i(x) C \}$ from E to the dual space E^* is compact [7]. The Banach spaces $\mathfrak{L}(E \times E;\, \mathrm{R})$ and $\mathfrak{L}(E;\, E^*)$ being isometric-

[7] In the above definition the compactness of C «with respect to the first argument» is merely postulated. However, the dual map $C_b^* \in \mathfrak{L}(E^{**};\, E^*)$ of the compact map C_b being compact and the linear map $C^b = \{ y \to C(.\,, y) \}$ being the restriction $C^b = C_b^*|\,E$ of C_b^* to $E \subset E^{**}$, also C^b is compact. Hence the above definition implies that C is in fact «compact with respect to both of its arguments».

ally isomorphic by the correspondence

$$\mathcal{L}(E\times E;\ \mathrm{R}) \ni B \to B_b = \{\, x \to i(x)B \,\}$$

and the set of all compact linear transformations $E \to E^*$ being a closed linear subspace in $\mathcal{L}(E;\ E^*)$, the set of all compact bilinear functions C of $\mathcal{L}(E\times E;\ \mathrm{R})$ is a closed linear subspace $\mathcal{C}(E\times E;\ \mathrm{R})$ in $\mathcal{L}(E\times E;\ \mathrm{R})$.

With the strongly regular bilinear form A we have on the other hand the linear homeomorphism

$$L_1^{-1} = A^{-1}:\ \mathcal{L}(E\times E;\ \mathrm{R}) \to \mathcal{L}(E;\ E),$$

given in number 21. If $T = A^{-1}B$ corresponds to $B \in \mathcal{L}(E\times E;\ \mathrm{R})$, then $A_b \circ T = B_b$, and since here $A_b = \{\, x \to i(x)A \,\}$ is a linear homeomorphism it follows that the bilinear function B is compact iff the corresponding linear transformation $T = A^{-1}B$ of E is.

23. Denoting

$$\mathcal{C}^2(E;\ \mathrm{R}) = \mathcal{C}^2 = \mathcal{C}(E\times E;\ \mathrm{R})$$

the space of all compact bilinear functions $E\times E \to \mathrm{R}$, the inverse image of $\mathcal{C}^2$ by the bounded linear map $i(h_1, \dots, h_m)$ given in number 21 is a closed linear subspace of $\mathcal{L}^{m+2}(E;\ \mathrm{R})$ for every $(h_1, \dots, h_m) \in E\times E\times\dots\times E$, and thus also the intersection

$$(20)\quad \mathcal{C}^{m+2} = \mathcal{C}^{m+2}(E;\mathrm{R}) = \bigcap_{(h_1,\dots,h_m)\in E\times\dots\times E} i(h_1, \dots, h_m)^{-1}\,\mathcal{C}^2$$

is a closed linear subspace in $\mathcal{L}^{m+2}(E;\ \mathrm{R})$, $m = 1, 2, \dots$. We say the elements $C \in \mathcal{C}^{m+2}$ are compact with respect to the pair $(m+1,\ m+2)$ of arguments [8].

[8] By the above definition a bounded $(m+2)$-linear form C belongs to $\mathcal{C}^{m+2}$ iff the bilinear function $i(h_1, \dots, h_m)C \in \mathcal{C}^2$, or - equivalently - iff the linear transformation

$$K = i(h_1, \dots, h_m)A^{-1}C \quad \text{with} \quad C(h_1, \dots, h_m, h, k) = \langle Kh, k\rangle,$$

for every $h, k \in E$, is compact for all $h_1, \dots, h_m \in E$. In [8], pages 6-7, there is given an analogous condition for multilinear forms C, which is called the *finiteness* with respect to a pair of arguments and where the above compactedness of K is replaced by the finiteness of the rank of K and the closure $\mathcal{F}^{m+2}$ of the set in question is taken. We have $\mathcal{F}^{m+2} \subset \mathcal{C}^{m+2}$.

24. In what follows we need skew-symmetric multilinear forms. We denote by $\mathcal{L}_a^m(E;F)$ the space of all bounded m-linear and skew-symmetric functions, with values in the Banach space F. $\mathcal{L}_a^m(E, F)$ is a closed subspace in $\mathcal{L}^m(E; F)$, and hence a Banach space.

We denote by $\mathcal{C}_a^m = \mathcal{C}_a^m(E, \mathrm{R})$ the space of all m-linear and skew-symmetric forms which are compact with respect to one pair and hence - because of the skew-symmetry - with respect to any pair of its arguments. As an intersection

$$\mathcal{C}_a^m(E; \mathrm{R}) = \mathcal{C}^m(E; \mathrm{R}) \cap \mathcal{L}_a^m(E; \mathrm{R})$$

of closed subspaces, $\mathcal{C}_a^m(E; \mathrm{R})$ is a closed linear subspace in $\mathcal{L}_a^m(E; \mathrm{R})$; we call the forms $C \in \mathcal{C}_a^m(E; \mathrm{R})$ shortly *compact*.

25. In the space $\mathcal{L}(E; E)$ of linear transformations of an infinite-dimensional Banach space E there is a unique representation $T = \lambda I + C$ for elements T which are compactly perturbed multiples of the identity I in $\mathcal{L}(E; E)$. Furthermore, the mappings

$$\lambda I + C \to \lambda \quad \text{and} \quad \lambda I + C \to C$$

are continuous. We are going to prove the following analogous property on trilinear and skew-symmetric forms:

PROPOSITION 1. *Let (E, A) be a real infinite-dimensional Banach space where there is given a strongly regular and skew-symmetric bilinear form A of $\mathcal{L}_a^2(E; \mathrm{R})$. Given a trilinear and skew-symmetric form $M \in \mathcal{L}_a^3(E; \mathrm{R})$, there exists at most one linear function $\lambda : E \to \mathrm{R}$ and at most one compact trilinear form $C \in \mathcal{C}_a^3(E; \mathrm{R})$ such that*

$$M = \lambda \wedge A + C \tag{21}$$

and in such a representation λ is necessarily bounded. Furthermore, the set $(E^* \wedge A)_p$ of all forms $M \in \mathcal{L}_a^3(E; \mathrm{R})$ admitting a decomposition (21) is a closed linear subspace in the space $\mathcal{L}_a^3(E; \mathrm{R})$ of all bounded skew-symmetric trilinear forms, the correspondence

$$tr: \lambda \wedge A + C \to \lambda \quad resp. \quad c: \lambda \wedge A + C \to C$$

being a bounded and linear map from $(E^* \wedge A)_p$ to $\mathcal{L}(E; \mathrm{R})$, resp. to

$\mathcal{C}_a^3(E;\mathrm{R})$. *For the bound* [9] *we have:*

$$(22) \qquad |tr| \leqslant |A^{-1}| \, .$$

REMARK. Suppose there is given a bounded set $\mathcal{P} \subset \mathcal{L}(E;E)$ consisting of projection operators, with finite rank and such that:

a) for all $n \in \mathrm{N}$ there exists a $P \in \mathcal{P}$ with $rank P > n$;

b) the restriction $A \,|\, P(E)\times P(E)$ of A to the finite-dimensional linear subspace $P(E) \subset E$ is regular for all $P \in \mathcal{P}$.

Given $h \in E$ the value $\lambda(h)$ of the one-form $\lambda = trM$ can be calculated by usual tensor contraction in the finite-dimensional spaces $P(E)$ if E is a Hilbert space. Indeed, contraction of $P_* i(h)M$ in $(P(E), A)$ gives

$$Trace\, P_* i(h)M = Trace\, T,$$

where T is the linear transformation of $P(E)$ with

$$[\, i(h)M](k,l) = A(Tk,l) \quad \text{for all } k,\ l \in P(E).$$

As in [7,8] it is verified then

$$\lambda(h) = (trM)(h) = \lim_{rank\, P \to \infty} \frac{Trace\, P_* i(h)M}{rank P} \, .$$

In particular this holds choosing the elements of $\mathcal{P}$ as orthogonal projections; in this case the operator tr is denoted in [8] by sp.

26. The proof of Proposition 1 is based on the analogous property for bilinear forms and linear transformations:

LEMMA 2. *Let $A,\ B \in \mathcal{L}(E\times E;\mathrm{R})$ be bounded bilinear functions in the infinite-dimensional Banach space E with $A_b = \{ x \to i(x)A \}$ bijective. Then there exists at most one representation*

$$(23) \qquad B = \lambda A + C$$

where $\lambda \in \mathrm{R}$ and C is compact. If B admits a representation (23), *then, denoting $|A_b^{-1}| = |A^{-1}|$,*

$$(23') \qquad |\lambda| \leqslant |A^{-1}|\,|B| \, .$$

PROOF. Write $O_\rho = \{ x \mid |x| \leqslant \rho \}$ and recall that in infinite-dimensional

[9] For the definition of A^{-1} see number 21.

Banach spaces balls O_ρ never are compact when the radius $\rho > 0$. A linear map $T: E \to E^*$ being compact iff $\overline{T(O_1)}$ is compact, μA_b , and hence μA , can be compact only for $\mu = 0$, since, by the postulated bijectivity,

$$\inf_{|x| = 1} |A_b(x)| \geqslant |A_b^{-1}|^{-1} > 0 ,$$

implying that the ball of radius $\mu |A_b^{-1}|^{-1}$ is contained in $(\mu A_b)(O_1)$. Hence from

$$\lambda' A + C' = B = \lambda A + C,$$

λ, λ' real and C, C' compact, we get with $\mu = \lambda - \lambda'$,

$$\mu A = C' - C = \text{compact}, \quad \text{so} \quad \mu = 0, \quad \lambda = \lambda' \quad \text{and} \quad C = C'.$$

The compactedness of $C_b = \{ x \to i(x)C \}$ implies furthermore that for given $\epsilon > 0$ there exists a unit vector h such that $|C_b(h)| < \epsilon$, since otherwise C_b would be invertible and - as we saw above - consequently not compact. Having $B = \lambda A + C$, we get $B_b = \lambda A_b + C_b$, hence

$$|B| = |B_b| = \sup_{|x| = 1} |B_b(x)| \geqslant |B_b(h)| = |\lambda A_b(h) + C_b(h)| \geqslant$$
$$\geqslant |\lambda A_b(h)| - |C_b(h)| \geqslant |\lambda| \inf_{|x| = 1} |A_b(x)| - \epsilon \geqslant |\lambda| \, |A_b^{-1}|^{-1} - \epsilon ,$$

so the inequality (23') follows.

PROPOSITION 2. *The bilinear function A being as in Lemma 2 there exist at most one representation* [10)]

$$M = \lambda \otimes A + C \tag{24}$$

of a bounded trilinear function $M \in \mathfrak{L}_a^3(E; \mathrm{R})$, where $\lambda \in E^*$ and where C is compact with respect to the pair $(2, 3)$ of its arguments. The set

$$E^* \otimes A + \mathcal{C}^3(E; \mathrm{R})$$

of all trilinear and bounded functions M admitting a decomposition (24) is a closed linear subspace in $\mathfrak{L}^3(E; \mathrm{R})$, the mapping

$$tr: M = \lambda \otimes A + C \to \lambda$$

10) Here $\lambda \otimes A$ denotes the trilinear form with values

$$\lambda(x)A(y, z) \quad \text{for all} \quad (x, y, z) \in E \times E \times E .$$

defining a bounded linear function from $E^*\otimes A + \mathcal{C}^3(E;\mathrm{R})$ *to* E^* *with:*

(24')
$$|tr| \leqslant |A^{-1}| \,.$$

PROOF. Since by assumption $i(x)C$ is a compact bilinear form for any $x \in E$, we get from $M = \lambda \otimes A + C$ the equation

$$i(x)M = \lambda(x)A + i(x)C$$

between bilinear forms where, by Lemma 2, $\lambda(x)$ is uniquely given and

$$|\lambda(x)| \leqslant |i(x)M|\,|A^{-1}| \leqslant |x|\,|M|\,|A^{-1}| \,.$$

The set $E^*\otimes A + \mathcal{C}^3(E;\mathrm{R})$ is a closed linear subspace in $\mathcal{L}^3(E;\mathrm{R})$ as a direct sum of the closed linear subspaces

$$E^*\otimes A = \{\, \lambda \otimes A \mid \lambda \in E^* \,\}$$

(which is homeomorphic with E^*, by the linear map $tr: \lambda \otimes A \to \lambda$) and $\mathcal{C}^3(E;\mathrm{R})$.

27. The proof of Proposition 1 goes back to Proposition 2 if we first show the boundedness of the linear function $\lambda: E \to \mathrm{R}$ in the representation $M = \lambda \wedge A + C$. The boundedness of λ in turn will follow from Lemma 3 below.

Given a unit vector $h \in E$, denote by

$$h^{\perp} = \{\, k \mid A(h,k) = 0 \,\}$$

the space orthogonal to h with respect to the strongly regular and skew-symmetric form A.

$$S = S_1 = \{\, x \mid |x| = 1 \,\}$$

being the unit sphere, we write

$$|A|_{h^{\perp}} = \sup_{k,\,l \in S \cap h^{\perp}} A(k,l), \quad |A| = \sup_{k,\,l \in S} A(k,l),$$

getting:

LEMMA 3. *There exists a positive constant* ρ *such that*

$$|A|_{h^{\perp}} > \rho\,|A| \quad \textit{for all} \quad h \in S.$$

The proof of Lemma 3 with a lower bound for ρ is given in Appendix, number 45.

Now, by definition of the skew-product $\wedge$, the equation

$$M = \lambda \wedge A + C$$

implies

(25) $$Mhkl = \lambda h A kl + \lambda k A lh + \lambda l A hk + Chkl,$$

for all $h, k, l \in E$. Given a unit vector $h \in S$ choose k, l to be A-orthogonal to h : $k, l \in h^{\perp}$. Then (25) becomes

(25') $$Mhkl = \lambda h A kl + Chkl.$$

There exists for a given ϵ $(0 < \epsilon < |A|_{h^{\perp}})$ unit vectors

$$k_0 , l_0 \in S \cap h^{\perp} \quad \text{such that} \quad A k_0 l_0 > |A|_{h^{\perp}} - \epsilon .$$

Put $k = k_0$, $l = l_0$ in (25'). Then we have, by Lemma 3,

$$(\rho |A| - \epsilon)|\lambda h| \leqslant (|A|_{h^{\perp}} - \epsilon)|\lambda h| \leqslant (A k_0 l_0)|\lambda h| =$$

$$= |(A k_0 l_0)\lambda h| = |Mhk_0 l_0 - Chk_0 l_0| \leqslant (|M| + |C|),$$

so for all unit vectors h:

$$\rho |A| |\lambda h| \leqslant (|M| + |C|),$$

the linear form λ thus being bounded.

28. Since λ is bounded, the trilinear function F given by

$$Fxyz = (\lambda y)A xz - (Axy)\lambda z , \quad \text{for all} \quad x, y, z \in E,$$

is bounded and $B = i(x)F$ has [11] finite rank $\leqslant 2$ for all x ; the image [12] $B_b(E)$ of

$$B_b = \{ y \to i(y)B = Fxy = \lambda y A x - (Axy)\lambda \}$$

is namely spanned by $Ax = i(x)A$ and λ in E^*. It follows that

$$i(x)C - i(x)F = i(x)\{ C - F \}$$

is compact, being a perturbation of the compact bilinear form $i(x)C$ by the

[11] The rank of a bilinear function B is the dimension of

$$Im B_b = \{ i(x)B \mid x \in E \} = B_b(E).$$

[12] For convenience we write $i(x,y)F = Fxy$, $i(x)A = Ax$.

finite rank form $-i(x)F$. This means that the trilinear form $\bar{C} = C - F$ is compact with respect to the pair $(2,3)$ of its arguments (see number 23). Having

$$M = \lambda \wedge A + C = \lambda \otimes A + \bar{C}, \tag{26}$$

we thus conclude from Proposition 2 that $\lambda = trM$ and $\bar{C}$, hence also C equal to $\bar{C} + F$, are uniquely defined with $|\lambda| < |A^{-1}||M|$. It follows immediately also that the set

$$(E^*\wedge A)_p = E^*\wedge A + \mathcal{C}^3_a(E;\, \mathrm{R}) = \{ \lambda \wedge A + C \mid \lambda \epsilon E^*,\ C \epsilon \mathcal{C}^3_a \}$$

is a closed linear subspace in $\mathcal{L}^3_a(E;\, \mathrm{R})$, and we have

$$(E^*\wedge A)_p \subset E^*\otimes A + \mathcal{C}^3(E;\, \mathrm{R}).$$

The proof of Proposition 1 is ended.

29. Suppose for an element

$$M = \lambda \wedge A + C \,\epsilon (E^*\wedge A)_p$$

the number $m = rank_{(2,3)}C$ (defined in number 30) is finite. Then for all given $h \,\epsilon\, E$ the rank of the linear operator $K = A^{-1}i(h)C$ (defined by

$$A(Kk, l) = C(h,k,l) \quad \text{for all} \quad k,\ l \,\epsilon\, E,$$

see number 21) also is finite with

$$rank K = rank\, i(h)C \leqslant m.$$

Consequently K has the usual trace given as the trace of the linear transformation K/ImK of the finite-dimensional vector space $ImK = K(E) \subset E$. We get thus in this case a new one-form $h \to Trace K$ which is denoted by $\delta M = Trace\, C$.

30. We come to the finite-dimensional «analogon» of a compactly perturbed multiple $\lambda\wedge A + C$ of $\lambda \wedge A$, namely the «small-rank perturbation». Recall that in any vector space E - finite-dimensional or not - the rank of a m-linear skew-symmetric function M can be defined as

$$rank M = codim\ ker M, \tag{27}$$

where $ker M = \{ x \mid i(x)M = 0 \}$. If M is trilinear we define

(27')
$$rank_{(2,3)}M = \max_{x \in E} rank\, i(x)M.$$

In the case where E is finite-dimensional, these numbers always exist.

LEMMA 4. *For all skew-symmetric forms* $(0 \neq) M \in \mathfrak{L}_a^3(E; \mathrm{R})$,

$$rank_{(2,3)}M < rank\, M.$$

Also we have:

LEMMA 5. *In a symplectic vector space* (E, A) *with* $\dim E = 2n \geqslant 4$,

$$rank_{(2,3)}(\mu \wedge A) = 2n - 2 \quad \text{for all} \quad 0 \neq \mu \in E^*.$$

The proofs of these lemmas is given in Appendix, number 47.

The following proposition can be viewed as an analogue to Proposition 1, number 25:

PROPOSITION 1'. *Let* (E, A) *be a symplectic vector space with*

$$\dim E = 2n \geqslant 4.$$

Given a trilinear and skew-symmetric form $M \in \mathfrak{L}_a^3(E; \mathrm{R})$, *there exists at most one linear function* $\lambda \in E^* = \mathfrak{L}(E; \mathrm{R})$ *and at most one trilinear form* $C \in \mathfrak{L}_a^3(E; \mathrm{R})$, *with*

(28)
$$rank_{(2,3)}C < n - 1,$$

such that

(28')
$$M = \lambda \wedge A + C.$$

In particular there exists at most one representation (28') *with*

(28")
$$rank\, C < n.$$

PROOF. The last consequence follows from Lemma 4 above. To prove the first part of the proposition, suppose we have

$$\lambda \wedge A + C = M = \lambda' \wedge A + C',$$

where C and C' satisfy the rank-condition (28). Then, with

$$\mu = \lambda' - \lambda, \quad K = C' - C, \quad 0 = \mu \wedge A + K,$$

hence

$$rank_{(2,3)}(\mu \wedge A) = rank_{(2,3)}K.$$

For all $x \in E$,

$$\begin{aligned} rank\, i(x)K &= rank\{ i(x)C' - i(x)C \} \leqslant rank\, i(x)C' + rank\, i(x)C \\ &\leqslant \max_{y \in E} rank\, i(y)C' + \max_{y \in E} rank\, i(y)C = \\ &= rank_{(2,3)}C' + rank_{(2,3)}C < 2n - 2 , \end{aligned}$$

so, having

$$rank_{(2,3)}K = \max rank\, i(x)K ,$$

we get

$$rank_{(2,3)}(\mu \wedge A) = rank_{(2,3)}K < 2n - 2.$$

By the previous lemma this is possible only with $0 = \mu = \lambda' - \lambda$, from which Proposition 1' follows.

31. (E, A) being a $2n$-dimensional symplectic vector space, denote

$$(29) \quad \mathcal{C}^3_a(E;\, \mathrm{R}) = \{ C \mid C \in \mathcal{L}^3_a(E;\, \mathrm{R}),\ rank_{(2,3)}C < n - 1 \}$$

by abbreviation $\mathcal{C}$; these skew-symmetric trilinear forms we call in the sequel the *forms «with small-rank»*. Since the rank of every skew-symmetric bilinear form is even, it follows that for every $C \in \mathcal{C}$ the number

$$rank_{(2,3)}C = \max_{x \in E} rank\ i(x)C$$

is even. Consequently we have:

LEMMA 6. *If the dimension $2n$ of E is 4 or 6, then* $\mathcal{C} = \{0\}$.

The set of all «small-rank perturbations» of multiples of A given by

$$(30) \quad (E^* \wedge A)_p = E^* \wedge A + \mathcal{C} = \{ \lambda \wedge A + C \mid \lambda \in E^*,\ C \in \mathcal{C} \}$$

reduces hence in the cases of the two lowest dimensions 4 and 6 to the linear subspace $E^* \wedge A \subset \mathcal{L}^3_a(E;\, \mathrm{R})$. In general, however, $(E^* \wedge A)_p$ is not a *linear* subspace of $\mathcal{L}^3_a(E;\, \mathrm{R})$. One easily verifies:

LEMMA 7. *If $M \in (E^* \wedge A)_p$, then $\mu M \in (E^* \wedge A)_p$ for all $\mu \in \mathrm{R}$. If N, $M + N$ are in $(E^* \wedge A)_p$, then the operators*

$$(31) \qquad tr: M = \lambda \wedge A + C \rightarrow \lambda \in E^*,$$

$$(31') \qquad \delta: M = \lambda \wedge A + C \rightarrow Trace\, C \in E^*$$

satisfy

$$tr(\mu M) = \mu\, trM, \qquad \delta(\mu M) = \mu(\delta M),$$
$$tr(M+N) = trM + trN, \quad \delta(M+N) = \delta M + \delta N.$$

Furthermore, the operators tr *and* δ *are related to the usual contraction operator* $Trace: \mathfrak{L}_a^3(E;\, \mathrm{R}) \to E^*$ *(defined in the next number) by*

(31") $$2(n-1)tr = Trace - \delta \quad (2n = dim\, E).$$

32. With the notations introduced in number 21 we denote by

$$TraceM = \{ h \to Trace\, A^{-1} i(h) M \} \in E^*$$

the one-form, derived from a trilinear form $M \in \mathfrak{L}_a^3(E;\, \mathrm{R})$ by means of tensor contraction with respect to the symplectic form $A \in \mathfrak{L}_a^2(E;\, \mathrm{R})$; for the expression of $TraceM$ in index notation, see Footnote 4 in number 4 where $(d\Omega)_{ijk}$ must be replaced by $M_{ijk} = M(e_i, e_j, e_k)$. It is easily verified that in a $2n$-dimensional space (E, A) we have for $M = \lambda \wedge A$:

(32) $$2(n-1)\lambda = Trace(\lambda \wedge A).$$

Consequently, the linear space

$$E^* \wedge A = \{ \lambda \wedge A \mid \lambda \in E^* \}$$

is the kernel $f^{-1}(0)$ of the linear transformation f of $\mathfrak{L}_a^3(E;\, \mathrm{R})$ which transforms M to the corresponding «Lee-form» (see (2') and (2), number 4)

$$L = f(M) = M - \frac{1}{2(n-1)} (TraceM) \wedge A.$$

Having, with $2(n-1)\lambda = TraceM$,

$$f^2 M = f(f(M)) = f(M - \lambda \wedge A) = f(M) - f(\lambda \wedge A) = f(M),$$

it follows that f is a projection operator and it projects the space $\mathfrak{L}_a^3(E;\, \mathrm{R})$ on the linear subspace

$$Im f = \{ M \mid f(M) = M \} = \{ M \mid TraceM = 0 \},$$

which is a complementary subspace to $E^* \wedge A = ker f$. In particular for a «small-rank perturbation»

$$M = \lambda \wedge A + C \in (E^* \wedge A)_p$$

the projection $L = f(M)$ becomes

$$L = f(\lambda \wedge A) + f(C) = f(C),$$

or explicitly

$$(32') \qquad L = C - \frac{1}{2(n-1)} (\operatorname{Trace} C) \wedge A .$$

This is the relation between the perturbation tensor C and the Lee-tensor L derived from a $M \in (E^* \wedge A)_p$.

4. DIFFERENTIABILITY OF THE TENSOR FIELD R.

33. We have now sufficiently information about the linear space theory in order to return to almost symplectic manifolds $(\mathfrak{M}, \Omega)$. Every tangent space $\mathfrak{M}_x$ gives rise to the linear symplectic Banach space $(\mathfrak{M}_x, \Omega(x))$, which is infinite-dimensional (resp. $2n$-dimensional) and we denote by

$$\mathcal{P}_x = (\mathfrak{M}_x^* \wedge \Omega(x))_p = \{ \lambda \wedge \Omega(x) + C \mid \lambda \in \mathfrak{M}_x^*, \ C \in \mathcal{C}(x) \}$$

the fiber of all compact (resp. small-rank) perturbations of multiples of $\Omega(x)$, whereby $\mathcal{C}(x) = \mathcal{C}_a^3(\mathfrak{M}_x ; \mathrm{R})$ is the set of all $C \in \mathcal{L}_a^3(\mathfrak{M}_x ; \mathrm{R})$ which are compact (resp. which have a small-rank, see number 24 resp. 31). We prove:

PROPOSITION 3. *Suppose that on an infinite-dimensional (resp. 2n-dimensional) almost symplectic C^{m+1}-manifold $(\mathfrak{M}, \Omega)$ with the C^m-form Ω there is given a 3-form M which is C^{m-1} and which admits at every point $x \in \mathfrak{M}$ a representation*

$$M(x) = R(x) \wedge \Omega(x) + C(x),$$

where $R(x)$ is a linear function $\mathfrak{M}_x \to \mathrm{R}$ and where $C(x) \in \mathcal{C}_a^3(\mathfrak{M}_x ; \mathrm{R})$ is compact (resp. has a small-rank). Then $R(x)$ and $C(x)$ are uniquely defined and $R(x)$ is bounded, so

$$M(x) \in (\mathfrak{M}_x^* \wedge \Omega(x))_p \quad \text{for all} \quad x \in \mathfrak{M}.$$

Furthermore, in the cases where the dimension of $\mathfrak{M}$ is 4, 6 or ∞, the tensor field $R : x \to R(x)$ and hence also $C = M - R \wedge \Omega$ is C^{m-1}. In the cases $6 < \dim \mathfrak{M} < \infty$ the tensor fields R and C are C^{m-1} under the extra hypothesis that $\operatorname{Trace} C$ is C^{m-1}.

Here $r = Trace\, C$ is the contracted tensor field with

$$r(x) = Trace\, C(x)$$

defined as indicated in number 32. The first part of the proposition expresses the fact that M defines a C^{m-1} section in the bundle $\mathcal{P} = \bigcup_{x \in \mathfrak{M}_x} \mathcal{P}_x$ (see number 37).

REMARK. I have tried, without success, to strengthen the above proposition in the following directions:

Claim 1. The differentiability condition for the tensor field $r = Trace\, C$ can be replaced by the rank-invariance condition

$$rank_{(2,3)} C(x) = constant \text{ on } \mathfrak{M},$$

Claim 2: The differentiability conditions for M and Ω do not alone ensure the differentiability of R and C, when $6 < dim\, \mathfrak{M} < \infty$, that is, in this case there may exist a one-form R and a 3-form C which are *not* C^{m-1}, such that $C(x)$ has small-rank and $M = R \wedge \Omega + C$ is C^{m-1}.

PROOF OF PROPOSITION 3. Fixing $x \in \mathfrak{M}$ replace the symplectic vector space (E, A) occuring in Section 3 by $(\mathfrak{M}_x, \Omega(x))$. By the propositions in the numbers 25, resp. 30, $R(x)$ and $C(x)$ are uniquely given, $R(x)$ is bounded and hence $M(x) \in (\mathfrak{M}_x^* \wedge \Omega(x))_p$. In the cases where the dimension $2n$ of $\mathfrak{M}$ is 4 or 6, we have by number 31:

$$C(x) = 0, \quad \text{so} \quad M(x) = R(x) \wedge \Omega(X),$$

and consequently by (32) in 32,

$$2(n-1) R(x) = Trace\, M(x),$$

so the tensor field R is C^{m-1} since M is. In the cases $6 < dim\, \mathfrak{M} < \infty$, we have $R \wedge \Omega = M - C$, so by (32)

$$2(n-1) R = Trace\, M - Trace\, C,$$

the one-form R and also $C = M - R \wedge \Omega$ being thus C^{m-1}, since M and $Trace\, C$ are.

34. Suppose now that $dim\, \mathfrak{M} = \infty$. By Proposition 1, number 25, the operator $\lambda \wedge \Omega(x) + C(x) \to \lambda$ is bounded; we denote it by $tr(x)$, so that

$R(x) = tr(x)M(x)$.

To prove the differentiability of R we take a local chart $(\Phi, \mathfrak{W})$ around x_0 with the Banach space E as parameter space, denoting by the same symbols $x, M, \Omega, R, \ldots$ the representants

$$\Phi(x),\ \Phi_*^{-1}M,\ \Phi_*^{-1}\Omega,\ \Phi_*^{-1}R, \ldots \quad \text{of } x, M, \Omega, R, \ldots \text{ resp.}$$

Furthermore, we write $\Phi_*^{-1}\Omega(x_0) = A$, so the Banach space E becomes a linear symplectic vector space (E, A).

For all $x \in \Phi(\mathfrak{W})$ we have

$$(33) \qquad M(x) = R(x) \wedge \Omega(x) + C(x),$$

where $C(x)$ is compact, $x \to \Omega(x)$ is C^m and $x \to M(x)$ is C^{m-1}. We prove that $x \to R(x) = tr(x)M(x)$ is C^{m-1} by showing that

$$R(x) = tr(x_0)N(x), \quad \text{where } tr(x_0) = \{\lambda \otimes A + \text{compact} \to \lambda\}$$

operates on the same Banach space $E^* \otimes A + \mathcal{C}^3(E, \mathrm{R})$ for all x, and $x \to N(x)$ is C^{m-1}.

35. The tangent spaces being identified with the parameter space E we have for each x in the parameter domain $\Phi(\mathfrak{W}) \subset E$ the representation

$$(34) \qquad \Omega(x)hk = A(U(x)h, k), \quad h, k \in E,$$

of $\Omega(x)$, where $U(x) = A^{-1}\Omega(x) \in \mathfrak{L}(E; E)$. Since $A = \Omega(x_0)$, so

$$U(x_0) = I = \text{identity operator of } E.$$

By the strong regularity of $\Omega(x)$ the linear transformation $U(x)$ is bijective, and since $x \to \Omega(x)$ is C^m it follows that the function

$$(35) \qquad x \to T(x) = \{U(x)\}^{-1}$$

is C^m as well. For brevity let us omit the symbol x for a moment. Defining a trilinear form F analogously to the discussion in number 28 by

$$Fhkl = Rk\Omega hl - (\Omega hk)Rl \quad (h, k, l \in E)$$

we have $R \wedge \Omega + C = R \otimes \Omega + \bar{C}$, with

$$(35') \qquad \bar{C} = C - F.$$

Since R is a bounded linear function it follows, as in number 28, that C is *compact* with respect to the pair $(2,3)$ of its arguments. The equation (33) becomes

(35")
$$M = R\otimes\Omega + \bar{C}$$

and having $T = U^{-1}$, we get here by (34) for the first term on the right

$$(R\otimes\Omega)(h, Tk, l) = Rh\,\Omega(Tk, l) = RhAkl = (R\otimes A)(h,k,l),$$

so

$$M(h, Tk, l) = (R\otimes A)(h,k,l) + \bar{C}(h, Tk, l)$$

for all $h, k, l \in E$. Defining trilinear forms $N = N(x)$ and $D = D(x)$ by

(36) $N(h,k,l) = M(h, Tk, l),\quad D(h,k,l) = \bar{C}(h, Tk, l)$

$(h,k,l \in E)$, we have for all $x \in \Phi(\mathfrak{W})$,

(37)
$$N(x) = R(x)\otimes A + D(x).$$

It follows that, according to Proposition 2, number 26, the linear function $R = R(x)$ has the representation $R(x) = tr(x_0)N(x)$ provided $D(x)$ is compact with respect to $(2,3)$. Since by Proposition 2, $tr(x_0)$ is a bounded linear operator operating in the Banach space $E^*\otimes A + \mathcal{C}^3(E;\mathrm{R})$, we conclude that $x \to R(x)$ is C^{m-1} if besides of the compactedness of D the differentiability of $x \to N(x)$ will be proved.

36. By definition given in number 23 the trilinear form D in (36) is compact with respect to the pair $(2,3)$ of its arguments if for every $h \in E$ the bilinear form $i(h)D = B$ is compact, that is, iff for every h the corresponding linear map

$$B_b = \{\, k \to i(h)B = Dhk \,\}$$

is compact; here

$$Dhk = \{\, l \to Dhkl \,\} \in E^*.$$

Writing

$$i(h)\bar{C} = \bar{B} \quad\text{and}\quad \bar{B}_b = \{\, k \to i(k)\bar{B} \,\}$$

we get by (36), for all $k \in E$, the equations between elements of E^*:

$$B_b(k) = Dhk = \bar{C}hTk = i((h)C)Tk = \bar{B}Tk = (\bar{B}_b \circ T)(k).$$

Accordingly $B_b = \bar{B}_b \circ T$, and because $\bar{C}$ is compact with respect to $(2,3)$, the factor $\bar{B}_b$ is compact. As a product of compact and bounded operators, B_b is compact.

In (36) the form N is a linear expression of M and T ; in fact we have the bounded and bilinear function

$$*: \mathfrak{L}^3(E;\, \mathrm{R}) \times \mathfrak{L}(E;\, E) \to \mathfrak{L}^3(E;\, \mathrm{R})$$

given for all $\alpha \in \mathfrak{L}^3(E;\, \mathrm{R})$, $\beta \in \mathfrak{L}(E;\, E)$ by

$$\alpha * \beta = \{ (h, k, l) \to \alpha(h, \beta k, l) \} \quad (h, k, l \in E),$$

and according to (36)

$$N(x) = M(x) * T(x).$$

Since both the factors $M(x)$ and $T(x)$ are here C^{m-1} functions of x, then by the product rule of Calculus, $x \to N(x)$ is C^{m-1}. Proposition 3 is proved.

37. With $\mathcal{P}_x = (\mathfrak{M}_x \wedge \Omega(x))_p$ as fibers, we have the bundle

$$\mathcal{P} = (T^*(\mathfrak{M}) \wedge \Omega)_p = \bigcup_{x \in \mathfrak{M}} \mathcal{P}_x$$

over $\mathfrak{M}$ with projection $\pi: (x, M) \to x$. If $(\mathfrak{W}, \Phi)$ is a chart of $\mathfrak{M}$ around $x_0 \in \mathfrak{M}$ then, writing with the derivative Φ', $f(x) = (\Phi'(x))^{-1} \circ \Phi'(x_0)$, a local trivialization

$$\tau: \pi^{-1}(\mathfrak{W}) \to \mathfrak{W} \times (\mathfrak{M}^*_{x_0} \wedge \Omega(x_0))$$

is defined: for $x \in \mathfrak{M}$ and

$$M = \lambda \wedge \Omega(x) + C \in (\mathfrak{M}^*_x \wedge \Omega(x))_p,$$

$\tau(x, M) = (x, N)$ with

$$N = (f_*(x)\lambda) \wedge \Omega(x_0) + f_*(x)C.$$

Here $f_*(x)\lambda$ and $f_*(x)C$ are the pullbacks of the one-form $\lambda \in \mathfrak{M}^*_x$ and the compact (small-rank) trilinear form $C \in \mathcal{C}^3_a(\mathfrak{M}_x;\, \mathrm{R})$, respectively. In the cases $dim\, \mathfrak{M} = 4, 6$ and ∞, any C^{m-1} section

$$\mathfrak{M} \ni x \to M(x) = R(x) \wedge \Omega(x) + C(x)$$

of $\mathcal{P}$ gives, according to Proposition 3, rise to C^{m-1} sections $x \to R(x)$ and $x \to C(x)$ on the bundles

$$T^*(\mathfrak{M}) = \mathfrak{L}(T(\mathfrak{M});\, \mathrm{R}) \quad \text{and} \quad \mathcal{C}^3_a(T(\mathfrak{M});\, \mathrm{R})$$

of one-forms and compact (small-rank) skew-symmetric trilinear forms, respectively, whereby for the dimensions 4 and 6, C is the zero section.

REMARK. In the cases $\dim \mathfrak{M} = 4, 6, \infty$ the typical fiber $(E^* \wedge A)_p$ of $\mathcal{P}$ is a closed *linear* subspace of $\mathfrak{L}_a^3(E; \mathrm{R})$ (see 25 and 31; (E, A) is a symplectic Banach space). Now it is known that the closed linear subspace $\mathcal{C}(E; E)$ consisting of all compact operators $C \in \mathfrak{L}(E; E)$ of an infinite-dimensional Banach space E is not a *direct* subspace of $\mathfrak{L}(E; E)$ that is it hasn't any *closed* supplementary space in $\mathfrak{L}(E; E)$. It follows that the typical fibre $(E^* \wedge A)_p$ is not in general a direct linear subspace in $\mathfrak{L}_a^3(E; \mathrm{R})$. The latter space being the typical fiber of the tensor bundle $\mathfrak{L}^3(T(\mathfrak{M}); \mathrm{R})$ of trilinear and skew-symmetric forms, it follows that in the case $\dim \mathfrak{M} = \infty$ the vector bundle $\mathcal{P} \subset \mathfrak{L}_a^3(T(\mathfrak{M}); \mathrm{R})$ is *not* a sub-bundle of $\mathfrak{L}_a^3(T(\mathfrak{M}); \mathrm{R})$ in the sense of [12], page 49 (see also [3]).

5. COMPACT AND SMALL-RANK PERTURBATIONS OF CONFORMALLY SYMPLECTIC STRUCTURES.

38. We say that the structure of an infinite-dimensional (resp. $2n$-dimensional) almost symplectic manifold $(\mathfrak{M}, \Omega)$ is a compact (resp. small-rank) perturbation of a conformally symplectic structure iff $d\Omega$ defines a section in

$$\mathcal{P} = \bigcup_{x \in \mathfrak{M}} (\mathfrak{M}_x^* \wedge \Omega)_p$$

(see number 37). By the first part of Proposition 3 this happens iff for every $x \in \mathfrak{M}$ there exists some linear function $R(x): \mathfrak{M}_x \to \mathrm{R}$ and some compact (resp. small-rank) form $C(x) \in \mathcal{C}(x)$ such that

$$d\Omega(x) = R(x) \wedge \Omega(x) + C(x).$$

Furthermore, by Proposition 3, in the case where the dimension of $\mathfrak{M}$ is 4, 6 or ∞, the tensor fields R and C are C^1 if Ω is C^2 and a sufficient condition for R and C to be C^1 in the cases $6 < \dim \mathfrak{M} < \infty$ is that Ω is C^2 and $\mathit{Trace}\, C$ is C^1. In what follows we denote by ρ the family of all almost symplectic C^3-manifolds $(\mathfrak{M}, \Omega)$ with a C^2 fundamental form Ω, which have the above mentioned structure, whereby the characteristic tensor fields R and C are supposed to be C^1.

39. By number 31 the perturbation tensor $C = 0$ in all spaces $(\mathfrak{M}, \Omega)$ of ρ with $dim \mathfrak{M} < 6$. By number 15 *all* almost symplectic 4-dimensional manifolds belong to ρ and by 16 all 6-dimensional members of ρ are just those which are conformally symplectic. By number 32 the connection between the «conformal curvature tensor» L given in (2'), number 4, and the perturbation tensor C is on a $2n$-dimensional manifold $(\mathfrak{M}, \Omega) \epsilon \rho$:

$$(38) \qquad L(x) = C(x) - \frac{1}{2(n-1)} (Trace\, C(x)) \wedge \Omega(x).$$

40. Let $(\mathfrak{M}, \Omega) \epsilon \rho$ be given, with

$$d\Omega = R \wedge \Omega + C, \quad \text{so} \quad R(x) = tr(x)\, d\Omega(x).$$

If $\lambda: x \to \lambda(x) = e^{\alpha(x)}$ is a given positive C^2-function on $\mathfrak{M}$, then the form

$$(39) \qquad \bar{\Omega}(x) = \lambda(x)\Omega(x)$$

satisfies

$$d\bar{\Omega} = \lambda\{ d\alpha \wedge \Omega + d\Omega \} = \lambda\{ d\alpha \wedge \Omega + R \wedge \Omega + C \} =$$
$$= \lambda\{ (d\alpha + R) \wedge \Omega + C \},$$

so we have

$$(39') \qquad d\bar{\Omega} = \bar{R} \wedge \bar{\Omega} + \bar{C}$$

with the laws of transformation

$$(39'') \qquad \bar{R}(x) = R(x) + d\,\alpha(x), \quad \bar{C}(x) = \lambda(x) C(x),$$

where $\lambda(x) = e^{\alpha(x)}$. In particular it follows that $\bar{C}(x) = \lambda(x) C(x)$ is compact and

$$rank_{(2,3)} \bar{C}(x) = rank_{(2,3)} C(x),$$

so $\bar{C}(x) \epsilon\, \mathcal{C}^3_a(\mathfrak{M}_x ; \mathrm{R})$. Consequently $(\mathfrak{M}, \bar{\Omega}) \epsilon \rho$.

41. From the first equation (39''), we get, taking outer derivatives on both sides,

$$(40) \qquad d\bar{R} = dR = w,$$

where w denotes the common value of the outer derivatives; w is a closed two-form defined on $\mathfrak{M}$.

Going in the second equation (39") by multiplication with $\Omega(x)^{-1}$ (see number 21) over to mixed tensor fields

$$(41) \qquad \bar{K} = \bar{\Omega}^{-1}\bar{C}, \quad K = \Omega^{-1}C,$$

and noting that because of the conformality condition (39) we have

$$\bar{\Omega}(x)^{-1} = \lambda(x)^{-1}\,\Omega(x)^{-1},$$

it follows

$$\bar{K}(x) = \bar{\Omega}(x)^{-1}\,\bar{C}(x) = \lambda(x)^{-1}\Omega(x)^{-1}\{\lambda(x)C(x)\} = \Omega(x)^{-1}\,C(x)$$

and hence

$$(41') \qquad \bar{K} = K.$$

Also taking outer derivatives on both sides of the second equation (39") and using then both equations (39"), one easily verifies the transformation law

$$(42') \qquad \bar{D}(x) = \lambda(x)D(x)$$

of the fourth order skew-symmetric tensor field D defined by

$$(42) \qquad dC = R \wedge C + D.$$

A conformally invariant one-form r is defined in finite-dimensional cases on $(\mathfrak{M}, \Omega) \epsilon \rho$ also by

$$r(x) = \delta\, d\Omega(x) = \mathit{Trace}\ C(x)$$

(see number 29). In the infinite-dimensional case r is defined in the same way provided $\mathit{rank}_{(2,3)}C(x)$ is finite for all $x \epsilon \mathfrak{M}$ (see number 29).

42. Let $\bar{\Omega}(x) = \lambda(x)\,\Omega(x)$ as above. Looking at

$$(\mathfrak{M}, \Omega) \quad \text{and} \quad (\mathfrak{M}, \bar{\Omega}) = (\bar{\mathfrak{M}}, \bar{\Omega}).$$

as two distinct almost symplectic manifolds, the identity map $y: x \to x$ of $\mathfrak{M}$ defines a conformal diffeomorphism $(\mathfrak{M}, \Omega) \to (\bar{\mathfrak{M}}, \bar{\Omega})$. If, on the other hand, $(\mathfrak{M}', \Omega')$ is an almost symplectic manifold, then a conformal C^3-diffeomorphism

$$y: (\mathfrak{M}, \Omega) \to (\mathfrak{M}', \Omega')$$

defines an almost symplectic form $\bar{\Omega} = y_*\Omega'$ on $\mathfrak{M}$ and by the conformality we have $\bar{\Omega}(x) = \lambda(x)\Omega(x)$. Taking instead of $\mathfrak{M}$ and $\mathfrak{M}'$ merely neighborhoods $\mathfrak{U}$ and $\mathfrak{U}'$ of any $x_0 \in \mathfrak{M}$ and $x_0' \in \mathfrak{M}'$ respectively, the discussions of the previous numbers can be applied to spaces $(\mathfrak{M}, \Omega)$ and $(\mathfrak{M}', \Omega')$ which are locally conformally diffeomorph, and they imply:

PROPOSITION 4. *Let $(\mathfrak{M}, \Omega)$ be a member of the class ρ of all almost symplectic manifolds which have a perturbed conformally symplectic structure, so $d\Omega = R \wedge \Omega + C$. Define a 4-form D on $\mathfrak{M}$ by (42). If an almost symplectic manifold $(\bar{\mathfrak{M}}, \bar{\Omega})$ is locally conformally diffeomorph with $(\mathfrak{M}, \Omega)$ then $(\bar{\mathfrak{M}}, \bar{\Omega}) \in \rho : d\bar{\Omega} = \bar{R} \wedge \bar{\Omega} + \bar{C}$. Furthermore, $y : (\mathfrak{M}, \Omega) \to (\bar{\mathfrak{M}}, \bar{\Omega})$ denoting a local conformal diffeomorphism, hence $e^{a(x)}\Omega(x) = y*(x)\bar{\Omega}$, and having $\bar{D} = d\bar{C} - \bar{R}$, the following transformation formulas are valid:*

$$(43) \qquad e^{a(x)}C(x) = y*(x)\bar{C}, \quad e^{a(x)}D(x) = y*(x)\bar{D},$$

for the corresponding mixed tensor fields $K = \Omega^{-1} C$ and $G = \Omega^{-1} D$:

$$(43') \qquad K = y * \bar{K}, \quad G = y * \bar{G}$$

and

$$(44) \qquad R + d a = y * \bar{R},$$

the closed 2-form $w = dR$ being hence conformally invariant:

$$(44') \qquad w = y * \bar{w}.$$

Also we have

$$rank_{(2,3)} C(x) = rank_{(2,3)} \bar{C}(y(x))$$

and if this number is finite (as it is especially for all finite-dimensional manifolds), then one more conformally invariant form $r = y * \bar{r}$ is given as a contraction of the perturbation tensor (see number 29):

$$r(x) = \delta(x)\, d\Omega(x) = Trace\ C(x).$$

Recall that for $dim\,\mathfrak{M} < 6$ we have $C = 0$, so all tensors K, D, G, r, and in the case $dim\,\mathfrak{M} = 6$, also $w = dR$, vanish.

43. The class ρ divides in conformal equivalence classes, two spaces $(\mathfrak{M}, \Omega)$ and $(\mathfrak{M}', \Omega')$ of ρ belonging to the same class iff for all couples $(x_0, x_0') \in \mathfrak{M} \times \mathfrak{M}'$ there exist conformally diffeomorphic neighborhoods of x_0 and x_0'. On every equivalence class the tensor fields w, K, G and r are

common in the sense of Proposition 4. In particular by Theorem 1, number 16, and Proposition 4, number 42, there are equivalence classes where $C = 0$ and they consist of conformally symplectic spaces.

44. Let us say that an almost symplectic infinite-dimensional (resp. $2n$-dimensional) manifold $(\bar{\mathfrak{M}}, \bar{\Omega})$ is a «perturbed symplectic space» if $d\bar{\Omega}(x) \epsilon \mathcal{C}^3_a(\mathfrak{M}_x ; \mathrm{R})$ is compact (resp. has small-rank). In this case $(\bar{\mathfrak{M}}, \bar{\Omega})$ is in ρ and the characteristic tensor fields $\bar{R}$ and $\bar{C}$ satisfy $\bar{R} = 0$ and $d\bar{\Omega} = \bar{C}$. This structure reduces to the symplectic one in all cases $dim \mathfrak{M} \leqslant 6$ since no non-zero perturbation then occurs (see Lemma 6, number 31). We have:

PROPOSITION 5. *An almost symplectic manifold* $(\mathfrak{M}, \Omega)$ *with* $dim \mathfrak{M} > 6$ *is locally conformally diffeomorphic with a perturbed symplectic space* $(\bar{\mathfrak{M}}, \bar{\Omega})$ *iff* $(\mathfrak{M}, \Omega) \epsilon \rho$ *(hence* $d\Omega = R \wedge \Omega + C$*) and iff one of the two equivalent conditions* (45) *and* (45') *holds:*

$$D = dC - R \wedge C = 0, \tag{45}$$

$$dR = 0. \tag{45'}$$

PROOF. The calculations in number 12 first give for all members $(\mathfrak{M}, \Omega)$ of ρ the tensor identity

$$dR \wedge \Omega = -D, \tag{46}$$

which shows by Lemma 9 in number 46 that the conditions (45) and (45') are equivalent. By $dR = 0$ there exist local solutions a of $da = -R$ and for $\bar{\Omega}(x) = e^{a(x)}\Omega(x)$, we have, by Proposition 4,

$$\bar{R} = 0, \quad \text{so} \quad d\bar{\Omega} = \bar{R} \wedge \bar{\Omega} + \bar{C} = \bar{C}.$$

Consequently $(\mathfrak{M}, \bar{\Omega})$ has a perturbed symplectic structure, the identity map $x \to x$ giving a local conformal diffeomorphism between $(\mathfrak{M}, \Omega)$ and $(\mathfrak{M}, \bar{\Omega})$. On the other hand, if $(\mathfrak{M}, \Omega)$ is locally conformally diffeomorph to a perturbed symplectic space $(\bar{\mathfrak{M}}, \bar{\Omega})$, then from $(\bar{\mathfrak{M}}, \bar{\Omega}) \epsilon \rho$ it follows $(\mathfrak{M}, \Omega) \epsilon \rho$, and y being a local conformal diffeomorphism, $y_* \bar{\Omega} = e^{a}\Omega$ implies by Proposition 4

$$R + da = y_* \bar{R} = 0, \quad \text{so} \quad dR = 0,$$

or equivalently $D = 0$, the conditions being hence necessary.

6. APPENDIX.

45. We denote by (E, A) a Banach space with a strongly regular skew-symmetric bilinear form $A \in \mathfrak{L}^2_a(E;\ \mathrm{R})$. For $X \in E$ write

$$X^{\perp} = \{\ Y \mid A X Y = 0,\ \ Y \in E\ \},\quad S = \{\ X \mid |X| = 1,\ \ X \in E\ \}.$$

When

$$L = A_b\colon X \to i(X)A = \{\ Y \to A X Y\ \}$$

is the corresponding linear homeomorphism $E \to E^* = \mathfrak{L}(E;\ \mathrm{R})$ we have

$$(47)\qquad |L| = |A| = \sup_{X,\, Y \in S} |A X Y|$$

and setting

$$(47')\qquad m = |L^{-1}|,\quad q = |L|\,|L^{-1}|$$

we prove the following

LEMMA 8. *There exists a positive constant* ρ *such that for all* $X_0 \in E$:

$$(48)\qquad \sup_{X,\, W \in X_0^{\perp} \cap S} |A X W| \geqslant \rho\, |A|\ ,$$

and we have

$$(48')\qquad \rho \geqslant \{\ q(1 + 2q)\ \}^{-1}.$$

PROOF. Since L is bijective we have for all $X \in S$:

$$0 \neq |L X| = \sup_{Z \in S} |A X Z|$$

and, having $|L X|^{-1} L X \in S$,

$$|L X|^{-1} = |X|\,|L X|^{-1} = |L^{-1}(|L X|^{-1} L X)| \leqslant m,$$

so

$$(49)\qquad |L X| = \sup_{Z \in S} |A X Z| \geqslant \frac{1}{m}\ .$$

Hence, given a small number s $(0 < s < \frac{1}{m})$ and a unit vector $X_0 \in S$, there exists a $Y_0 \in S$ such that $|A X_0\, Y_0| > \frac{1}{m} - s$, that is, such that

$$(50)\qquad |A X_0\, Y_0|^{-1} < m(1 - m s)^{-1}.$$

The symplectic vector space (E, A) divides in a direct sum $E = F \otimes F^{\perp}$ of mutually A-orthogonal complementary spaces

$$F = (X_0\, , Y_0\,) = \{\ \lambda X_0 + \mu Y_0 \mid \lambda,\ \mu \in \mathrm{R}\ \}$$

$$\text{and } F^{\perp} = \{ X \mid AXY = 0 \text{ for all } Y \in F \},$$

so every $Z \in S$ is represented as a corresponding sum $Z = U + V$, with $U \in F$, $V \in F^{\perp}$, where

$$(51) \qquad (AX_0 Y_0)U = (AZY_0)X_0 + (AX_0 Z)Y_0 .$$

For the length $|V|$ of the «normal vector» V we get from (51):

$$\begin{aligned} |AX_0 Y_0||V| &= |AX_0 Y_0||Z-U| \leqslant |AX_0 Y_0|(1+|U|) = \\ &= |AX_0 Y_0| + |AX_0 Y_0||U| \leqslant |AX_0 Y_0| + |AZY_0| + |AX_0 Z| \\ &\leqslant |AX_0 Y_0| + 2|A| , \end{aligned}$$

so, by (50),

$$(52) \qquad |V| \leqslant k(s),$$

where

$$(52') \qquad k(s) = (1-ms)^{-1}(1-ms+2m|A|).$$

Suppose first that the normal component V of $Z = U + V$ does not vanish, and denote by $W = |V|^{-1} V$ the corresponding unit vector; thus $W \in S \cap F^{\perp}$, and choosing also $X \in S \cap F^{\perp}$, we have by $U \in F$, $AXU = 0$, hence

$$|AXZ| = |AX(U+V)| = |AXU + AXV| = |AXV| = |V||AXW| ,$$

so by (52),

$$(53) \qquad |AXZ| \leqslant k(s)|AXW| .$$

This inequality holds in fact for all unit vectors Z, because if the normal component V of Z vanishes, then $Z = U \in F$ and, by $X \in F^{\perp}$, $AXZ = 0$. By (49) we thus get from (53), having $X \in S \cap F^{\perp}$,

$$\frac{1}{m} \leqslant \sup_{Z \in S} |AXZ| \leqslant k(s) \sup_{Z \in S} |AXW| \leqslant k(s) \sup_{\bar{X}, \bar{W} \in F^{\perp} \cap S} |A\bar{X}\bar{W}| ,$$

so

$$(54) \qquad \sup_{\bar{X}, \bar{W} \in F^{\perp} \cap S} |A\bar{X}\bar{W}| \geqslant (mk(s))^{-1} .$$

This inequality holds for all values $0 < s < \frac{1}{m}$ of s and, letting $s \to 0$, we get $k(0) = 1 + 2m|A|$, and by (47'),

$$(54') \qquad \sup_{X, W \in F^{\perp} \cap S} |AXW| \geqslant [m(1+2m|A|)]^{-1} = [q(1+2q)]^{-1}|A| .$$

Since here $F^{\perp} = X_0^{\perp} \cap Y_0^{\perp}$, we have $X_0^{\perp} \cap S \supset F^{\perp} \cap S$, so (54') is a stronger result than (48), the lemma being hence proved.

46. Recall that in the space (E, A) with the strongly regular skew-symmetric bilinear form A there can be constructed «A-orthogonal systems» by means of the well-known Schmidt orthogonalization procedure. In particular, for $\dim E \geqslant 4$ there exist two pairs (X_0, Y_0) and (U_0, V_0) of vectors such that (X_0, Y_0, U_0, V_0) is a linearly independent set and the «orthonormality conditions»

$$A X_0 Y_0 = A U_0 V_0 = 1 \quad \text{and} \quad A X_0 U_0 = A X_0 V_0 = A Y_0 U_0 = A Y_0 V_0 = 0$$

hold. We have the following

LEMMA 9. *Let (E, A) be a Banach space with a strongly regular and skew-symmetric bilinear form A and with $\dim E \geqslant 6$: If for a bilinear and skew-symmetric form $B \in \mathfrak{L}_a^2(E, \mathrm{R})$*

$$A \wedge B = 0, \tag{55}$$

then $B = 0$.

PROOF. We have by definition of the skew-product $\wedge$:

$$\begin{aligned} 0 &= (A \wedge B)(X, Y, U, V) = \\ &= (AXY)(BUV) + (AYU)(BXV) + (AUX)(BYV) \\ &\quad + (BXY)(AUV) + (BYU)(AXV) + (BUX)(AYV) \end{aligned} \tag{55'}$$

for all $X, Y, U, V \in E$. Because of the strong regularity of A, there exists a bounded linear transformation $T = A^{-1} B \in \mathfrak{L}(E; E)$ such that

$$B(Z, W) = A(TZ, W) \tag{56}$$

for all $Z, W \in E$. Substitution of (56) to the previous equation gives, taking V as «common factor»,

$$0 = A(\{(AXY)TU + TXAYU - TYAXU + (BXY)U + XBYU - YBXU\}, V)$$

for all $V \in E$, so by the regularity of A,

$$0 = \{\ \} = [(AXY)T + TXAY - TYAX + (BXY)I + XBY - YBX]\, U,$$

for all $U \in E$, I denoting the identity operator of E. Here the linear transformation in the parenthesis [ ] operating on U vanishes, so for all $X, Y \in E$

(57) $$(AXY)T = -(BXY)I + K(X, Y),$$

where $K(X, Y)$ is the linear transformation

(57') $$K(X, Y) = TYAX - TXAY + YBX - XBY,$$

whose range is spanned by the four vectors X, Y, TX, TY, so for all X, Y in E its rank $\dim K(X, Y)E \leqslant 4$.

Suppose first that $8 < \dim E \leqslant \infty$. An equation $\mu I = \bar{C}$ between linear transformations of E being impossible for $\mu \neq 0$, $\operatorname{rank} \bar{C} < \dim E$ (I = identity operator), every linear transformation $T \in \mathfrak{L}(E; E)$ has at most one representation

$$T = \lambda I + \bar{C} \quad \text{with} \quad \tfrac{1}{2} \dim E > \operatorname{rank} \bar{C}.$$

Having $\operatorname{rank} K(X, Y) \leqslant 4$ we apply this fact to (57). If $AXY \neq 0$, by (57)

$$T = -(BXY)(AXY)^{-1} I + (AXY)^{-1} K(X, Y) = -\lambda I + C,$$

so it follows, λ is a constant such that, for all X, Y with $AXY \neq 0$,

(58) $$BXY = \lambda AXY.$$

If $AXY = 0$, then, by (57), $BXY = 0$, since $\operatorname{rank} K(X, Y) < \dim E$, so (58) holds for all X, $Y \in E$. In fact, it holds also in the cases

$$4 < \dim E = 2n \leqslant 8 :$$

by taking the *Trace* in (57), one gets

$$0 = (2n-4)BXY + (AXY) \operatorname{Trace} T,$$

so (58) holds with $\lambda = -(\operatorname{Trace} T)(2n-4)^{-1}$. Substitution of

$$B = \lambda A \quad \text{and} \quad (X, Y, U, V) = (X_0, Y_0, U_0, V_0)$$

to (55') gives

$$0 = (A \wedge B)(X_0, Y_0, U_0, V_0) = 2\lambda, \quad \text{so} \quad B = 0.$$

47. Denote by $M \in \mathfrak{L}_a^3(E; \mathrm{R})$ a trilinear and skew-symmetric form in a real vector space E. Then

$$\ker M = \{ x \mid x \in E,\ i(x)M = 0 \}, \quad \operatorname{rank} M = \operatorname{codim} \ker M,$$

and, for a bilinear form $B \in \mathfrak{L}_a^2(E; \mathrm{R})$, we have analogously

$$rank\, B = codim\ ker\ B,$$

which can also be written with help of the linear function $B_b = \{ y \to i(y)B \}$:

$$rank\, B = rank\, B_b = dim\ Im\, B_b$$

(where $Im\, B_b = \{ B_b y \mid y \in E \}$). Having

$$rank_{(2,3)} M = \max_{x \in E} rank\, i(x)M,$$

we get

LEMMA 10. *If* $0 \neq M \in \mathfrak{L}_a^3(E; \mathrm{R})$, *then* $rank_{(2,3)} M < rank\, M$.

PROOF. It is sufficient to show that, for all $x \in E$,

$$(59) \qquad ker M \subset ker\ i(x)M$$

properly. Since

$$-M(x, y, z) = (i(y)M)(x, z),$$

we first have

$$ker M = \{ x \mid i(x)M = 0 \} = \{ x \mid M(x, y, z) = 0 \ \text{ for all } y, z \in E \}$$

$$= \{ x \mid (i(y)M)(x, z) \ \text{ for all } y, z \in E \}$$

$$= \{ x \mid i(x)(i(y)M) = 0 \ \text{ for all } y \in E \},$$

so

$$(60) \qquad ker M = \bigcap_{y \in E} ker\ i(y)M.$$

Given $x \in E$, if $ker\ i(x)M = E$, then (59) is true since otherwise $ker M = E$ which means that $M = 0$. If again $ker\ i(x)M \neq E$, then there exists $x \in E$ with

$$0 \neq (i(x)M)z = \{ y \to M(x, z, y) \} = \{ y \to -M(z, x, y) \} = -(i(z)M)x,$$

so $x \notin ker\ i(z)M$ and, by the skew-symmetry of M, $x \in ker\ i(x)M$. Consequently

$$ker\ i(x)M \ni x \notin \bigcap_{y \in E} i(y)M = ker M,$$

from which (59) follows.

LEMMA 11. *In a 2n-dimensional* ($n \geqslant 2$) *linear symplectic space* (E, A),

$$rank_{(2,3)}(\mu \wedge A) = 2n - 2, \quad \text{for all} \quad 0 \neq \mu \in E^*.$$

Indeed, since A is regular, that is, the corresponding linear map

$A_b = \{ h \to i(h)A \}$ is bijective, there exists a vector

$$0 \neq v = A_b^{-1}(\mu) \quad \text{with} \quad i(v)A = \mu .$$

For all $h, k, l \in E$, the value of $\mu \wedge A$ is

$$(\mu \wedge A)(h,k,l) = \mu(h)A(k,l) - \mu(k)A(h,l) + \mu(l)A(h,k),$$

and denoting

$$B = i(h)(\mu \wedge A) \quad \text{and} \quad B_b = \{ k \to i(k)B \} \in \mathfrak{L}(E; E^*),$$

we get

$$\begin{aligned} ker B = ker B_b &= \{ k \mid \mu(h)i(k)A - \mu(k)i(h)A + A(h,k)i(v)A = 0 \} \\ &= \{ k \mid A(v,h)k = A(v,k)h - A(h,k)v \} \end{aligned}$$

so, since A_b is a bijection $E \to E^*$,

$$\begin{aligned} ker B = ker\, i(h)(\mu \wedge A) &= \{ k \mid \mu(h)k = \mu(k)h - A(h,k)v \} \\ &= \{ k \mid \mu(h)A_b k = \mu(k)A_b h - A(h,k)A_b v \}. \end{aligned}$$

If (h, v) is a linearly dependent pair, then one verifies directly that

$$i(h)(\mu \wedge A) = 0 , \quad \text{so} \quad rank\, i(h)(\mu \wedge A) = 0.$$

If, on the other hand, one takes (v, h) linearly independent and such that $A(v, h) \neq 0$, then from the last equation it follows that

$$k = \alpha h + \beta v \quad (\alpha, \beta \in \mathrm{R}) \quad \text{for all} \quad k \in ker\, i(h)(\mu \wedge A),$$

and one verifies easily that all such linear combinations belong to the kernel in question. Hence $dim\, ker\, i(h)(\mu \wedge A) = 2$, so

$$rank\, i(h)(\mu \wedge A) = 2n - 2.$$

The last possibility is to take h such that $A(v, h) = 0$ with (v, h) linearly independent. In this case,

$$k \in ker\, i(h)(\mu \wedge A) \quad \text{iff} \quad A(v,k) = 0 = A(h,k),$$

so $dim\, ker\, i(h)(\mu \wedge A) = 2n - 2$ and hence $rank\, i(h)(\mu \wedge A) = 2$. Consequently

$$rank_{(2,3)}(\mu \wedge A) = \max_{x \in E} rank\, i(h)(\mu \wedge A) = 2n - 2.$$

48. LEMMA 12. *Let E and F denote two real Hilbert spaces with a*

common dimension [13] $\dim E = \dim F$. *Given strongly regular skew-symmetric forms* $A \in \mathfrak{L}_a^2(E;\mathrm{R})$ *and* $B \in \mathfrak{L}_a^2(F;\mathrm{R})$, *the linear symplectic space* (E, A) *is isomorph with* (F, B).

PROOF. Following the idea given in [22], Section 5, we first have:

LEMMA 13. *Suppose that in a Hilbert space* $(\mathfrak{H}, G)$ *there is given two symplectic forms* A *and* $\bar{A}$ *and a pair of closed mutually G-orthogonal and complementary subspaces* U *and* V, *which are maximal isotropic with respect to the two forms* A *and* $\bar{A}$. *Then the symplectic spaces* $(\mathfrak{H}, A)$ *and* $(\mathfrak{H}, \bar{A})$ *are isomorph.*

PROOF. Since V is maximal isotropic with respect to A we have

$$v \in V \quad \text{iff} \quad A(v, u) = 0 \quad \text{for all } u \in V.$$

Indeed, every $v \in V$ satisfies this condition because of the isotropy of V, and conversely from the condition it follows that the linear subspace spanned by V and v is isotropic, which is possible only if $v \in V$, since V is maximal. Denoting with $\langle h, k\rangle = G(h, k)$ the values of the Hilbert scalar product, we have

$$A(h, k) = \langle Th, k\rangle \quad \text{for all } h, k \in \mathfrak{H}.$$

Here T is (because of the strong regularity of A) a linear homeomorphism $\mathfrak{H} \to \mathfrak{H}$. Given $h \in V$, by the isotropy

$$\langle Th, v\rangle = A(h, v) = 0 \quad \text{for all } v \in V,$$

so $Th \in V^{\perp} = U$ and, given $u \in U$, by the orthogonality of U and V,

$$A(T^{-1}u, v) = \langle u, v\rangle = 0 \quad \text{for all } v \in V,$$

so $T^{-1}u \in V$. Thus $T(V) = U$, and equally, with

$$\bar{A}(h, k) = \langle \bar{T}h, k\rangle \quad (h, k \in \mathfrak{H}),$$

$\bar{T}(V) = U$. Denoting by P and $Q = I - P$ the complementary orthogonal projection operators on U and V resp., it is easily verified that the linear transformation

$$X = P + T^{-1}\bar{T}Q \quad \text{of } \mathfrak{H}$$

[13] The dimension of a Hilbert space is defined as the cardinality of its orthonormal basis (see [2], IV-4-15).

is toplinear isomorphism $(\mathcal{H}, \bar{A}) \to (\mathcal{H}, A)$.

LEMMA 14. *Suppose in the Hilbert space E there is given a strongly regular symplectic form A and a maximal A-isotropic linear subspace $U \subset E$. Then there exists a maximal isotropic subspace V and a Hilbert scalar product $G = \langle .,. \rangle$ such that U and V are G-orthogonal complementary subspaces with equal dimensions and G defines the Hilbert topology of E.*

PROOF. Denoting by $\langle\langle h, k \rangle\rangle$ the original Hilbert scalar product of the E-vectors h and k, we first have in the space $(E, \langle\langle .,. \rangle\rangle)$ the representation

$$\langle\langle Sh, k \rangle\rangle = A(h, k) = -A(k, h) = -\langle\langle Sk, h \rangle\rangle$$

of A by means of the corresponding skew-adjoint linear homeomorphism $S = -S^* \in \mathfrak{L}(E; E)$. In the canonical representation (see for instance [20], 110) $S = JH = HJ$ of the linear homeomorphism S as a product of a positive operator $H = H^*$ with $H^2 = S^*S$ and a unitary one J, the two factors are linear homeomorphisms, and from

$$-JH = -S = S^* = (JH)^* = (HJ)^* = J^{-1}H,$$

we get

$$J^2 = -I, \tag{61}$$

the scalar product $G = \langle .,. \rangle$ with values

$$G(h, k) = \langle h, k \rangle = \langle\langle Hh, k \rangle\rangle$$

defining the same topology in E as does $\langle\langle .,. \rangle\rangle$. Furthermore,

$$A(h, k) = \langle Jh, k \rangle, \quad \text{for all } h, k \in E, \tag{62}$$

and because of (61), J is isometric with respect to both G and A. Since $U \subset E$ is A-isotropic and A is bounded, also the closure $\bar{U}$ is isotropic, so by the maximality $U = \bar{U}$ is closed and we have the representation of E as a direct sum $E = U + V$ of mutually G-orthogonal Hilbert spaces, with

$$V = \{ v \mid v \in E, \langle v, u \rangle = 0 \text{ for all } u \in U \}.$$

To prove Lemma 13 we show that

$$V = J(U) \tag{63}$$

from which the equality of the dimensions of U and V and the maximal A-isotropy of V follow, the former by the G-isometry and the latter by the A-isometry of the linear homeomorphism J. Indeed, we have by (61)and by the maximal A-isotropy of U the following equivalent relations

$$v_0 \in J(U) \Longleftrightarrow v_0 = J u_0 , \quad u_0 \in U \Longleftrightarrow J v_0 = - u_0 \in U$$
$$\Longleftrightarrow A(J v_0 , u) = 0 \text{ for all } u \in U$$
$$\Longleftrightarrow < J^2 v_0 , u> = 0 \text{ for all } u \in U \quad \Longleftrightarrow \quad v_0 \in U^{\perp} = V,$$

so (63) holds.

To prove at last Lemma 12, note that by Zorn's Lemma there exist in the symplectic Hilbert spaces (E, A) and (F, B) maximal isotropic linear subspaces $U \subset E$ and $U' \subset F$. By Lemma 14 a Hilbert scalar product G, resp. $\bar{G}$, can be defined in E, resp. in F, in such a way that the complementary orthogonal space $V = U^{\perp}$, resp. $V' = U'^{\perp}$, is maximal A-isotropic, resp. maximal B-isotropic, and

$$dim\, V = dim\, U \ , \quad \text{resp. } dim\, V' = dim\, U' .$$

Since the dimensions of the Hilbert spaces E and F are by hypothesis equal, also

$$dim\, U = dim\, U', \quad dim\, V = dim\, V',$$

and there exists consequently an isomorphism $T: (E, G) \to (F, \bar{G})$ of the Hilbert spaces such that $T(U) = U'$, $T(V) = V'$. We now have in the Hilbert space (E, G) two strongly regular skew-symmetric forms A and $\bar{A} = T_* B$ which satisfy the conditions of Lemma 13, since by construction U and V are closed complementary and Hilbert-orthogonal linear subspaces of E, which are maximal isotropic with respect to both A and $\bar{A}$. Consequently there exists a symplectic homomorphism $X: (E, \bar{A}) \to (E, A)$ and the linear homeomorphism $X T^{-1} \in \mathfrak{L}(F; E)$ satisfies

$$(X T^{-1})_* A = (T^{-1})_* X_* A = (T_*)^{-1} \bar{A} = B,$$

the lemma being thus proved.

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