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EXTENSIONS OF FUNCTORS AND THEIR APPLICATIONS

by Jiří ROSICKÝ

This paper is, in fact, devoted to the search of functors F making the following diagram commute

$$(1) \qquad \begin{tikzcd} & M \arrow[dl, "K"'] \arrow[dr, "T"] & \\ A \arrow[rr, dashed, "F"] \arrow[dr, "U"'] & & B \arrow[dl, "V"] \\ & X & \end{tikzcd}$$

This problem was investigated by M. Hušek (see [11]) who introduced a construction producing a solution F which is the greatest one. His construction works for functors V of the topological type (in this case the problem (1) was dealt with in [1] and [2], too). In general this construction gives only a functor $L: A \to B$ and a natural transformation $\lambda: U \to VL$.

We propose another construction which tests the solvability of (1) in the following sense: whenever a solution F exists, then our construction yields a solution (which is the smallest one). This construction consists in a transfinite modification of the functor $L: A \to B$ (for algebraic V one step suffices) and it was introduced in [15] in a special case.

The search for a functor F is the same thing as the study of extensions in the 2-category E_X consisting of categories over X. It is advantageous to work in a more general 2-category D_X. Then, roughly speaking, the Hušek's construction corresponds to the situation that a left extension of T along K in D_X sits in E_X, while our construction computes a left extension of T along K in D_X and modifies it into E_X.

Left extensions in D_X were investigated by R. Guitart [5], where it was given one sufficient condition for their existence. This condition was generalized in [22] (see also [6]). We present another sufficient condition which describes «pointwise» left extensions in D_X. These extensions are not pointwise in the sense of Street [20] but they seem to be the right ones. For instance, they induce a good notion of density. As in any 2-category left extensions in D_X lead to the concept of a density comonad. In addition, in D_X the construction of a density comonad can be parametrized by comonads in X. Many of these results will be stated in their dual version.

If we put in (1) $A = B$ and $U = PV$, where $P: X \to X$ is a given functor, then functors F are lifting of P along V and our results bring a general point of view to questions investigated by M. Sekanina in [18, 19]. We touch also liftings of monads using a variation on parametrized codensity monads. Extensions of functors presented here cover techniques of extensions of full and faithful functors developed in [15, 16, 17].

The Appendix of this paper is devoted to the study of 2-categories E_X^C and D_X^C arising from a 2-category C in the same way as E_X and D_X from the 2-category CAT of categories. There are touched their properties (comma objects, 2-completeness) depending on those of C. Especially, it is investigated what they have from the structure of a cosmos, when C is a cosmos in the sense of Street [21]. For $C = CAT$ these questions are related to the construction of the initial completion of a faithful functor (cf. [8, 10, 23]).

I am indebted to M. Sekanina who stimulated the origin of this investigation, to R. Guitart who hinted me at the possibility to work in D_X, and to both of them for many valuable discussions. A part of this paper has rised during my stay in Paris and I would like to express my gratitude to Prof. Charles and Andrée Ehresmann for their encouragement and interest in my work.

CONTENTS.

1. EXTENSIONS IN D_X.

1.1. DEFINITION. Let X be a category. Denote by D_X the 2-category whose objects are couples (A, U) where $U: A \to X$ is a functor, arrows are couples $(F, \phi): (A, U) \to (B, V)$ where $F: A \to B$ is a functor and $\phi: U \to VF$ a natural transformation, and 2-cells $\alpha: (F, \phi) \to (F', \phi')$ are natural transformations $\alpha: F \to F'$ such that $\phi' = V\alpha . \phi$.

Arrows of D_X are composed as follows:

$$(F', \phi').(F, \phi) = (F'F, \phi'F.\phi).$$

Further denote by D_X^* the 2-category which has the same objects as D_X, arrows $(F, \phi): (A, U) \to (B, V)$ where $\phi: VF \to U$, and 2-cells $\alpha: (F, \phi) \to (F', \phi')$ where $\alpha: F \to F'$ is a natural transformation such that $\phi'. V\alpha = \phi$.

Such 2-categories D_X^C and D_X^{*C} can be defined for every 2-category C and for every object X of it. Then

$$D_X = D_X^{CAT} \quad \text{and} \quad D_X^* = D_X^{*CAT}.$$

Clearly $D_1 = CAT$ where 1 is the one-morphism category. Denote by C^{op} the 2-category which arises from C by the reversing of 2-cells.

1.2. LEMMA. $D_X^* \cong (D_{X^{op}})^{op}$.

PROOF. The isomorphism $D_X^* \to (D_{X^{op}})^{op}$ is given by

$$\begin{tikzcd}
A \arrow[rr, bend left, "F'"] \arrow[rr, bend right, "F"'] \arrow[rr, phantom, "\alpha\Uparrow"] \arrow[dr, "U"'] & \arrow[d, phantom, "\overset{\phi'}{\longleftarrow}\;\underset{\phi}{\longleftarrow}"] & B \arrow[dl, "V"] \\
& X &
\end{tikzcd}
\longmapsto
\begin{tikzcd}
A^{op} \arrow[rr, bend left, "F'^{op}"] \arrow[rr, bend right, "F^{op}"'] \arrow[rr, phantom, "\alpha^{op}\Downarrow"] \arrow[dr, "U^{op}"'] & \arrow[d, phantom, "\overset{\phi'^{op}}{\longrightarrow}\;\underset{\phi^{op}}{\longrightarrow}"] & B^{op} \arrow[dl, "V^{op}"] \\
& X^{op} &
\end{tikzcd}$$

Categories D_X are investigated in [5] and [6]. We will be interested in left extensions in D_X and therefore, with respect to 1.2, in right extensions in D_X^*. We recall that having arrows $K: M \to A$ and $T: M \to B$ in a

2-category C, then a *left extension of* T *along* K *in* C is a couple L, α consisting of an arrow $L: A \to B$ and a 2-cell $\alpha: T \to LK$ such that, for any extension of T along K, i.e. for any couple $S: A \to B$, $\beta: T \to SK$, there is a unique 2-cell

$$\gamma: L \to S \quad \text{such that} \quad \gamma K . \alpha = \beta .$$

All basic concepts concerning 2-categories can be found in [14].

In the sequel we will suppose that we have the following situation in CAT:

$$\text{(2)} \qquad \begin{tikzcd} & M \arrow[dl, "K"'] \arrow[dr, "T"] \arrow[dd, "W"] & \\ A \arrow[dr, "U"'] \arrow[r, phantom, "\kappa"] & {} \arrow[r, phantom, "\tau"] & B \arrow[dl, "V"] \\ & X & \end{tikzcd}$$

1.3. LEMMA. *Let* $L: A \to B$ *be a functor and* $\lambda: U \to VL$, $\bar{\lambda}: T \to LK$ *natural transformations such that* $\lambda K . \kappa = V\bar{\lambda} . \tau$. *Then* $(L, \lambda), \bar{\lambda}$ *is a left extension of* $(T, \tau): (M, W) \to (B, V)$ *along* $(K, \kappa): (M, W) \to (A, U)$ *in* D_X *iff, for every functor* $S: A \to B$ *and for every natural transformations*

$$\sigma: U \to VS, \ \bar{\sigma}: T \to SK \quad \textit{such that} \quad \sigma K . \kappa = V\bar{\sigma} . \tau ,$$

there is a unique natural transformation $\alpha: L \to S$ *such that*

$$\sigma = V\alpha . \lambda \quad \textit{and} \quad \bar{\sigma} = \alpha K . \bar{\lambda} .$$

PROOF is evident.

1.4. EXAMPLE. Consider the following special case of (2)

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "F"] \arrow[dd, "G"] & \\ A \arrow[dr, "H"'] \arrow[r, phantom, "\alpha"] & {} \arrow[r, phantom, "\beta"] & X \arrow[dl, equal] \\ & X & \end{tikzcd}$$

Then (L,λ), $\bar{\lambda}$ is a left extension of (F,β) along $(1_A,\alpha)$ in D_X iff

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\beta"'] & H \arrow[d, "\lambda"] \\ F \arrow[r, "\bar{\lambda}"] & L \end{tikzcd}$$

is a pullback.

1.5. DEFINITION (see [9], 26.3). Let $V: B \to X$ be a functor. A morphism $f: x \to Vb$ is said to be *V-generating* provided that

$$V(r).f = V(s).f \text{ implies } r = s.$$

Dually, we define a *V-cogenerating morphism.*

1.6. REMARK. The characterization 1.3 of left extensions in D_X is more simple when r is V-generating. Namely, the second equality demanded for α is a consequence of the first. Indeed,

$$V\bar{\sigma}.r = \sigma K.\kappa = V\alpha K.\lambda K.\kappa = V(\alpha K.\bar{\lambda}).r$$

and thus $\bar{\sigma} = \alpha K.\bar{\lambda}$.

1.7. DEFINITION (see [22]). A functor $V: B \to X$ is called a *q-functor* if every diagram in X of the form

(3)
$$\begin{tikzcd} x \arrow[r, "v"] \arrow[d, "u"'] & y \\ Vb & \end{tikzcd}$$

has a universal solution. This means that there are

$$\bar{b} \in B,\ \bar{v}: b \to \bar{b} \text{ and } \bar{u}: y \to V\bar{b} \text{ such that } \bar{u}.v = V(\bar{v}).u,$$

and for every

$$b' \in B,\ v': b \to b' \text{ and } u': y \to Vb' \text{ such that } u'.v = V(v').u,$$

there is a unique $t: \bar{b} \to b'$ such that $t.\bar{v} = v'$ and $V(t).\bar{u} = u'$.

1.8. DEFINITION (see [3]). Let $v: Vb \to y$ be a morphism in X. A morphism $\bar{v}: b \to \bar{b}$ in the universal solution of a diagram (3) in which we put $u = 1_{Vb}$ is called a *V-quasiquotient of* v. We say that V *is a functor with quasiquotients* if a V-quasiquotient exists for every $v: Vb \to y$.

1.9. REMARK. Any q-functor is a functor with quasiquotients. If B has pushouts, then the converse implication holds (see [22]). If B has pushouts and V has a left adjoint, then V has quasiquotients (see [4]). If B has finite limits, intersections and coequalizers and V preserves finite limits and intersections, then each regular epimorphism has a V-quasiquotient.

In the sequel we will need to know how properties of a V-quasiquotient of v depend on properties of v. Clearly a V-quasiquotient of an epimorphism is an epimorphism. If X is an (extremally epi-mono)-category and V preserves monomorphisms, then a V-quasiquotient of an extremal epi is extremally epi. If B has and V preserves kernel pairs, then a V-quasiquotient of a regular epimorphism is a regular epimorphism.

1.10. THEOREM (see [5,6,22]). *Let there exist left Kan extensions of T and W along K and V be a q-functor. Then a left extension of (T,τ) along (K,κ) in D_X exists.*

SKETCH OF THE PROOF. Let

$$L_1, \epsilon_1: T \to L_1 K \quad \text{or} \quad L_2, \epsilon_2: W \to L_2 K$$

be left Kan extensions of T or W resp. along K. We get natural transformations $\alpha: L_2 \to VL_1$, $\beta: L_2 \to U$ such that

$$\alpha K.\epsilon_2 = V\epsilon_1.\tau \quad \text{and} \quad \beta K.\epsilon_2 = \kappa.$$

Following [22] A^V is a q-functor. Thus the diagram

$$\begin{tikzcd} L_2 \arrow[r, "\alpha"] \arrow[d, "\beta"'] & VL_1 \\ U & \end{tikzcd}$$

has a universal solution

$$L, \bar{\alpha}: U \to VL, \quad \bar{\beta}: L_1 \to L.$$

Now, $(L, \bar{a})$, $\bar{\beta} K . \epsilon_1$ is the desired left extension.

The same theorem holds in every 2-category C with a 2-terminal object. Now, we are going to state another condition for the existence of left extensions in D_X.

1.11. CONSTRUCTION. Suppose that for every $a \in A$ there exist $L a \in B$, $\lambda_a : U a \to V L a$ and a natural transformation $\beta^a : A(K-, a) \to B(T-, La)$ such that, for any $f : Km \to a$, it holds

$$\lambda_a . U f . \kappa_m = V \beta^a_m(f) . \tau_m ,$$

with the following universal property: for every

$$b \in B, \quad u : U a \to V b \quad \text{and} \quad a : A(K-, a) \to B(T-, b)$$

such that $u . Uf . \kappa_m = V a_m(f) . \tau_m$ there is a unique morphism $t : L a \to b$ such that

$$V t . \lambda_a = u \quad \text{and} \quad B(T, t) . \beta^a = a .$$

Clearly $L : A \to B$ is a functor and $\lambda : U \to V L$ a natural transformation. Namely, $L g$ is defined, if $g : a \to a'$, by the universal property as:

$$\begin{tikzcd}
Wm \arrow[r, "U f . \kappa_m"] \arrow[d, "\tau_m"'] & Ua \arrow[d, "\lambda_a"'] \arrow[dr, "\lambda_{a'} . U g"] & \\
VTm \arrow[r, "V\beta^a_m(f)"] \arrow[drr, "V\beta^{a'}_m(g . f)"'] & VLa \arrow[dr, dashed, "VLg"'] & \\
 & & VLa'
\end{tikzcd}$$

Furthermore, the equality $\bar{\lambda}_m = \beta^{Km}_m(1_{Km})$ defines a natural transformation $\bar{\lambda} : T \to L K$.

1.12. THEOREM. *$(L, \lambda), \bar{\lambda}$ is a left extension of (T, τ) along (K, κ), in the 2-category D_X.*

PROOF. (L, λ), $\bar{\lambda}$ is an extension because

$$V\bar{\lambda}_m . \tau_m = V \beta^{Km}_m(1_{Km}) . \tau_m = \lambda_{Km} . U(1_{Km}) . \kappa_m = \lambda_{Km} . \kappa_m .$$

Let (S, σ), $\bar{\sigma}$ be another extension. The universal property defining La produces a natural transformation $\alpha : L \to S$ as follows:

$$\begin{tikzcd}
Wm \arrow[r, "Uf.\kappa_m"] \arrow[d, "\tau_m"'] & Ua \arrow[d, "\lambda_a"'] \arrow[dr, "\sigma_a"] & \\
VTm \arrow[r, "V\beta^a_m(f)"] \arrow[rr, bend right, "V(Sf.\bar{\sigma}_m)"'] & VLa \arrow[r, dashed, "V\alpha_a"] & VSa
\end{tikzcd}$$

From 1.3 we obtain that (L, λ), $\bar{\lambda}$ is the desired left extension.

1.13. DEFINITION. The left extension constructed in 1.11 will be called *pointwise*.

1.14. REMARK. If X has pullbacks, then D_X is a representable 2-category (in the sense of [20]). It means that D_X has comma objects and 2-pullbacks. A comma object for an op-span

$$\begin{tikzcd}
A \arrow[dr, "F"] \arrow[ddr, "U"'] & & B \arrow[dl, "G"'] \arrow[ddl, "V"] \\
& C \arrow[d, "W"] \arrow[d, phantom, shift right=4, "\alpha"'] \arrow[d, phantom, shift left=4, "\beta"] & \\
& X &
\end{tikzcd}$$

in D_X is the span $((D_0, \phi), (F/G, Z), (D_1, \psi))$ together with the 2-cell

$$\lambda : (F, \alpha).(D_0, \phi) \to (G, \beta).(D_1, \psi),$$

where the span $(D_0, F/G, D_1)$ together with the 2-cell $\lambda : FD_0 \to GD_1$ is a comma object for the op-span (F, C, G) in CAT and the following diagram is a pullback in $CAT(F/G, X)$:

$$\begin{tikzcd}
Z \arrow[r, "\psi"] \arrow[d, "\phi"'] & VD_1 \arrow[d, "\beta D_1"] \\
UD_0 \arrow[r, "W\lambda . \alpha D_0"] & WGD_1
\end{tikzcd}$$

Pointwise left extensions in D_X in our sense do not agree with the pointwise left extensions in the sense of [20] ; the last one are too strong. Our pointwise left extensions are precisely those having the preservation property defining pointwise left extensions in [20] for arrows

$$(G,\gamma):(C,Z)\to(A,U) \text{ such that } \gamma \text{ is iso.}$$

Hence both concepts agree in the case $X = 1$.

1.15. LEMMA. *Let τ be pointwise V-generating (it means that each component of τ is V-generating). Then (L,λ), $\bar{\lambda}$ is a pointwise left extension of (T,τ) along (K,κ) in D_X iff the morphisms*

$$\lambda_a : Ua \to VLa, \quad Lf.\bar{\lambda}_m : Tm \to La \text{ for } f: Km \to a$$

have the following universal property: For any

$$b \in B, \quad u: Ua \to Vb \text{ and } \bar{f}: Tm \to b$$

such that $u.Uf.\kappa_m = V\bar{f}.\tau_m$ for every $f: Km \to a$, there is a unique morphism $t: La \to b$ such that $Vt.\lambda_a = u$.

PROOF. The assignment $a_m(f) = \bar{f}$ defines a natural transformation

$$a: A(K-,a) \to B(T-,b)$$

because τ_m is V-generating. Namely, for every $g: m \to n$, it holds

$$V(a_n(f).Tg).\tau_m = V\bar{f}.\tau_n.Wg = u.Uf.\kappa_n.Wg =$$
$$= u.U(f.Kg).\kappa_m = V(\overline{f.Kg}).\tau_m = Va_m(f.Kg).\tau_m.$$

Then the assertion follows from 1.6 and 1.12.

1.16. PROPOSITION. *Let τ be pointwise V-generating. Then each of the following two conditions ensures the existence of a pointwise left extension of (T,τ) along (K,κ) in D_X :*

1° B is complete, well-powered, has a cogenerating set of objects and V preserves limits.

2° M is small, B and X have sums and V is a q-functor.

PROOF. It is sufficient to produce a universal solution from 1.15. Supposing 1, one can show that the category of all possible solutions is complete, well-powered and has a cogenerating set of objects. Thus it has an initial object which is the desired universal solution. If M is small and B, X have sums, then instead of searching a universal solution from 1.15, one can search a universal solution of the diagram

$$\begin{tikzcd} \sum\limits_{f: Km \to a} Wm \arrow[r, "v"] \arrow[d, "u"] & Ua \\ V \sum\limits_{f: Km \to a} Tm & \end{tikzcd}$$

But now the property of being a q-functor makes the job.

1.17. PROPOSITION. *Let κ be an isomorphism, K full and (L, λ), $\bar{\lambda}$ a pointwise left extension of (T, τ) along (K, κ) in D_X. Then each of the following two conditions ensures that $\bar{\lambda}$ is an isomorphism:*

1º K is faithful;

2º τ is pointwise V-generating.

PROOF. Put $\beta_m^{Kn}(f) = Tf'$ for every $f: Km \to Kn$ such that $f = Kf'$. It holds

$$\tau_n . \kappa_n^{-1} . Uf . \kappa_m = \tau_n . \kappa_n^{-1} . UKf' . \kappa_m = \tau_n . Wf' =$$
$$= VTf' . \tau_m = V\beta_m^{Kn}(f) . \tau_m .$$

In the case of 2, it implies that $\beta^{Kn}: A(K-, Kn) \to B(T-, Tn)$ is a natural transformation (compare with the proof of 1.15). Let K be faithful and $g: m' \to m$. Then

$$\beta_m^{Kn}(f) . Tg = T(f' . g) = T((f . Kg)') = \beta_{m'}^{Kn}(f . Kg).$$

So β^{Kn} is again a natural transformation. It suffices to show that

$$Tn, \quad \tau_n . \kappa_n^{-1} \quad \text{and} \quad \beta^{Kn}$$

have the universal property from 1.11. Let us have b, u, a from 1.11. Then $a_n(1_{Kn}): Tn \to b$ is the desired morphism t.

1.18. LEMMA. *Let τ be (pointwise) V-generating and (L, λ), $\bar{\lambda}$ a (pointwise) left extension of (T, τ) along (K, κ) in D_X. Then λ is (pointwise) V-generating.*

PROOF is evident.

1.19. LEMMA. *Let* $(L,\lambda), \bar{\lambda}$ *be a left extension of* (T,τ) *along* (K,κ), *in* D_X. *Then each of the following two conditions ensures that* λ *is pointwise mono:*

1° κ *and* $\bar{\lambda}$ *are iso,* τ *is pointwise mono and*

$$\{\, Uh \mid h: a \to Km,\ m \in M \,\}$$

is mono for every $a \in A$.

2° There is an extension $(S,\sigma), \bar{\sigma}$ *of* (T,τ) *along* (K,κ) *in* D_X *such that* σ *is pointwise mono.*

PROOF is evident.

2. EXTENSIONS IN C_X.

Let C_X be the sub-2-category of D_X which has the same objects as D_X, arrows (F,ϕ) such that ϕ is an isomorphism and any 2-cell in D_X between arrows of C_X belongs to C_X.

We will be interested in left and right extensions in C_X. In this Part we will work in the situation (2) considered in C_X (i.e. κ and τ will be isomorphisms). Evidently, if $(L,\lambda), \bar{\lambda}$ is a left extension of (T,τ), along (K,κ) in D_X and λ is an isomorphism, then $(L,\lambda), \bar{\lambda}$ is a left extension of (T,τ) along (K,κ) in C_X. Similarly, one can treat right extensions.

2.1. PROPOSITION. *If* $(R,\rho), \bar{\rho}$ *is a right extension of* (T,τ^{-1}) *along* (K,κ^{-1}) *in* D^*_X *and* ρ *is an isomorphism, then* $(R,\rho^{-1}), \bar{\rho}$ *is a right extension of* (T,τ) *along* (K,κ) *in* C_X.

PROOF follows from the fact that the assignment

$$a:(F,\phi)\to(F',\phi') \mapsto a:(F,\phi^{-1})\to(F',\phi'^{-1})$$

gives an isomorphism $C_X \to C^*_X$.

If the extension from 2.1 is pointwise, then we obtain the construction from [11]. A typical situation in which right extensions in C_X are described in 2.1 is given in the following theorem. Firstly we recall some definitions from [7].

A source in X is a class of morphisms $\{ f_i : x \to x_i \}_{i \in I}$ in X. It is called *mono* if

$$f_i \cdot r = f_i \cdot s \quad \text{for each } i \in I \quad \text{implies} \quad r = s.$$

X is called a *(E, M)-category* if E is a class of epimorphisms in X closed under compositions with isomorphisms, M is a class of sources in X closed under compositions with isomorphisms, and the following conditions hold:

(a) X is (E, M)-factorizable, i. e. for every source $\{ f_i \}_{i \in I}$ in X there exist $e \in$ E and $\{ g_i \}_I \in$ M such that $f_i = g_i \cdot e$ for each $i \in I$.

(b) X has the (E, M)-diagonalization prpoerty, i. e. whenever f and e are morphisms and $\{ g_i \}_I$ and $\{ f_i \}_I$ are sources in X such that

$$e \in \mathrm{E}, \quad \{ g_i \}_I \in \mathrm{M} \quad \text{and } f_i \cdot e = g_i \cdot f \text{ for each } i \in I,$$

then there exists a morphism g such that

$$g \cdot e = f \quad \text{and} \quad g_i \cdot g = f_i \quad \text{for each } i \in I.$$

Let X be an (E, M)-category. We say that a functor $V : B \to X$ is (E, M)*-topological* if for every source $\{ f_i : x \to V b_i \}_I \in$ M there is a source $\{ \bar{f}_i : b \to b_i \}_I$ in B and an isomorphism $f : V b \to x$ such that

$$f_i \cdot f = V \bar{f}_i \quad \text{for each} \quad i \in I,$$

with the following universal property: For every source $\{ g_i : b' \to b_i \}_I$ in B and every morphism $g : Vb' \to x$ in X such that $f_i \cdot g = V g_i$ for each $i \in I$ there is a unique morphism $k : b' \to b$ such that

$$f \cdot V k = g \quad \text{and} \quad \bar{f}_i \cdot k = g_i \quad \text{for each } i \in I.$$

Further, V is called *absolutely topological* if it is (E, M)-topological for any (E, M)-structure on X.

2.2. THEOREM. *Let* V *be an* (E, M)*-topological functor and*

$$\{ U f \mid f : a \to K m, \ m \in M \} \in \mathrm{M} \quad \textit{for every} \quad a \in A.$$

Then there exists a right extension of (T, τ^{-1}) *along* (K, κ^{-1}) *in* C_X.

PROOF. Since V is (E, M)-topological and

$$\{\tau_m \cdot \kappa_m^{-1} \cdot Uf \mid f: a \to Km,\ m \in M\} \in \mathrm{M},$$

the following diagram has a universal solution

$$Ra,\quad \rho_a : VRa \to Ua,\quad \bar{f}: Ra \to Tm$$

such that ρ_a is an isomorphism.

$$\begin{tikzcd} Ua \arrow[r, "\kappa_m^{-1} Uf"] & Wm \\ & VTm \arrow[u, "\tau_m^{-1}"'] \end{tikzcd}$$

Since any (E, M)-topological functor is faithful (see [7]), τ is pointwise V-generating and the assertion follows from the dual of 1.15 and from 2.1.

2.3. COROLLARY. *If V is absolutely topological, then there exists a left extension of (T, τ) along (K, κ) and a right extension of (T, τ^{-1}) along (K, κ^{-1}) in C_X.*

PROOF follows from 2.2 and from the fact that any absolutely topological functor is absolutely co-topological (see [7]).

In [2] it is shown that this property is characteristic for absolutely topological functors.

2.4. PROPOSITION. *Let any morphism of* E *be extremally epi and V be* (E, M)*-topological. If there exists an extension of (T, τ) along (K, κ) in C_X, then there exists a left extension, too.*

PROOF. Following Lemma 6.1 from [7] there is a left extension (L, λ), $\bar{\lambda}$ of (T, τ) along (K, κ) in D_X such that λ is pointwise extremally epi. Hence the assertion follows from 1.19, 2°.

If V is not (E, M)-topological, then the Hušek's construction usually does not yield a left extension in C_X. We shall state a more general construction of left extensions in C_X which was introduced in a special case in [15] (compare with 2.13).

2.5. CONSTRUCTION. Let X be an (extremally epi-monosource)-category and $L: A \to B$ be a functor. Take $a \in A$ and consider an (extremally epi-monosource) factorization of $\{ VLh \mid h: a \to Km,\ m \in M \}$.

$$\begin{tikzcd}
VLa \arrow[rr, "VLh"] \arrow[dr, "v_a"'] & & VLKm \\
& x_a \arrow[ur, "\hat{h}"'] &
\end{tikzcd}$$

Let γ_a^L be a V-quasiquotient of v_a

$$\begin{tikzcd}
VLa \arrow[rr, "v_a"] \arrow[dr, "V\gamma_a^L"'] & & x_a \arrow[dl, "k_a"] \\
& V\bar{L}a &
\end{tikzcd}$$

Let $g: a' \to a$ be a morphism in A. Then

$$\widehat{h.g}.\, v_{a'} = VL(h.g) = \hat{h}.\, v_a .\, VLg.$$

Since v_a is extremally epi and

$$\{ \hat{h} \mid h: a \to Km,\ m \in M \}$$

is a monosource, the (extremally epi-monosource)-diagonalization property of X provides a unique morphism $g': x_{a'} \to x_a$ such that

$$g'.\, v_{a'} = v_a .\, VLg \quad \text{and} \quad \hat{h}.\, g' = \widehat{h.g}\ .$$

Hence there is a unique morphism $\bar{L}g: \bar{L}a' \to \bar{L}a$ such that

$$\bar{L}g.\gamma_{a'}^L = \gamma_a^L . Lg \quad \text{and} \quad V\bar{L}g.\, k_{a'} = k_a . g'.$$

Thus $\bar{L}: A \to B$ is a functor and $\gamma^L: L \to \bar{L}$ a natural transformation. Further, $\gamma^L K$ is iso because

$$\{ VLh \mid h: Kn \to Km,\ m \in M \}, \text{ for any } n \in M,$$

is mono. Namely, 1_{VLKn} belongs to this source.

2.6. LEMMA. *Let X be an (extremally epi-monosource)-category and $V\gamma^L$ be pointwise extremally epi. Then $\{ V\bar{L}h \mid h: a \to Km,\ m \in M \}$ is a monosource for every $a \in A$.*

PROOF. Since

$$V\gamma^L_{Km}.\hat{h}.v_a = V\gamma^L_{Km}.VLh = V(\bar{L}(h).\gamma^L_a),$$

$\{V\gamma^L_{Km}.\hat{h}\}_h$ is mono and $V\gamma^L_a$ extremally epi, there is a unique morphism $t: V\bar{L}a \to x_a$ such that

$$t.V\gamma^L_a = v_a \text{ and } V\gamma^L_{Km}.\hat{h}.t = V\bar{L}h \text{ for any } h: a \to Km.$$

Clearly t is iso with the inverse k_a. Thus $\{V\bar{L}h\}_h$ is mono.

2.7. CONSTRUCTION. Let $L: A \to B$ be a functor. Put $L_0 = L$. Let α be an ordinal number. Suppose that for any $\beta < \alpha$ we have a functor $L_\beta: A \to B$ and for any $\beta' \leq \beta'' < \alpha$ a natural transformation $\gamma^{\beta',\beta''}: L_{\beta'} \to L_{\beta''}$ such that

$$\gamma^{\beta,\beta} = 1_{L_\beta} \text{ and } \gamma^{\beta'',\beta'''}.\gamma^{\beta',\beta''} = \gamma^{\beta',\beta'''}.$$

If α is isolated, we put

$$L_\alpha = \bar{L}_{\alpha-1} \text{ and } \gamma^{\beta,\alpha} = \gamma^L.\gamma^{\beta,\alpha-1} \text{ for every } \beta < \alpha.$$

For α limit, $\gamma^{\beta,\alpha}: L_\beta \to L_\alpha$ are components of a colimit cone of a diagram having L_β, $\beta < \alpha$ as objects and

$$\gamma^{\beta',\beta''}, \quad \beta' \leq \beta'' < \alpha$$

as morphisms.

Suppose that for any $a \in A$ there exists α_a such that $\gamma_a^{\alpha_a,\beta}$ is iso for every $\beta \geq \alpha_a$. Put $L_*a = L_{\alpha_a}a$ and

$$L_*g = L_\alpha g \text{ for } g: a \to a', \text{ where } \alpha = max\{\alpha_a, \alpha_{a'}\}.$$

In this way we obtain a functor $L_*: A \to B$ and the equality $\gamma^\alpha_a = \gamma_a^{\alpha,\alpha_a}$ defines a natural transformation $\gamma^\alpha: L_\alpha \to L_*$. By 2.5, $\gamma^\alpha K$ is iso for any α.

2.8. PROPOSITION. *Let X be an (extremally epi-monosource)-category and V a functor with quasiquotients. Then each of the following conditions ensures the existence of L_* for any functor L:*

1° V preserves monomorphisms and extremal epimorphisms.

2° B has kernel pairs and V preserves kernel pairs and regular epimorphisms.

3° M is a small and B a co-well-powered category with colimits of functors $D: S \to B$ where S is a well-ordered set.

4° In comparison with 3, B is only extremally co-well-powered, but in addition B has and V preserves kernel pairs.
In cases 1 and 2, it holds $L_* = \bar{L}$.

PROOF. 1° By 1.9, $V\gamma^L$ is pointwise extremally epi. Following 2.6, $L_*=\bar{L}$ exists.

2° Similarly, $L_* = \bar{L}$ exists by 1.9 and 2.6.

3° Clearly L_α exists for every a. Further any $\gamma_a^{\beta,\alpha}$ is an epimorphism. Since B is co-well-powered, $L_{*}a$ exists for every a.

4° By 1.9 any $\gamma^{\alpha-1,\alpha}$ is pointwise regularly epi. Thus $\{ \gamma_a^{\beta,\alpha} \}_{\beta,\alpha}$ is a right multistrict analysis, in the sense of [12]. Following 2.3 of [12] any $\gamma_a^{\beta,\alpha}$ is extremally epi. Since B is extremally co-well-powered, $L_{*}a$ exists for every a.

2.9. THEOREM. *Let X be an (extremally epi-monosource)-category and*

$$\{ Uh \mid h: a \to Km,\ m \in M \}, \quad \text{for every } a \in A,$$

be mono. Let there exist the functor L_* where $(L,\lambda), \bar{\lambda}$ is a left extension of (T,τ) along (K,κ) in D_X such that λ is an isomorphism. Suppose that there exists an extension of (T,τ) along (K,κ) in C_X. Then

$$(L_*,\ V\gamma^0.\lambda), \gamma^0 K.\bar{\lambda}$$

is a left extension of (T,τ) along (K,κ) in C_X.

PROOF. Evidently, $(L_*, V\gamma^0.\lambda), \gamma^0 K.\bar{\lambda}$ is an extension of (T,τ) along (K,κ) in D_X. Let $(S,\sigma), \bar{\sigma}$ be an extension of (T,τ) along (K,κ) in C_X. By 1.3 there is a unique natural transformation $\delta^0: L_0 = L \to S$ such that:

$$\sigma = V\delta^0.\lambda \quad \text{and} \quad \bar{\sigma} = \delta^0 K.\bar{\lambda}\ .$$

Let a be an ordinal number and suppose that, for any $\beta < a$ there is defined $\delta^\beta: L_\beta \to S$ such that

$$\delta^{\beta''}.\gamma^{\beta',\beta''} = \delta^{\beta'} \quad \text{for any } \beta' \leqslant \beta'' < a\ .$$

Let a be isolated, $a \in A$, and consider the diagram

$$\begin{tikzcd}
 & VSa \arrow[r, "VSh"] & VSKm \\
VL_a a \arrow[ur, dashed, "V\delta_a^a"] & VL_{a-1}a \arrow[l, "V\gamma_a^{a-1,a}"'] \arrow[u, "V\delta_a^{a-1}"] \arrow[r, "VL_{a-1}h"] \arrow[d, "v_a"] & VL_{a-1}Km \arrow[u, "V\delta_{Km}^{a-1}"'] \\
 & x_a \arrow[ul, "k_a"] \arrow[ur, "\hat{h}"'] \arrow[uu, dashed, bend left=100, "t"] &
\end{tikzcd}$$

where the triangles at the bottom express the definition of $L_a = \bar{L}_{a-1}$. Since $\{Uh\}_h$ is mono and σ_a iso, $\{VSh\}_h$ is mono. Thus there is a unique morphism t such that

$$t.v_a = V\delta_a^{a-1} \quad \text{and} \quad VSh.t = V\delta_{Km}^{a-1}.\hat{h}.$$

So there is a unique morphism δ_a^a such that

$$\delta_a^a.\gamma_a^{a-1,a} = \delta_a^{a-1} \quad \text{and} \quad V\delta_a^a.k_a = t.$$

Since $\gamma_a^{a-1,a}$ is epi, $\delta^a: L_a \to S$ is a natural transformation. For a limit, the existence of δ^a follows immediately from the construction of L_a as a colimit. Now, $\delta_a = \delta_a^{a_a}$ defines a natural transformation

$$\delta: L* \to S \quad \text{such that} \quad \delta.\gamma^0 = \delta^0.$$

δ is a morphism of the corresponding extensions and the unicity of δ follows from 2.1 and from the fact that γ^0 is epi. It remains to prove that $V\gamma^0.\lambda$ is iso. By the supposition there is an extension (S,σ), $\bar{\sigma}$ of (T,τ) along (K,κ) in C_X. We will show that $\sigma^{-1}.V\delta$ is the inverse of $V\gamma^0.\lambda$. Clearly it is a left inverse. Further it holds

$$V(\delta_{Km}.L*h).(V\gamma_a^0.\lambda_a).(\sigma_a^{-1}.V\delta_a) = V(\delta_{Km}.L*h).$$

Since $L*a = \bar{L}*a$,

$$\{VL*h \mid h: a \to Km,\ m \in M\}$$

is mono. Since $\bar{\sigma}$, $\bar{\lambda}$ and $\gamma^0 K$ are isomorphisms, δK is iso as well. Hence $\sigma_a^{-1}.V\delta_a$ is a right inverse of $V\gamma_a^0.\lambda_a$.

If $(L*, V\gamma^0.\lambda)$, $\gamma^0 K.\bar{\lambda}$ is a left extension of (T,τ) along (K,κ) in C_X, then $\bar{\lambda}$ has to be iso. Further, if there exists an extension of (T,τ)

along (K, κ) in C_X, then λ is extremally mono.

Theorem 2.9 says that, under certain assumptions, $L*$ tests the extendability of (T, r) along (K, κ) in C_X. We are going to give conditions ensuring that $L*$ is really a left extension.

2.10. LEMMA. *Let X be an (extremally epi-mono)-category, $L: A \to B$ a functor and $\lambda: U \to VL$ a pointwise mono natural transformation such that λK is iso. Let for every $a \in A$ be $n \in M$ and $h: a \to Kn$ such that VLh is mono and the monomorphism k in an extremally epi-mono factorization $Uh = k.e$ is an intersection of equalizers of pairs Ur, Us such that $r.h = s.h$. Then λ is iso.*

PROOF. Let $r, s: Kn \to a'$ equalize h. Then $\lambda_{a'}.Ur, \lambda_{a'}.Us$ equalize $\lambda^{-1}_{Kn}.VLh$ and thus Ur, Us equalize $\lambda^{-1}_{Kn}.VLh$ because $\lambda_{a'}$ is mono. Hence there is a unique morphism t such that $k.t = \lambda^{-1}_{Kn}.VLh$. But this implies that $t.\lambda_a = e$ and t is mono because VLh is mono. Hence t is iso, λ_a extremally epi and thus iso.

2.11. LEMMA. *Let $X = Set$, $L: A \to B$ be a functor such that*

$$\{ VLh \mid h: a \to Km,\ m \in M \}, \text{ for every } a \in A,$$

is mono and $\lambda: U \to VL$ be a natural mono transformation such that λK is iso. Let for every $f: Km \to a$ such that $m \in M$ and $a \in A - K(M)$ there exist $n \in M$ and $h_f: a \to Kn$ with the following properties:

(a) For any $y \in UKn - U(h_f.f)(UKm)$ there are r, s such that

$$r.h_f.f = s.h_f.f \quad \text{and} \quad Ur(y) \neq Us(y).$$

(b) for any $m' \in M$ and $h: a \to Km'$ the source

$$\{ Ut \mid t \text{ has domain } Km' \text{ and there is } t' \text{ such that } t.h.f = t'.h_f.f \}$$

is mono.

Further let there exist a functor $V': B \to Set$ such that

$$\{ V'Lf \mid f: Km \to a,\ m \in M \}, \text{ for any } a \in A,$$

is epi and a natural mono transformation $\delta: V \to V'$. Then δ is iso.

PROOF. Let $a \in A - K(M)$ and $x \in VLa$. Since $\{ V'Lf \mid f: Km \to a \}$ is epi one can find

$$m \in M, \quad z \in V'LKm \quad \text{and} \quad f: Km \to a$$

such that $\delta_{La}(x) = V'Lf(z)$. Suppose that

$$y = \lambda^{-1}_{Kn} . VLh_f(x) \notin U(h_f . f)(UKm).$$

Considering $r, s: Kn \to a'$, from (a) we get that

$$\delta_{La'} . \lambda_{a'} . Ur(y) = \delta_{La'} . VLr . \lambda_{Kn}(y) = \delta_{La'} . VL(r.h_f)(x) =$$
$$= V'L(r.h_f) . \delta_{La}(x) = V'L(r.h_f.f)(z) = V'L(s.h_f.f)(z) =$$
$$= \delta_{La'} . \lambda_{a'} . Us(y).$$

Since $\delta_{La'}$ and $\lambda_{a'}$ are mono, it holds $Ur(y) = Us(y)$, which is a contradiction. Thus $y \in U(h_f . f)(UKm)$ and so there is a $w' \in UKm$ such that $y = U(h_f . f)(w')$. Hence

$$VLh_f(x) = VL(h_f . f)(w), \quad \text{where} \quad w = \lambda_{Km}(w').$$

Consider $h: a \to Km'$ and take $t: Km' \to a'$ such that $t.h.f = t'.h_f.f$ for a suitable t'. Successively it holds

$$\delta_{La'} . VL(t.h)(x) = V'L(t.h) . \delta_{La}(x) = V'L(t.h.f)(z) =$$
$$= V'L(t'.h_f.f)(z) = \delta_{La'} . VL(t'.h_f)(x) =$$
$$= \delta_{La'} . VL(t'.h_f.f)(w) = \delta_{La'} . VL(t.h.f)(w),$$

$VL(t.h)(x) = VL(t.h.f)(w)$ and

$$Ut . \lambda^{-1}_{Km'} . VL(h)(x) = Ut . \lambda^{-1}_{Km'} . VL(h.f)(w).$$

Following (b),

$$VL(h)(x) = VL(h.f)(w).$$

Since $\{VLh\}_h$ is mono, $x = VLf(w)$. We have proved that

$$\{ VLf \mid f: Km \to a, \ m \in M \}$$

is epi. It immediately implies that λ_a is epi and thus iso.

The condition (a) in 2.11 says that the monomorphism k in an extremally epi-mono factorization $U(h_f . f) = k.e$ is an intersection of equalizers of pairs Ur, Us such that $r.h_f.f = s.h_f.f$. This supposition is considerably less restrictive than the corresponding condition in 2.10.

2.12. THEOREM. *Let X be an (extremally epi-monosource)-category and*

$$\{\, Uh \mid h: a \to Km,\ m \in M \,\}, \quad \textit{for any } a \in A,$$

be mono. Let for every $a \in A$ there exist $n \in M$ and $h_0 : a \to Kn$ with the following properties:

(a) A monomorphism k in an extremally epi-mono factorization $Uh_0 = k.e$ is an intersection of equalizers of pairs Ur, Us such that $r.h_0 = s.h_0$.

(b) The source

$$\{\, Ut \mid t \textit{ has domain } Km \textit{ and there exists } t' \textit{ such that } t.h = t'.h_0 \,\}$$

is mono for any $m \in M$ and $h : a \to Km$.
Let there exist a functor L_*, where (L, λ), $\bar\lambda$ is a left extension of (T, τ) along (K, κ) in D_X such that $\bar\lambda$ is iso. Then $(L_*, V\gamma^0.\lambda)$, $\gamma^0 K.\bar\lambda$ is a left extension of (T, τ) along (K, κ) in C_X.

PROOF. With respect to 2.9 it suffices to show that $V\gamma^0.\lambda$ is iso. Similarly as in 1.19, 1, one deduces that $V\gamma^0.\lambda$ is pointwise mono. Since $\bar\lambda$ is iso, $(V\gamma^0.\lambda)K$ is iso as well. This fact, (b) and the fact that

$$\{\, VL_*h \mid h: a \to Km,\ m \in M \,\}$$

is mono imply that VLh_0 is mono. By 2.10, $V\gamma^0.\lambda$ is iso.

Analogously it is possible to formulate a sufficient condition for L_* to be a left extension in C_X based on 2.11.

2.13. EXAMPLE. Let M be a small category and consider the following special case of (2):

$$\begin{tikzcd}
& M \arrow[dl, "K"'] \arrow[dr, "T"] \arrow[dd, "M(M,-)"] & \\
A \arrow[dr, "A(KM,-)"'] \arrow[r, phantom, "\swarrow\kappa"] & & B \arrow[dl, "B(TM,-)"] \arrow[l, phantom, "\searrow\tau"] \\
& Set &
\end{tikzcd}$$

where $A(KM,-)$ assigns to each $a \in A$ the set of all morphisms $Km \to a$

where $m \in M$. The effect on morphisms is defined by composition and $M(M,-)$ and $B(TM,-)$ are defined in the same way. Natural transformations κ and τ are defined by the assignments

$$\kappa_m(f) = Kf, \quad \tau_m(f) = Tf.$$

It can be shown that (L,λ), $\bar{\lambda}$ is a left extension of (T,τ) along (K,κ) in D_X iff L, $\bar{\lambda}$ is a left Kan extension of T along K. If K and T are full and faithful, then κ and τ are isomorphisms and extensions of (T,τ) along (K,κ) in C_X correspond to left M-full and left M-faithful functors in the sense of [15]. Now, the construction 2.7 provides just the functor L_* from [15]. Then Theorem 2.9 is a generalization of Proposition 2 from [15]. Similarly 2.10 and 2.12 go out from Theorem 3.5 of [16].

3. PARAMETRIZED CODENSITY MONADS.

3.1. DEFINITION. Let $V: A \to X$ be a functor and (P,η,μ) a monad in X. If $(S,\bar{\eta},\bar{\mu})$ is a monad in A and $\sigma: VS \to PV$ a natural transformation such that

$$\sigma . V\bar{\eta} = \eta V \quad \text{and} \quad \sigma . V\bar{\mu} = \mu V . P\sigma . \sigma S,$$

then we say that (S,σ) (more precisely $((S,\bar{\eta},\bar{\mu}),\sigma))$ *is a lax lifting of* (P,η,μ) *along* V.

Morphisms of liftings $\alpha: (S,\sigma) \to (S',\sigma')$ are taken as morphisms of monads

$$\alpha: S \to S' \quad \text{such that } \sigma'. V\alpha = \sigma.$$

In this way we get the category $Z(P,V)$ of lax liftings of a monad P along the functor V.

Liftings of (P,η,μ) along V are lax liftings (S,σ) such that: $\sigma = 1_{PV}$. Any lax lifting (S,σ) determines a functor $\bar{V}: A_S \to X_P$ between Kleisli categories and a natural transformation $\bar{\sigma}: VG_S \to G_P\bar{V}$ where

$$G_S: A_S \to A \quad \text{and} \quad G_P: X_P \to X$$

are the underlying functors.

Let us have functors

$$K : M \to A, \quad V : A \to X \quad \text{and} \quad W : M \to X,$$

a natural transformation $\kappa : VK \to W$ and a monad (P, η, μ) in X. Consider the following special case of the dual of (2):

$$\begin{tikzcd}
 & M \arrow[dl, "K"'] \arrow[dr, "K"] \arrow[dd, "PW"] & \\
A \arrow[dr, "PV"'] \arrow[r, phantom, "P\kappa \Rightarrow"] & {} & A \arrow[dl, "V"] \arrow[l, phantom, "\Leftarrow P\kappa . \eta VK"'] \\
 & X &
\end{tikzcd}$$

3.2. THEOREM. *Let (R, ρ), $\bar{\rho}$ be a right extension of $(K, P\kappa . \eta VK)$ along $(K, P\kappa)$ in D^*_X. Then there are natural transformations $\tilde{\eta}$, $\tilde{\mu}$ exhibiting $(R, \tilde{\eta}, \tilde{\mu})$ as a monad in A such that (R, ρ) is a lax lifting of P along V and $\bar{\rho}$ is an action of R on K. In addition, for any monad $(S, \bar{\eta}, \bar{\mu})$ in A, any lax lifting (S, σ) of P along V and any action $\bar{\sigma}$ of S on K such that*

$$P\kappa . \eta VK . V\bar{\sigma} = P\kappa . \sigma K,$$

there exists a unique morphism of liftings $a : (S, \sigma) \to (R, \rho)$ such that: $\bar{\sigma} = \bar{\rho} . aK$ (i. e. a is a morphism of actions, too).

PROOF. Clearly $(1_A, \eta V)$, 1_K is an extension of $(K, P\kappa . \eta VK)$ along $(K, P\kappa)$ in D^*_X. Thus there is a unique natural transformation $\tilde{\eta} : 1_A \to R$ such that $\rho . V\tilde{\eta} = \eta V$ and $\bar{\rho} . \tilde{\eta}K = 1_K$. Further, $(RR, \mu V . P\rho . \rho R)$, $\bar{\rho} . R\bar{\rho}$ is such an extension as well, because the following diagram commutes:

$$\begin{tikzcd}
VRRK \arrow[r, "VR\bar{\rho}"] \arrow[d, "\rho RK"'] & VRK \arrow[r, "V\bar{\rho}"] \arrow[d, "\rho K"] & VK \arrow[r, "\eta VK"] & PVK \arrow[dddd, "P\kappa"] \\
PVRK \arrow[r, "PV\bar{\rho}"] \arrow[dd, "P\rho K"'] & PVK \arrow[d, "P\eta VK"'] \arrow[dr, equal] & & \\
 & PPVK \arrow[r, "\mu VK"'] \arrow[d, "PP\kappa"'] & PVK \arrow[ddr, "P\kappa"] & \\
PPVK \arrow[r, "PP\kappa"'] \arrow[dr, "\mu VK"'] & PPW \arrow[drr, "\mu W"] & & \\
 & PVK \arrow[rr, "P\kappa"] & & PW
\end{tikzcd}$$

Thus there is a unique natural transformation $\tilde{\mu}: RR \to R$ such that

$$\rho . V\tilde{\mu} = \mu V . P\rho . \rho R \quad \text{and} \quad \bar{\rho} . \tilde{\mu} K = \bar{\rho} . R\bar{\rho} .$$

If we show that $(R, \tilde{\eta}, \tilde{\mu})$ is a monad, then (R, ρ) will be a lax lifting and $\bar{\rho}$ an action. The commutative diagrams

(3)

$$\begin{tikzcd}
VRRR \arrow[rrr, "V\tilde{\mu}R"] \arrow[dr, "\rho RR"] \arrow[ddd, "VR\tilde{\mu}"'] & & & VRR \arrow[rr, "V\tilde{\mu}"] \arrow[d, "\rho R"] & & VR \arrow[dddd, "\rho"] \\
& PVRR \arrow[r, "P\rho R"] \arrow[dd, "PV\tilde{\mu}"'] & PPVR \arrow[r, "\mu VR"] \arrow[d, "PP\rho"'] & PVR \arrow[dr, "P\rho"] & & \\
& & PPPV \arrow[rr, "\mu PV"] \arrow[d, "P\mu V"'] & & PPV \arrow[ddr, "\mu V"'] & \\
VRR \arrow[r, "\rho R"] \arrow[d, "V\tilde{\mu}"'] & PVR \arrow[r, "P\rho"] & PPV \arrow[drrr, "\mu V"] & & & \\
VR \arrow[rrrrr, "\rho"'] & & & & & PV
\end{tikzcd}$$

$$\begin{tikzcd}
RRRK \arrow[rr, "\tilde{\mu}RK"] \arrow[dr, "RR\bar{\rho}"'] \arrow[dd, "R\tilde{\mu}K"'] & & RRK \arrow[r, "\tilde{\mu}K"] \arrow[d, "R\bar{\rho}"] & RK \arrow[ddd, "\bar{\rho}"] \\
& RRK \arrow[r, "\tilde{\mu}K"] \arrow[d, "R\bar{\rho}"'] & RK \arrow[ddr, "\bar{\rho}"] & \\
RRK \arrow[r, "R\bar{\rho}"] \arrow[d, "\tilde{\mu}K"'] & RK \arrow[drr, "\bar{\rho}"] & & \\
RK \arrow[rrr, "\bar{\rho}"] & & & K
\end{tikzcd}$$

and the universality of (R, ρ), $\bar{\rho}$ imply the associativity of $(R, \tilde{\eta}, \tilde{\mu})$. Similarly the commutative diagrams

(4)

$$\begin{tikzcd}
VR \arrow[rrr, "\rho"] \arrow[dr, "\eta VR"] \arrow[dd, "V\tilde{\eta}R"'] & & & PV \arrow[dl, "\eta PV"'] \arrow[dd, equal] \\
& PVR \arrow[r, "P\rho"] & PPV \arrow[dr, "\mu V"'] & \\
VRR \arrow[ur, "\rho R"] \arrow[r, "V\tilde{\mu}"] & VR \arrow[rr, "\rho"] & & PV
\end{tikzcd}$$

$$\begin{tikzcd}
RK \arrow[rrr, "\tilde{\eta} RK"] \arrow[dr, "\bar{\rho}"] \arrow[ddd, "\bar{\rho}"'] & & & RRK \arrow[ddl, "R\bar{\rho}"'] \arrow[ddd, "\tilde{\mu} K"] \\
& K \arrow[dr, "\eta K"] \arrow[ddl, equal] & & \\
& & RK \arrow[dll, "\bar{\rho}"'] & \\
K \arrow[rrr, "\bar{\rho}"'] & & & RK
\end{tikzcd}$$

(5)
$$\begin{tikzcd}
VR \arrow[rrr, "\rho"] \arrow[ddd, "VR\tilde{\eta}"'] & & & PV \arrow[dll, "PV\tilde{\eta}"] \arrow[ddl, "P\eta V"'] \arrow[ddd, equal] \\
& PVR \arrow[dr, "P\rho"] & & \\
& & PPV \arrow[dr, "\mu V"] & \\
VRR \arrow[uur, "\rho R"] \arrow[r, "V\tilde{\mu}"'] & VR \arrow[rr, "\rho"'] & & PV
\end{tikzcd}$$

$$\begin{tikzcd}
RK \arrow[rr, "R\tilde{\eta}K"] \arrow[dr, equal] \arrow[dd, "\bar{\rho}"'] & & RRK \arrow[dl, "R\bar{\rho}"'] \arrow[dd, "\tilde{\mu} K"] \\
& RK \arrow[dl, "\bar{\rho}"'] & \\
K & & RK \arrow[ll, "\bar{\rho}"]
\end{tikzcd}$$

complete the proof that $(R, \tilde{\eta}, \tilde{\mu})$ is a monad.

Let $(S, \bar{\eta}, \bar{\mu})$ be a monad in A, (S, σ) be a lax lifting of P along V, $\bar{\sigma}$ be an action of S on K and suppose that

$$P_K . \eta VK . V\bar{\sigma} = P_K . \sigma K .$$

Thus $(S, \sigma), \bar{\sigma}$ is an extension of $(K, P_K . \eta VK)$ along (K, P_K), which provides a unique natural transformation $\alpha : S \to R$ such that $\sigma = \rho . V\alpha$ and $\bar{\sigma} = \bar{\rho} . \alpha K$. It remains to show that α is a morphism of monads. This follows from the commutative diagrams

(6)
$$\begin{tikzcd}
V \arrow[r, "V\bar{\eta}"] \arrow[d, "V\tilde{\eta}"'] \arrow[drr, "\eta V"'] & VS \arrow[r, "V\alpha"] \arrow[dr, "\sigma"] & VR \arrow[d, "\rho"] \\
VR \arrow[rr, "\rho"'] & & PV
\end{tikzcd}$$

$$\begin{tikzcd}
K \arrow[r, "\bar\eta K"] \arrow[d, "\tilde\eta K"'] \arrow[drr, equal] & SK \arrow[r, "\alpha K"] \arrow[dr, "\bar\sigma"] & RK \arrow[d, "\bar\rho"] \\
RK \arrow[rr, "\bar\rho"'] & & K
\end{tikzcd}$$

(7)
$$\begin{tikzcd}
VSS \arrow[rrr, "V(\alpha\circ\alpha)"] \arrow[dr, "VS\alpha"] \arrow[ddr, "\sigma S"'] \arrow[ddd, "V\bar\mu"'] & & & VRR \arrow[dl, "\rho R"'] \arrow[dd, "V\tilde\mu"] \\
& VSR \arrow[urr, "V\alpha R"] \arrow[r, "\sigma R"'] & PVR \arrow[dd, "P\rho"] & \\
& PVS \arrow[ur, "PV\alpha"] \arrow[dr, "P\sigma"'] & & VR \arrow[dd, "\rho"] \\
VS \arrow[drrr, "\sigma"] \arrow[d, "V\alpha"'] & & PPV \arrow[dr, "\mu V"] & \\
VR \arrow[rrr, "\rho"'] & & & PV
\end{tikzcd}$$

$$\begin{tikzcd}
SSK \arrow[rrr, "(\alpha\circ\alpha)K"] \arrow[dr, "\alpha SK"] \arrow[ddr, "S\bar\sigma"'] \arrow[ddd, "\bar\mu K"'] & & & RRK \arrow[dl, "R\bar\rho"'] \arrow[dd, "\tilde\mu K"] \\
& RSK \arrow[urr, "R\alpha K"] \arrow[r, "R\bar\sigma"'] & RK \arrow[dddr, "\bar\rho"] & \\
& SK \arrow[ur, "\alpha K"] \arrow[ddrr, "\bar\sigma"] & & RK \arrow[dd, "\bar\rho"] \\
SK \arrow[drrr, "\bar\sigma"] \arrow[d, "\alpha K"'] & & & \\
RK \arrow[rrr, "\bar\rho"'] & & & K
\end{tikzcd}$$

3.3. PROPOSITION. *Let $V: A \to X$ be a functor, (P, η, μ) a monad in X, $S: A \to A$ a functor and $\bar\eta: 1_A \to S$, $\bar\mu: SS \to S$ natural transformations. Let $\sigma: VS \to PV$ be a V-cogenerating natural transformation such that there are satisfied the conditions for a lax lifting from 3.1. Then $(S, \bar\eta, \bar\mu)$ is a monad.*

PROOF. The assertion follows from the diagrams (3), (4) and (5) from the proof of 3.2 (considered for $(S, \bar\eta, \bar\mu)$ instead of $(R, \tilde\eta, \tilde\mu)$). Here the universality of ρ, $\bar\rho$ is replaced by the fact that σ is V-cogenerating.

If (P, η, μ) is the identity monad in X, then the monad $(R, \tilde\eta, \tilde\mu)$ from 3.2 is the codensity monad induced by (K, κ) in D^*_X. So Theorem 3.2 tells us that the construction of a codensity monad in D^*_X admits a parametrization by monads in X. In the case $X = 1$ there is possible only the parametrization by the identity monad and we get the usual codensity monad in CAT.

3.4. DEFINITION. We say that an arrow $(K,\kappa):(M,W)\to(A,V)$ *is codense in* D^*_X if $(1_A, 1_V)$, 1_K is a pointwise right extension of (K,κ) along (K,κ) in D^*_X.

3.5. PROPOSITION. *Let V be faithful. Then an arrow*

$$(K, 1_{VK}):(M, VK)\to(A,V)$$

is codense in D^*_X iff for a morphism $f: Va\to Va'$, $f = Vf'$ holds for a morphism $f': a\to a'$ whenever for any $z\in A$ and any morphism $h: a'\to z$ there exists a morphism $h': a\to z$ such that $Vh' = Vh.f$.

PROOF follows from 1.15.

The condition from the last proposition was often considerated in the litterature (e.g. [11,8 or 16]) and it means that the codensity in our sense is a right one (at least for the full subcategory of D^*_X consisting of faithful functors $V: A\to X$). Results 1.5 and 1.8 from [16] can be generalized to the following statements concerning «the reflection of a codensity along a change of base».

3.6. LEMMA. *Let $G: X\to Y$ be a functor and suppose that*

$$(K, G\kappa):(M, GW)\to(A, GV)$$

is codense in D^*_Y. Then, each of the two following conditions ensures that (K,κ) is codense in D^*_X:

1° G is faithful.

2° κ is pointwise mono and $\{Vf \mid f: Km\to a,\ m\in M\}$ epi for any $a\in A$.

PROOF is straightforward.

4. LIFTINGS OF FUNCTORS AND MONADS.

Let E_X be the sub-2-category of C_X having the same objects as C_X and such that an arrow $(F,\phi):(A,U)\to(B,V)$ in C_X belongs to E_X

iff $VF = U$ and $\phi = 1_U$, and any 2-cell in C_X between arrows of E_X belongs to E_X.

Under a mild supposition arrows (and thus also extensions) in E_X are given by those in C_X. We say that a functor $V: B \to X$ *has the property of transfer* if for every object b of B and every isomorphism $g: x \to Vb$ there is an isomorphism f' of B such that $Vf' = f$. Now, if f has the property of transfer, then for every arrow $(F, \phi): (A, U) \to (B, V)$ in C_X there exists an arrow in E_X isomorphic with it.

E_X is a representable 2-category. A comma object for an opspan

$$(A,U) \xrightarrow{F} (C,W) \xleftarrow{G} (B,V)$$

in E_X is a full subcategory of a comma object F/G in CAT determined by all $f: Fa \to Gb$ such that $Wf = 1$. This description shows that pointwise left extensions in the sense of 1.13 (considered for arrows in E_X) do not agree with those in E_X in the sense of [20]; the last ones are again too strong.

Let us have functors $U: A \to X$ and $V: B \to X$ and denote briefly the category of arrows $E_X((A, U), (B, V))$ by $E(U, V)$. By the definition $E(U, V)$ is the category of liftings of U through V (i.e. $VF = U$). Recall that this category has as morphisms natural transformations

$$a : F \to F' \text{ such that } Va = 1 .$$

We will be interested in the structure of this category.

V has the property of unicity if every isomorphism f of B such that $Vf = 1$ is the identity.

4.1. LEMMA. *If V is faithful and has the property of unicity, then the category of arrows $E(U, V)$ is an ordered class for every object U of E_X. If, in addition, V reflects isomorphisms, $E(U, V)$ is a discrete category.*

PROOF is evident.

Any functor $K: M \to A$ induces a functor

$$E(K, V): E(U, V) \to E(UK, V).$$

If a right extension $R_K(T)$ of T along K in E_X exists for each functor $T: M \to B$ such that $VT = UK$, then we obtain a functor

$$R_K(-): E(UK, V) \to E(U, V)$$

which is right adjoint to $E(K, V)$. We are going to show that it enables us to partly recognize the structure of $E(U, V)$ from the structure of $E(UK, V)$, i.e. to recognize the structure of liftings of U through V on a suitable full subcategory M of A.

4.2. THEOREM. *Let V be an* (E, M)*-topological functor having the property of unicity and transfer,*

$$\{ Uf \mid f: a \to Km,\ m \in M \} \in \mathrm{M} \quad \textit{for every} \quad a \in A,$$

and K be a full functor. Then $E(K, V)$ is an isotone map of the ordered class $E(U, V)$ onto the ordered class $E(UK, V)$ inducing a bijection between maximal elements of these classes.

PROOF. Following 2.2 there is an isotone map

$$R_K(-): E(UK, V) \to E(U, V).$$

By 1.17, 2, $R_K(-)$ is a right inverse to $E(K, V)$. Thus $E(K, V)$ is surjective and since $R_K(-)$ is a right adjoint to $E(K, V)$, $E(K, V)$ induces a bijection between the classes of maximal elements of $E(UK, V)$ and of $E(U, V)$.

4.3. THEOREM. *Let X be an (extremally epi-monosource)-category with sums, M be small and B cocomplete and co-well-powered. Let V be a faithful q-functor having the property of unicity and transfer,*

$$\{ Uf \mid f: a \to Km,\ m \in M \}, \quad \textit{for any} \quad a \in A,$$

be mono and K be a full functor. Then, $E(K, V)$ induces a bijection between minimal elements of $E(U, V)$ and those of $\{ FK \mid F \in E(U, V) \}$. In addition, if V reflects isomorphisms, then $E(K, V)$ is injective.

PROOF follows similarly as above from 1.16, 2, 1.17, 2, 2.8, 3, 2.9 and 4.1.

4.4. EXAMPLE. Let A be the category of distributive lattices, *Ord* the category of ordered sets, and $V: Ord \to Set$ be the forgetful functor. Let $U: A \to Set$ be the functor which assigns to each distributive lattice the set of all its sublattices and to each homomorphism f the mapping Uf carrying a sublattice to its image in f. Functors $F \in E(U, V)$ correspond to functorial orderings of the set of all sublattices. Such functors are investigated by M. Sekanina in [19]. He has shown that

$$\{ Uf \mid f: a \to 4 \}, \quad \text{for any} \quad a \in A,$$

is mono, where 4 is the four-element Boolean algebra. Since V is (extremally epi-monosource)-topological, following 4.2 there is only finitely many maximal F. 4.3 makes possible to study liftings of U along the forgetful functor $A \to Set$.

Let $P: X \to X$ be a functor. If we put $U = PV$, then $E(U, V)$ is the category of all liftings of P along V.

4.5. EXAMPLE. Let $P^+: Set \to Set$ be the covariant power-set functor. It is easy to show that $\{ P^+ Vf \mid f: b \to 3 \}$ is mono for any ordered set b, where 3 is the three-element chain. Let K be the inclusion of the full subcategory of Ord generated by 3. By 4.2, $E(K, V)$ induces a bijection between maximal elements of $E(U, V)$ and $E(UK, V)$. Hence maximal elements of $E(U, V)$ form a finite set. These maximal liftings are dealt with in [18].

Now, let (P, η, μ) be a monad in X. We will be interested in liftings of the monad P along V, i.e. in monads $(S, \bar{\eta}, \bar{\mu})$ in B such that:

$$VS = PV, \quad V\bar{\eta} = \eta V \quad \text{and} \quad V\bar{\mu} = \mu V.$$

These liftings form a full subcategory $Z_E(P, V)$ of the category $Z(P, V)$ from 3.1.

4.6. LEMMA. *Let V be faithful. Then $Z_E(P, V)$ is a full subcategory of $E(PV, V)$.*

PROOF. Let

$$(S_i, \eta_i, \mu_i) \in Z_E(P, V) \quad \text{for} \quad i = 1, 2,$$

and $\alpha : S_1 \to S_2$ be a morphism in $E(PV, V)$. It holds

$$V(\alpha . \eta_1) = V\eta_1 = \eta V = V\eta_2$$

and similarly

$$V(\alpha . \mu_1) = V(\mu_2 . (\alpha \circ \alpha)).$$

Since V is faithful, α is a morphism of monads.

The study of the structure of $Z_E(P, V)$ analogous to the preceding examination of liftings of functors is based on the following result. Consider the diagram

$$\begin{tikzcd}
 & M \arrow[dl, "K"'] \arrow[dr, "T"] & \\
A \arrow[dr, "PV"'] & \arrow[d, phantom, "\overset{\delta}{\Leftarrow}"] & A \arrow[dl, "V"] \\
 & X &
\end{tikzcd}$$

and regard the following special case of the dual of (2):

$$\begin{tikzcd}
 & M+M \arrow[dl, "{[K, T]}"'] \arrow[dr, "{[T, T]}"] \arrow[dd, "{[1_{PVK}, \mu VK . P\delta]}"'] & \\
A \arrow[dr, "PV"'] & \arrow[l, phantom, "\Rightarrow"] \arrow[r, phantom, "\overset{[\delta,\delta]}{\Leftarrow}"] & A \arrow[dl, "V"] \\
 & X &
\end{tikzcd} \qquad (8)$$

Here $M+M$ is the sum of two copies of the category M and $[K, T]$, ..., $[\delta, \delta]$ are induced functors and natural transformations.

4.7. PROPOSITION. *Let us have a natural transformation* $\nu : K \to T$ *such that* $\eta VK = \delta . V\nu$ *and suppose that* δ *is V-cogenerating. Let* $(R, \rho), [\rho_0, \rho_1]$ *be a right extension of* $([T, T], [\delta, \delta])$ *along*

$$([K, T], [1_{PVK}, \mu VK . P\delta])$$

in D^*_X. *Then there exist natural transformations* $\tilde{\eta}, \tilde{\mu}$ *such that* $(R, \tilde{\eta}, \tilde{\mu})$ *is a monad and* (R, ρ) *a lax lifting of* P *along* V. *Further for any monad* $(S, \bar{\eta}, \bar{\mu})$ *in* A, *any lax lifting* (S, σ) *of* P *along* V *and any natural trans-*

formations $\sigma_0 : SK \to T$ *and* $\sigma_1 : ST \to T$ *such that*

$$\delta . V\sigma_0 = \sigma K \quad \textit{and} \quad \delta . V\sigma_1 = \mu VK . P\delta . \sigma T,$$

there exists a unique morphism of liftings $\alpha : (S, \sigma) \to (R, \rho)$.

PROOF. Denote

$$\Delta = ([T, T], [\delta, \delta]) \quad \text{and} \quad \Delta' = ([K, T], [1_{PVK}, \mu VK . P\delta]).$$

Since $\eta VK = \delta . V\nu$ and

$$\mu VK . P\delta . \eta VT = \mu VK . \eta PVK . \delta = \delta,$$

$(1_A, \eta V), [\nu, 1_T]$ is an extension of Δ along Δ' in D^*_X. Thus there is a unique natural transformation

$$\tilde{\eta} : 1_A \to R \quad \text{such that} \quad \rho . V\tilde{\eta} = \eta V$$

(see the dual of 1.6). The diagrams

$$\begin{tikzcd}
VRRK \arrow[r, "VR\rho_0"] \arrow[dd, "\rho RK"'] & VRT \arrow[r, "V\rho_1"] \arrow[d, "\rho T"] & VT \arrow[dd, "\delta"] \\
 & PVT \arrow[d, "P\delta"] & \\
PVRK \arrow[ur, "PV\rho_0"] \arrow[r, "P\rho K"'] & PPVK \arrow[r, "\mu VK"'] & PVK
\end{tikzcd}$$

$$\begin{tikzcd}
VRRT \arrow[r, "VR\rho_1"] \arrow[d, "\rho RT"'] & VRT \arrow[rr, "V\rho_1"] \arrow[d, "\rho T"'] & & VT \arrow[ddd, "\delta"] \\
PVRT \arrow[r, "PV\rho_1"] \arrow[d, "P\rho T"'] & PVT \arrow[dr, "P\delta"] & & \\
PPVT \arrow[r, "PP\delta"] \arrow[d, "\mu VT"'] & PPPVK \arrow[r, "P\mu VK"] \arrow[d, "\mu PVK"'] & PPVK \arrow[dr, "\mu VK"] & \\
PVT \arrow[r, "P\delta"'] & PPVK \arrow[rr, "\mu VK"'] & & PVK
\end{tikzcd}$$

imply that

$$(RR, \mu V . P\rho . \rho R), [\rho_1 . R\rho_0, \rho_1 . R\rho_1]$$

is an extension of Δ along Δ' in D^*_X. Thus there is a natural transformation

$$\tilde{\mu} : RR \to R \quad \text{such that} \quad \rho \,.\, V\tilde{\mu} = \mu V \,.\, P\rho \,.\, \rho R .$$

Following 3.3 and the dual of 1.18, $(R, \tilde{\eta}, \tilde{\mu})$ is a monad and (R, ρ) is a lax lifting because $[\delta, \delta]$ is V-cogenerating.

Consider S from the theorem. Hence (S, σ), $[\sigma_0, \sigma_1]$ is an extension of Δ along Δ' in D^*_X. Thus there is a natural transformation

$$\alpha : S \to R \quad \text{such that} \quad \sigma = \rho \,.\, V\alpha .$$

The diagrams (6) and (7) from the proof of 3.2 imply that α is a morphism of monads.

4.8. REMARK. (a) Following the dual of 1.6, ρ_1 is an action of R on T and it holds

$$\rho_0 \,.\, \tilde{\eta} K = \nu \quad \text{and} \quad \rho_0 \,.\, \tilde{\mu} K = \rho_1 \,.\, R\rho_0 .$$

(b) For the existence of a unique natural transformation $\alpha : S \to R$ preserving unit and multiplication it is not necessary to suppose that $(S, \bar{\eta}, \bar{\mu})$ satisfies the monad axioms.

(c) If we do not assume that δ is V-cogenerating, then the assertion of the above proposition holds except the fact that η is a right unit.

4.9. LEMMA. *Let $(S, \bar{\eta}, \bar{\mu})$ be a monad in A, (S, σ) a lax lifting of P along V and $\sigma_0 : SK \to T$ a natural isomorphism such that $\delta \,.\, V\sigma_0 = \sigma K$. If we put $\sigma_1 = \sigma_0 \,.\, \bar{\mu} K \,.\, S\sigma_0^{-1}$, it holds $\delta \,.\, V\sigma_1 = \mu VK \,.\, P\delta \,.\, \sigma T$.*

PROOF.

$$\begin{tikzcd}
VST \arrow[r, "VS\sigma_0^{-1}"] \arrow[dd, "\sigma T"'] & VSSK \arrow[r, "V\bar{\mu}K"] \arrow[d, "\sigma SK"'] & VSK \arrow[r, "V\sigma_0"] \arrow[ddr, "\sigma K"] & VT \arrow[dd, "\delta"] \\
 & PVSK \arrow[d, "P\sigma K"] & & \\
PVT \arrow[ur, "PV\sigma_0^{-1}"] \arrow[r, "P\delta"'] & PPVK \arrow[rr, "\mu VK"'] & & PVK
\end{tikzcd}$$

4.10. THEOREM. *Let the suppositions of 4.7 be satisfied and, in addition,*

K be full and R pointwise. Then the following statements are equivalent:

1° ρ_0 is iso (and one can choose R such that $\rho_0 = 1_T$).

2° There is a monad $(S, \bar{\eta}, \bar{\mu})$ in A, a lax lifting (S, σ) of P along V and a natural isomorphism $\sigma_0 : SK \to T$ such that $\delta . V\sigma_0 = \sigma K$.

3° For any $m, n \in M$ and any $g : Kn \to Tm$ there exists

$$\bar{g} : Tn \to Tm \quad \textit{such that} \quad \delta_m . V\bar{g} = \mu_{VKm} . P\delta_m . PVg . \delta_n .$$

PROOF. $1 \Longrightarrow 2$ by 4.7. Let 2 hold and consider $g : Kn \to Tm$. Put

$$\bar{g} = \sigma_{0,m} . \bar{\mu}_{Km} . S\sigma^{-1}_{0,m} . Sg . \sigma^{-1}_{0,n} .$$

Then 3 follows from the diagram

$$\begin{tikzcd}
VTn \arrow[r, "V\sigma^{-1}_{0,n}"] \arrow[dd, "\delta_n"'] & VSKn \arrow[r, "VSg"] \arrow[ddl, "\sigma_{Kn}"] & VSTm \arrow[r, "VS\sigma^{-1}_{0,m}"] \arrow[dd, "\sigma_{Tm}"] & VSSKm \arrow[r, "V\bar{\mu}_{Km}"] \arrow[d, phantom, "4.9"] & VSKm \arrow[d, "V\sigma_{0,m}"] \\
 & & & {} & VTm \arrow[d, "\delta_m"] \\
PVKn \arrow[rr, "PVg"'] & & PVTm \arrow[r, "P\delta_m"'] & PPVKm \arrow[r, "\mu_{VKm}"'] & PVKm
\end{tikzcd}$$

Let 3 hold. Then

$$Tn, \delta_n : VTn \to PVKn, \quad Tf : Tn \to Tm, \quad \bar{g} : Tn \to Tm,$$

is a universal solution of the diagram

$$\begin{tikzcd}
 & PVKn \arrow[d, bend right, "PVf"'] \arrow[d, bend left, "\mu_{VKm} . P\delta_m . PVg"] \\
VTm \arrow[r, "\delta_m"'] & PVKm
\end{tikzcd}$$

where f ranges over morphisms $Kn \to Km$ and g over $Kn \to Tm$ with $m \in M$. Namely it is a solution by 3 and the naturality of δ and for another solution

$$a, u : Va \to PVKn, \quad \hat{f} : a \to Tm, \quad \hat{g} : a \to Tm$$

the desired morphism $a \to Tn$ is equal to $\hat{1}_{Kn}$. Hence $RKn = Tn$ by the dual of 1.15, and thus 1 holds.

Denote by $Z(P, V, \delta)$ the subcategory of $Z(P, V)$ consisting of lax liftings

$$(S, \sigma) \text{ such that } SK = T \text{ and } \sigma K = \delta$$

and morphisms of liftings α such that $\alpha K = 1_T$.

4.11. THEOREM. *Let the suppositions of 4.10 be satisfied. Then $Z(P, V, \delta)$ is non-void iff the condition 3 from 4.10 holds and in this case (R, ρ) from 4.7 is a terminal object of $Z(P, V, \delta)$.*

PROOF. The first assertion follows from 4.10 and the second one from 4.7, 4.9 and 4.10.

Let $Z_E(P, V, T)$ be the full subcategory of $Z_E(P, V)$ consisting of all liftings $S \in Z_E(P, V)$ such that $SK = T$.

4.12. COROLLARY. *Let us have functors*

$$K: M \to A, \quad T: M \to A, \quad V: A \to X$$

and a monad (P, η, μ) in X such that $VT = PVK$. Let V be an (E,M)-*topological functor having the property of unicity and transfer, K be a full functor and*

$$\{ PVf \mid f: a \to Km,\ m \in M \} \in \mathrm{M} \quad \textit{for any} \quad a \in A.$$

Then $Z_E(P, V, T) \neq \emptyset$ iff there is a natural transformation $\nu: K \to T$ such that $V\nu = \eta VK$ and for any $m, n \in M$ and any $g: Kn \to Tm$ there is

$$\hat{g}: Tn \to Tm \quad \textit{such that} \quad V\hat{g} = \mu_{VKm} \cdot PVg.$$

In this case a right extension of

$$([T, T], [1_{VT}, 1_{VT}]) \quad \textit{along} \quad ([K, T], [1_{VT}, \mu VK])$$

in D_X^* is the greatest element of $Z_E(P, V, T)$.

PROOF. Since X is an (E, M)-category, $\{PVf\}_f \in \mathrm{M}$ implies that

$$\{PVf\}_f \cup \{ \mu_{VKm} \cdot PVg \mid g: a \to Tm,\ m \in M \} \in \mathrm{M}$$

for any $a \in A$. By 2.2 there exists a pointwise right extension

$$(R, \rho), [\rho_0, \rho_1] \quad \text{of} \quad ([T, T], [1_{VT}, 1_{VT}])$$

along $([K, T], [1_{VT}, \mu VK])$ in D_X^* and one can choose $\rho = 1_{PV}$. Now, the assertion follows from 4.7, 4.9 and 4.10 for the case $\delta = 1_{VT}$.

The construction of diagram (8) is not functorial in T. So we had to state Theorem 4.11 locally, i.e. as a result asserting the existence of a terminal lifting for each δ and not the existence of a right adjoint for the functor from $Z(P, V)$ to the category of T's. Consequently Corollary 4.12 admits the case where S is maximal in $Z_E(P, V)$ but SK is not maximal among restrictions of functors from $Z_E(P, V)$ on M.

Liftings of the monad P^+ on Ord (compare 4.5) were dealt with by M. Sekanina.

APPENDIX. PRESHEAF CONSTRUCTION IN E_X.

We have indicated that 2-categories D_X^C and E_X^C can be defined for any 2-category C. Formally E_X^C is the comma object $1_C/X$ of the opspan

$$\begin{tikzcd} C \arrow[dr, equal] & & 1 \arrow[dl, "X"] \\ & C & \end{tikzcd}$$

in $2\text{-}CAT$ (see [20]) and D_X^C is the lax comma object $1_C//X$, in $2\text{-}CAT^{\sim}$ (in the sense of [13]). We have also indicated that certain previous results hold in this general context (e.g. Theorems 1.10 and 3.2). Now, we will be interested in properties of D_X^C and E_X^C.

Let C have comma objects. Then (as for $C = CAT$) any opspan

$$(A_0, U) \xrightarrow{(F,a)} (A_2, W) \xleftarrow{(G,1)} (A_1, GW)$$

has a comma object

$$\begin{tikzcd} & (F/G, UD_0) \arrow[dl, "{(D_0,1)}"'] \arrow[dr, "{(D_1, W\lambda . aD_0)}"] & \\ (A_0, U) \arrow[dr, "{(F,a)}"'] \arrow[rr, phantom, "\lambda"] & & (A_1, GW) \arrow[dl, "{(G,1)}"] \\ & (A_2, W) & \end{tikzcd}$$

where

$$\begin{tikzcd} & F/G \arrow[dl, "D_0"'] \arrow[dr, "D_1"] & \\ A_0 \arrow[dr, "F"'] \arrow[rr, phantom, "\overset{\lambda}{\curvearrowright}"] & & A_1 \arrow[dl, "G"] \\ & A_2 & \end{tikzcd}$$

is a comma object in C.

Let C be representable and X be an object of C such that $C(A, X)$ has pullbacks for all $A \epsilon C$ and $C(F, X)$ preserves them for any arrow F of C. Then D_X^C is representable and comma objects in D_X^C are described in the same way as in 1.14. The description of 2-pullbacks follows from the next consideration about 2-limits.

Let C be 2-complete and $X \epsilon C$ be complete in the previous sense. It means that $C(-, X)$ is a functor into the category of complete categories and limit preserving functors. Then D_X^C is 2-complete and the 2-limit of a diagram $G: S \to D_X^C$ is $(lim \Delta_X^C G,\ lim \bar{G})$ where $\Delta_X^C: D_X^C \to C$ is the underlying functor and $\bar{G}: S \to C(lim \Delta_X^C . G,\ X)$ is defined as follows:

$$\bar{G}(s) = U_s . T_s \quad \text{where} \quad s \,\epsilon\, S,$$

if: $G(s) = (A_s, U_s)$, $T_s: lim \Delta_X^C . G \to A_s$ is the component of a limit cone and $\bar{G}(g) = \bar{g} T_s$, where $g: s \to s'$ is a morphism in S and $G(g) = (Fg, \bar{g})$. The limit cone in D_X^C has components

$$(T_s, \tau_s): lim \bar{G} \to Gs, \quad \text{where} \quad \tau_s: lim \bar{G} \to \bar{G} s$$

is a component of a limit cone in $C(lim \Delta_X^C . G,\ X)$.

If C is representable, then E_X^C is representable for any X. 2-pullbacks of E_X^C are those of C and the comma object of an opspan

$$(A_0, F W) \xrightarrow{\ F\ } (A_2, W) \xleftarrow{\ G\ } (A_1, G W)$$

is a 2-enriched op-localization (compare [21] page 167) of F/G at W/λ (in the notation (9)).

In the sequel we will work in a representable 2-category C, endowed with a 2-functor $P: C^{coop} \to C$ (i.e. both arrows and 2-cells are reversed). Further, there is specified a subclass of objects of C, which will be called *legitimate objects* and for any such legitimate object it is given an arrow $Y_A : A \to PA$. Finally, we demand that the assignment

$$\phi^{A,B}(F) = Y_A / F$$

yields a full and faithful functor

$$\phi^{A,B} : C(B, PA) \to Cov(A, B)$$

and these functors form the components of a pseudo-natural transformation (we recall that $Cov(A,B)$ is the category of all covering spans from A to B, i.e. of all comma objects of opspans from A to B). All these structures are present in a precosmos (see Street [21]). With size conditions aside (i.e. all objects are legitimate) such a C is precisely a uniform precosmos. C models the 2-category CAT of categories. Namely, PA is the category of all functors from A^{op} to Set, legitimate objects are categories with small hom-sets and Y_A is the Yoneda embedding.

Suppose that X is a legitimate object of C and consider D_X^C. Take $(A, U) \in D_X^C$ such that A is legitimate and define

$$\mathsf{P}_X(A,U) = (P_X(A,U),\ D_1^{(A,U)}),$$

where

$$\begin{tikzcd} & P_X(A,U) \arrow[dl, "D_0^{(A,U)}"'] \arrow[dr, "D_1^{(A,U)}"] & \\ PA \arrow[dr, equal] \arrow[rr, phantom, "\lambda_{(A,U)}"] & & X \arrow[dl, "PU.Y_X"] \\ & PA & \end{tikzcd}$$

is a comma object in C. Using the universal property of a comma object, one can complete P_X to a 2-functor $\mathsf{P}_X : (D_X^C)^{coop} \to E_X^C$. The 2-cell «effect of U on homs» (see [21] page 144) $Y_A \to {}^{D}U . Y_X . U$ induces an arrow of spans

$$Y^X_{(A,U)} : (Y_A, A, U) \to (D_0^{(A,U)}, P_X(A,U),\ D_1^{(A,U)}),$$

i.e. an arrow

$$(Y^X_{(A,U)}, 1_U) : (A,U) \to (P_X(A,U),\ D_1^{(A,U)})$$

in D^C_X. The assignment

$$\Phi_X^{(A,U),(B,V)}(F,\phi) = (Y^X_{(A,U)}, 1_U)/_{D^C_X}(F,\phi)$$

is neither pseudo-natural in (A,U), nor faithful, nor full. But one has the following result.

THEOREM.

$$\Phi_X^{(A,U),(B,V)} : E^C_X((B,V),\ P_X(A,U)) \to Cov_{D^C_X}((A,U),(B,V))$$

are fully faithful and form the components of a pseudo-natural transformation.

The proof is rather long and the computations lean on the fact that

$$Y_A/D_0^{(A,U)} \simeq Y^X_{(A,U)}/1.$$

The author does not know whether D^C_X or E^C_X resp. is a precosmos. But it seems not to be so because comma objects in D^C_X and E^C_X are bad. For instance, the prescription $P_X(A,U) = PA \times X$ does not work. Good comma objects are those which appear in the just stated theorem and these are the same which yield pointwise left extensions in the sense of 1.13. Namely, we mean comma objects in D^C_X of opspans from E^C_X.

Let the 2-functor $P : C^{coop} \to C$ have a left 2-adjoint $P^* : C \to C^{coop}$ (i.e. C is a cosmos). Then one can construct a left 2-adjoint

$$P^*_X : E^C_X \to (E^C_X)^{coop} \quad \text{to} \quad P_X : (E^C_X)^{coop} \to E^C_X$$

as follows:

$$\begin{tikzcd}
& P^*_X(A,U) \arrow[dl, "D^*_0"'] \arrow[dr, "D^*_1"] & \\
X \arrow[dr, "P^*U . Y^*_X"'] \arrow[rr, phantom, "\overset{\lambda^*}{\curvearrowright}"] & & P^*A \arrow[dl, equal] \\
& P^*A &
\end{tikzcd}$$

Here Y^*_X is the image of Y_X in the adjunction isomorphism

$$C(B,PA) \simeq C(A,P^*B)^{op}.$$

However, P^*_X cannot be extended on D^C_X.

Consider the case $C = CAT$. Then objects of $P_X(A, U)$ are triples (F, δ, x), where

$$F: A^{op} \to Set, \quad x \in X \quad \text{and} \quad \delta: F \to X(U-, x)$$

is a natural transformation. Morphisms

$$(F, \delta, x) \to (F', \delta', x') \quad \text{of} \quad P_X(A, U)$$

are couples (a, f) where

$$a: F \to F' \quad \text{and} \quad f: x \to x' \quad \text{such that} \quad \delta'.a = X(U, f).\delta.$$

The arrow $Y^X_{(A,U)}$ is given by

$$Y^X_{(A,U)}(a) = (A(-.a), \delta, Ua), \quad \text{where} \quad \delta(h) = Uh.$$

This construction is closely related to the initial completion of a faithful functor (see [8, 10, 23]). Namely consider the full sub-2-category $\hat{E}_X$ of E^{CAT}_X having objects (A, U) such that U is faithful. Then the functor $D^{(A,U)}_1: P_X(A, U) \to X$ need not be faithful, but it is faithful on a full subcategory $\hat{P}_X(A, U)$ of $P_X(A, U)$ consisting of the (F, δ, x) such that δ is mono. Then the arrow

$$Y^X_{(A,U)}: (A, U) \to (\hat{P}_X(A, U), D^{(A,U)}_1)$$

is precisely the initial completion E^{-2} of (A, U) in the sense of [8]. Similarly $P^*_X(A, U)$ yields E^2 from [8].

The same construction is given in [21], page 175. There is considered the full sub-2-category *Simp* of $\hat{E}_{Set}$ consisting of all the functors U from A to Set such that any constant mapping underlies a morphism of A and the 2-functor $\bar{P}: Simp^{coop} \to Simp$ such that $\bar{P}(A, U)$ is the full subcategory of $\hat{P}_{Set}(A, U)$ consisting of all (F, δ, x) such that $\delta_a(Fa)$ contains all constant mappings $Ua \to x$. But $\bar{P}$ does not make, from *Simp*, a cosmos, and similarly for $\hat{P}_X$ and $\hat{E}_X$ (though the first assertion is stated in [21]).

EXAMPLE. Let A be the subcategory of Set having one object $2 = \{0, 1\}$

and the identity and constant mappings as morphisms. Let $U: A \to Set$ be the inclusion. Then $\bar{P}(A, U)$ has objects (x, ρ), where ρ is a reflexive binary relation on a set x, and morphisms are relation preserving mappings. Let B be the subcategory of Set having one object 3 and the identity and constant mappings as morphisms. Let $V: B \to Set$ be the inclusion. There are functors

$$F, G: (B, V) \to (\bar{P}(A, U), \bar{D}_1)$$

such that there is no 2-cell $F \to G$ in *Simp* because there are two incomparable reflexive relations on 3. But

$$\bar{Y}_A/_{Simp}F = \bar{Y}_A/_{Simp}G = \emptyset,$$

and thus $\bar{Y}_A/_{Simp}$- is not full.

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