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ET GEOMETRIE DIFFERENTIELLE

BUILDING CLOSED CATEGORIES

by Michael BARR *

I am interested here in examining and generalizing the construction of the cartesian closed category of compactly generated spaces of Gabriel-Zisman [1967]. Suppose we are given a category $\mathfrak{A}$ and a full subcategory $\mathfrak{C}$. Suppose $\mathfrak{C}$ is a symmetric monoidal category in the sense of Eilenberg-Kelly [1966], Chapter II, section 1 and Chapter III, section 1. Suppose there is, in addition, a «Hom» functor $\mathfrak{C}^{op} \times \mathfrak{C} \to \mathfrak{C}$ which satisfies the axioms of Eilenberg-Kelly for a closed monoidal category insofar as they make sense. Then we show that, granted certain reasonable hypotheses on $\mathfrak{C}$ and $\mathfrak{A}$, this can be extended to a closed monoidal structure on the full subcategory of these objects of $\mathfrak{A}$ which have a $\mathfrak{C}$-presentation. One additional hypothesis suffices to show that when the original product in $\mathfrak{C}$ is the cartesian product, then the resultant category is even cartesian closed. As a result, we derive the cartesian closedness not only of compactly generated spaces but also of simplicially generated, sequentially generated, etc...

1. STRUCTURE ON $\mathfrak{C}$.

(1.1) We suppose that $\mathfrak{C}$ is equipped with a symmetric monoidal structure. This consists of a tensor product functor $-\otimes- : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$, an object I and coherent commutative, associative and unitary isomorphisms. See Eilenberg-Kelly for details.

We suppose also a functor $(-,-) : \mathfrak{C}^{op} \times \mathfrak{A} \to \mathfrak{A}$ and natural equivalences

$$hom(I,(C,A)) \cong hom(C,A), \quad (I,A) \cong A,$$
$$(C_1 \otimes C_2, A) \cong (C_1,(C_2,A)).$$

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Here C, $C_1, C_2 \in \mathfrak{C}$, $A \in \mathfrak{A}$. The first and third combine to give also

$$hom(C_1 \otimes C_2, A) \cong hom(C_1, (C_2, A)).$$

(1.2) We make one further hypothesis about the relationship between $\mathfrak{C}$ and $\mathfrak{A}$. Namely that for every object $A \in \mathfrak{A}$ there be a set $\{C_\omega \to A\}$ of maps with domain in $\mathfrak{C}$ such that every $C \to A$ with $C \in \mathfrak{C}$ factors through at least one $C_\omega \to A$. I am, in effect, demanding that each category of $\mathfrak{C}$-objects over $\mathfrak{A}$ have a weak initial family. Such a family is called *a terminal $\mathfrak{C}$-sieve for* $\mathfrak{A}$.

(1.3) We suppose, finally, that $\mathfrak{A}$ has regular epimorphism/monomorphism factorizations of its morphisms.

2. $\mathfrak{C}$-PRESENTED OBJECTS.

(2.1) I define $A \in \mathfrak{A}$ to be *$\mathfrak{C}$-generated* if there is a regular epimorphism:

$$\Sigma C_\psi \to A \quad \text{with each} \quad C_\psi \in \mathfrak{C}.$$

Because of the cancellation properties of regular epimorphisms, it follows that if there is any regular epimorphism of that kind we may suppose the $\{C_\psi \to A\}$ is a terminal $\mathfrak{C}$-sieve for A. For each $C_\psi \to A$ factors through some $C_\omega \to A$ and so the regular epimorphism factors:

$$\Sigma C_\psi \longrightarrow \Sigma C_\omega \longrightarrow A.$$

(2.2) We say that A *is $\mathfrak{C}$-presented* provided there is a coequalizer diagram

$$\Sigma C_\xi \rightrightarrows \Sigma C_\psi \longrightarrow A.$$

Assuming that A is $\mathfrak{C}$-generated by $\Sigma C_\psi \to A$ and that B is the kernel pair of that map it is sufficient that there be an epimorphism $\Sigma C_\xi \to B$. In any case if there is such a diagram it is clear that it suffices to take for $\{C_\xi\}$ a terminal $\mathfrak{C}$-sieve for the kernel pair B.

(2.3) It often happens that $\mathfrak{C}$ - or even I alone - is a generator for $\mathfrak{A}$. In that case, there is no distinction between $\mathfrak{C}$-generated and $\mathfrak{C}$-presented. However, as will be seen, being $\mathfrak{C}$-presented is the important thing.

(2.4) One word of warning is in order. It is entirely possible that an object be $\mathfrak{C}$-presented in a full subcategory of $\mathfrak{A}$ which contains $\mathfrak{C}$ but not in $\mathfrak{A}$.

This is because the coequalizer condition is less exigent in a subcategory.

(2.5) Suppose $\{C_\omega \to A\}$ is a terminal $\mathfrak{C}$-sieve for A, $A_0 = \Sigma C_\omega$, $A_1 = \Sigma C_\psi$ and $\{C_\psi \to A_0 \times_A A_0\}$ is a terminal $\mathfrak{C}$-sieve. Then we have a diagram:

$$A_1 \rightrightarrows A_0 \longrightarrow A\,.$$

This diagram is a coequalizer iff A is $\mathfrak{C}$-presentable and in that case we call it a $\mathfrak{C}$-presentation of A.

(2.6) Let $f: A \to B$. Whether or not A or B is $\mathfrak{C}$-presentable we can find diagrams

$$A_1 \rightrightarrows A_0 \longrightarrow A\,, \qquad B_1 \rightrightarrows B_0 \longrightarrow B$$

of the above type where

$$B_0 = \Sigma C_\xi, \quad B_1 = \Sigma C_\zeta.$$

For all ω we get that the map $C_\omega \to A \to B$ factors through some $C_\xi \to B$ and hence we have an $f_0 : \Sigma C_\omega \to \Sigma C_\xi$ such that

$$\begin{array}{ccc}
A_0 = \Sigma C_\omega & \longrightarrow & A \\
{\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\
B_0 = \Sigma C_\xi & \longrightarrow & B
\end{array}$$

commutes. For any ψ the two maps

$$C_\psi \rightrightarrows A_0 \longrightarrow B_0 \longrightarrow B$$

are equal and hence determine a map $C_\psi \to B_0 \times_B B_0$ which has a lifting through some C_ζ and results in a commutative diagram

$$\begin{array}{ccccc}
A_1 & \rightrightarrows & A_0 & \longrightarrow & A \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\
B_1 & \rightrightarrows & B_0 & \longrightarrow & B
\end{array}$$

If $g_0 : A_0 \to B_0$ is another choice for f_0, the two maps together determine a map $A_0 \to B_0 \times_B B_0$ which similarly has a lifting $h: A_0 \to B_1$. The upshot is that the map f induces a map, «unique up to homotopy» from $A_1 \rightrightarrows A_0$ to

$B_1 \rightrightarrows B_0$; if we let πA and πB be the respective coequalizers, there is induced a unique map, naturally denoted $\pi f: \pi A \to \pi B$.

(2.7) It is now a standard argument to see that π is a well defined endofunctor on the category $\mathfrak{A}$ which lands in the full subcategory of $\mathfrak{C}$-presented objects. It is clear also that there is a natural $\pi A \to A$ which is an isomorphism iff A is $\mathfrak{C}$-presented and, finally, that π determines a right adjoint to the inclusion of that subcategory. We let $\pi\mathfrak{A}$ denote the full subcategory and let π denote also the retraction $\mathfrak{A} \to \pi\mathfrak{A}$.

(2.8) PROPOSITION. *Let $f: A \to B$ be a map such that*

$$hom(C,f) : hom(C,A) \to hom(C,B)$$

is an isomorphism for all $C \in \mathfrak{C}$. Then πf is an isomorphism.

PROOF. Let $\{C_\omega \to A\}$ be a terminal $\mathfrak{C}$-sieve for A. Then the hypotheses imply that $\{C_\omega \to A \to B\}$ is one for B. Let $A_0 = \Sigma C_\omega$. Now both

$$hom(C, A_0 \times_A A_0) \quad \text{and} \quad hom(C, A_0 \times_B A_0)$$

are the kernel pairs for isomorphic maps, namely

$$hom(C, A_0) \longrightarrow hom(C,A) \xrightarrow{\ \approx\ } hom(C,B)$$

and hence they are also isomorphic for all $C \in \mathfrak{C}$. Hence, $A_0 \times_A A_0$ and $A_0 \times_B A_0$ have the same terminal $\mathfrak{C}$-sieves and it follows immediately that πf is an isomorphism.

(2.9) COROLLARY. *Under the same hypotheses, if A and B are $\mathfrak{C}$-presented, f is an isomorphism.*

3. INDUCED STRUCTURE ON $\pi\mathfrak{A}$.

(3.1) We define

$$[-,-] = \pi(-,-) : \mathfrak{C}^{op} \times \pi\mathfrak{A} \to \pi\mathfrak{A}.$$

There is a 1-1 correspondence between maps

$$I \to [C,A], \quad I \to (C,A), \quad C \to A.$$

With $A \in \pi\mathfrak{A}$,

$$[I, A] = \pi(I, A) \cong \pi A \cong A.$$

Finally, we have

$$(C_1, [C_2, A]) \to (C_1, (C_2, A)) \cong (C_1 \otimes C_2, A)$$

which gives, upon application of π, a map

$$[C_1, [C_2, A]] \to [C_1 \otimes C_2, A].$$

We prove it is an isomorphism by using (2.9). To a map $C \to [C_1 \otimes C_2, A]$ corresponds uniquely, using right adjointness of π several times:

$$C \to (C_1 \otimes C_2, A), \quad C \otimes C_1 \otimes C_2 \to A, \quad C \otimes C_1 \to (C_2, A);$$
$$C \otimes C_1 \to [C_2, A], \quad C \to (C_1, [C_2, A]), \quad C \to [C_1, [C_2, A]].$$

(3.2) Since a regular image of a $\mathfrak{C}$-presented object is $\mathfrak{C}$-presented, the regular factorization is inherited by $\pi\,\mathfrak{A}$. The continued existence of terminal $\mathfrak{C}$-sieves is evident. Hence we are able to conclude:

PROPOSITION. *The category $\pi\,\mathfrak{A}$ satisfies all the hypotheses of Section 1.*

4. INDUCED STRUCTURE ON $\pi\,\mathfrak{A}$.

(4.1) We will now extend the given tensor and internal hom to a closed monoidal structure on $\pi\,\mathfrak{A}$. In order to simplify notation we will, for the purposes of this Section, suppose that

$$\pi\,\mathfrak{A} = \mathfrak{A} \quad \text{and} \quad (-,-) = [-,-].$$

In view of (3.2) this is permissible.

(4.2) Now choose a $\mathfrak{C}$-presentation (see (2.5))

$$\Sigma C_\psi \rightrightarrows \Sigma C_\omega \longrightarrow A,$$

and let (A, B) be defined as the equalizer of

$$\Pi(C_\omega, B) \rightrightarrows \Pi(C_\psi, B).$$

The same argument as in (2.6) shows that any map $A \to A'$ induces a map $(A', B) \to (A, B)$ and this is well defined and functorial.

(4.3) Similarly define $A \otimes B$ by choosing $\mathfrak{C}$-presentations

$$\Sigma C_\psi \rightrightarrows \Sigma C_\omega \longrightarrow A\,, \qquad \Sigma C_\zeta \rightrightarrows \Sigma C_\xi \longrightarrow B$$

and then $A \otimes B$ is the coequalizer of

$$\Sigma C_\psi \otimes C_\zeta \rightrightarrows \Sigma C_\omega \otimes C_\xi .$$

(4.4) PROPOSITION. *For fixed $C \epsilon \mathfrak{C}$, the functor $(C,-)$ commutes with projective limits.*

PROOF. We use (2.9). Suppose

$$B = projlim\, B_\omega \quad \text{and} \quad C' \to proj\ lim(C, B_\omega).$$

We get

$$C' \to (C, B_\omega), \quad C' \otimes C \to B_\omega ,$$
$$C' \otimes C \to projlim\, B_\omega , \quad C' \to (C, projlim\, B_\omega)$$

and conversely.

(4.5) PROPOSITION. *Let $C \epsilon \mathfrak{C}$. Then $(A,(C,B)) \cong (C,(A,B))$.*

PROOF. Let

$$\Sigma C_\psi \rightrightarrows \Sigma C_\omega \longrightarrow A$$

be a $\mathfrak{C}$-presentation. Then

$$(A,(C,B)) \longrightarrow \Pi(C_\omega,(C,B)) \rightrightarrows \Pi(C_\psi,(C,B))$$

is an equalizer. But

$$\Pi(C_\omega,(C,B)) \cong \Pi(C,(C_\omega,B)) \quad \text{and} \quad \Pi(C_\psi,(C,B)) \cong \Pi(C,(C_\psi,B)).$$

Since

$$(A,B) \longrightarrow \Pi(C_\omega,B) \rightrightarrows \Pi(C_\psi,B)$$

is an equalizer, so are

$$(C,(A,B)) \to (C,\Pi(C_\omega,B)) \rightrightarrows (C,\Pi(C_\psi,B)),$$
$$(C,(A,B)) \to \Pi(C,(C_\omega,B)) \rightrightarrows \Pi(C,(C_\psi,B)).$$

(4.6) PROPOSITION. *For A fixed, $(A,-)$ commutes with projective limits.*

PROOF. We use (2.9). Let $B = projlim\, B_\omega$. If

$$C \epsilon \mathfrak{C} \quad \text{and} \quad C \to projlim(A, B_\omega),$$

$$C \to (A, B_\omega), \quad A \to (C, B_\omega),$$

$$A \to projlim(C, B_\omega), \quad A \to (C, projlim\, B_\omega), \quad C \to (A, projlim\, B_\omega),$$

and conversely.

(4.7) PROPOSITION. *For any* $A_1, A_2, B \in \mathfrak{A}$,

$$(A_1, (A_2, B)) \cong (A_2, (A_1, B)).$$

PROOF. Let

$$\Sigma C_\psi \rightrightarrows \Sigma C_\omega \longrightarrow A_1$$

be a $\mathfrak{C}$-presentation of A_1. Then we have equalizers

$$\begin{array}{ccc} (A_1,(A_2,B)) \to \Pi(C_\omega,(A_2,B)) & \rightrightarrows & \Pi(C_\psi,(A_2,B)) \\ \| \wr & & \| \wr \\ \Pi(A_2,(C_\omega,B)) & \rightrightarrows & \Pi(A_2,(C_\psi,B)) \\ \| \wr & & \| \wr \\ (A_2,(A_1,B)) \to (A_2,\Pi(C_\omega,B)) & \rightrightarrows & (A_2,\Pi(C_\psi,B)) \end{array}$$

from which the isomorphism follows.

(4.8) PROPOSITION. *For* B *fixed, the functor* $(-, B) : \mathfrak{A}^{op} \to \mathfrak{A}$ *commutes with projective limits.*

PROOF. Let $A = indlim\, A_\omega$. We have for any $C \in \mathfrak{C}$, $C \to projlim(A_\omega, B)$,

$$C \to (A_\omega, B), \quad A_\omega \to (C, B),$$

$$indlim\, A_\omega \to (C, B), \quad C \to (indlim\, A_\omega, B),$$

and the result follows from (2.9).

(4.9) PROPOSITION. *For* $C \in \mathfrak{C}$, $(C \otimes A, B) \cong (C, (A, B))$.

PROOF. Let

$$\Sigma C_\psi \rightrightarrows \Sigma C_\omega \longrightarrow A$$

be a $\mathfrak{C}$-presentation. Then

$$\Sigma\, C \otimes C_\psi \rightrightarrows \Sigma\, C \otimes C_\omega \longrightarrow C \otimes A$$

is a coequalizer so that

$$\begin{array}{ccccc}
(C\otimes A, B) & \longrightarrow & (\Sigma\, C\otimes C_\omega, B) & \rightrightarrows & (\Sigma\, C\otimes C_\psi, B) \\
 & & \| \wr & & \| \wr \\
 & & \Pi(C\otimes C_\omega, B) & \rightrightarrows & \Pi(C\otimes C_\psi, B) \\
 & & \| \wr & & \| \wr \\
 & & \Pi(C,(C_\omega, B)) & \rightrightarrows & \Pi(C,(C_\psi, B)) \\
 & & \| \wr & & \| \wr \\
(C,(A,B)) & \longrightarrow & (C,\Pi(C_\omega, B)) & \rightrightarrows & (C,\Pi(C_\psi, B))
\end{array}$$

are equalizers.

(4.10) PROPOSITION. *Let A, $B \in \mathfrak{A}$ and $\Sigma\, C_\psi \rightrightarrows \Sigma\, C_\omega \longrightarrow A$ be a $\mathfrak{C}$-presentation. Then*

$$\Sigma\, C_\psi \otimes B \rightrightarrows \Sigma\, C_\omega \otimes B \longrightarrow A\otimes B$$

is a coequalizer.

PROOF. The identity maps $\Sigma\, C_\omega \rightrightarrows \Sigma\, C_\omega$ are coequalized by $\Sigma\, C_\omega \to A$ and hence lift to a single map $\Sigma\, C_\omega \to \Sigma\, C_\psi$. Thus

$$\Sigma\, C_\psi \rightrightarrows \Sigma\, C_\omega \longrightarrow A$$

is a reflexive coequalizer.The same is true of a $\mathfrak{C}$-presentation of B,

$$\Sigma\, C_\zeta \rightrightarrows \Sigma\, C_\xi \longrightarrow B .$$

Now the rows and the diagonal of

$$\begin{array}{ccccc}
\Sigma\, C_\psi \otimes C_\zeta & \rightrightarrows & \Sigma\, C_\psi \otimes C_\xi & \longrightarrow & \Sigma\, C_\psi \otimes B \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
\Sigma\, C_\omega \otimes C_\zeta & \rightrightarrows & \Sigma\, C_\omega \otimes C_\xi & \longrightarrow & \Sigma\, C_\omega \otimes B \\
 & & & & \downarrow \\
 & & & & A\otimes B
\end{array}$$

are coequalizers, from which it is a standard diagram chase to see that the column is. (Hom it into a fixed object and consider the resultant diagram of equalizers of sets.)

(4.11) PROPOSITION. *For A_1, A_2, $B \in \mathfrak{A}$, $(A_1 \otimes A_2, B) \cong (A_1, (A_2, B))$.*

PROOF. Let

$$\Sigma C_\psi \rightrightarrows \Sigma C_\omega \longrightarrow A_1$$

be a $\mathfrak{C}$-presentation of A_1. Then

$$\Sigma C_\psi \otimes A_2 \rightrightarrows \Sigma C_\omega \otimes A_2 \longrightarrow A_1 \otimes A_2$$

is a coequalizer. From (4.8) and (4.9), we have that

$$\begin{array}{ccccc}
(A_1 \otimes A_2, B) \longrightarrow & (\Sigma C_\omega \otimes A_2, B) & \rightrightarrows & (\Sigma C_\psi \otimes A_2, B) \\
& \| ! & & \| ! \\
& \Pi(C_\omega \otimes A_2, B) & \rightrightarrows & \Pi(C_\psi \otimes A_2, B) \\
& \| ! & & \| ! \\
(A_1, (A_2, B)) \longrightarrow & \Pi(C_\omega, (A_2, B)) & \rightrightarrows & \Pi(C_\psi, (A_2, B))
\end{array}$$

are equalizers, from which the isomorphism follows.

(4.12) PROPOSITION. *There is a 1-1 correspondence between :*

maps $I \to (A, B)$ *and maps* $A \to B$.

PROOF. Let

$$\Sigma C_\psi \rightrightarrows \Sigma C_\omega \longrightarrow A$$

be a $\mathfrak{C}$-presentation. Since it is a coequalizer, it follows that the first line of

$$\begin{array}{ccccc}
hom(A, B) \longrightarrow & hom(\Sigma C_\omega, B) & \rightrightarrows & hom(\Sigma C_\psi, B) \\
& \| ! & & \| ! \\
& \Pi\, hom(C_\omega, B) & \rightrightarrows & \Pi\, hom(C_\psi, B) \\
& \| ! & & \| ! \\
hom(I, (A, B)) \longrightarrow & \Pi\, hom(I, (C_\omega, B)) & \rightrightarrows & \Pi\, hom(I, (C_\psi, B))
\end{array}$$

is an equalizer. That the third is follows from applying $hom(I, -)$ to an equalizer.

(4.13) THEOREM. *Given the hypotheses of Section 1, the category* $\pi\mathfrak{A}$, *equipped with* $-\otimes-$ *and* $(-,-)$ *is a closed monoidal category.*

5. CARTESIAN CLOSED CATEGORIES.

(5.1) Suppose the functor $-\otimes-: \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$ is the cartesian product, denoted as usual by $-\times-$. We would like to know when the induced tensor on $\pi\mathfrak{A}$ is the cartesian product. Of course we know by example that products in $\pi\mathfrak{A}$ may well be different from those in $\mathfrak{A}$. As above, we replace $\pi\mathfrak{A}$ by $\mathfrak{A}$ for purposes of exposition.

(5.2) Let $A \in \mathfrak{A}$ and let

$$\Sigma C_\psi \rightrightarrows \Sigma C_\omega \longrightarrow A$$

be a $\mathfrak{C}$-presentation. Then for $C \in \mathfrak{C}$,

$$\Sigma(C_\psi \times C) \rightrightarrows \Sigma(C_\omega \times C) \to A \otimes C$$

is a coequalizer. The projections $C_\omega \times C \to C$ give a map $\Sigma(C_\omega \times C) \to C$, which coequalizes the maps from $\Sigma(C_\psi \times C)$ and hence induces a map from $A \otimes C$ to C. The map

$$\Sigma(C_\omega \times C) \longrightarrow \Sigma C_\omega \longrightarrow A$$

also coequalizes the two maps from $\Sigma C_\psi \times C$ and thus induces a map from $A \otimes C$ to A. This gives a map $A \otimes C \to A \times C$. If $C' \to A \times C$ is a map with $C' \in \mathfrak{C}$, the map

$$C' \longrightarrow A \times C \longrightarrow A$$

factors through some $C_\omega \to A$ and determines a map

$$C' \longrightarrow C_\omega \times C \longrightarrow A \otimes C$$

whose projections on A and C are the given ones. This implies that the map $A \otimes C \to A \times C$ is a regular epimorphism since otherwise there would have to be a map $C' \to A \times C$ which doesn't factor through the image.

(5.3) Thus for any $\mathfrak{C}$-presentation

$$\Sigma C_\psi \rightrightarrows \Sigma C_\omega \longrightarrow A,$$

$\Sigma(C_\omega \times C) \to A \times C$ is a regular epimorphism. By adjointness

$$\Sigma(C_\omega \times C) \cong (\Sigma C_\omega) \otimes C,$$

so that $\Sigma(C_\omega \times C) \to (\Sigma C_\omega) \times C$ is a regular epimorphism. Let us now add the following

HYPOTHESIS. For any set of objects $\{C_\omega\}$ of $\mathfrak{C}$ and any $A \in \mathfrak{A}$, the canonical map $\Sigma(C_\omega \times A) \to (\Sigma C_\omega) \times A$ is a monomorphism.

(5.4) This hypothesis allows us to conclude immediately that

$$\Sigma(C_\omega \times C) \cong (\Sigma C_\omega)\times C.$$

Let $A_0 = \Sigma C_\omega$ in the above presentation. The sieve $\{C_\psi \to A_0 \times_A A_0\}$ are a terminal $\mathfrak{C}$-sieve so that $\Sigma C_\psi \to A_0 \times_A A_0$ is a regular epimorphism and hence so is

$$(\Sigma C_\psi)\times C \longrightarrow A_0 \times_A A_0 \times C \cong (A_0 \times C)\times_{A\times C}(A_0 \times C).$$

Since also $A_0 \times C \to A \times C$ is a regular epimorphism, it follows that

$$(\Sigma C_\psi)\times C \rightrightarrows (\Sigma C_\omega)\times C \longrightarrow A\times C$$

is a coequalizer, from which $A \otimes C \cong A \times C$ is immediate.

(5.5) Now simply repeat the argument with an object B in place of C, using (4.10) to initiate the argument used in (5.2).

(5.6) One easy way in which the hypothesis of (5.3) may be satisfied is if $\mathfrak{C}$ has finite sums, if any map from an object of $\mathfrak{C}$ to a sum in $\mathfrak{C}$ factors through a finite sum and in $\mathfrak{C}$ the injections to a sum are monomorphic. First consider the map

$$\Sigma(C_\omega \times C) \longrightarrow (\Sigma C_\omega)\times C \quad \text{for } C \in \mathfrak{C}.$$

If this fails to be a monomorphism, there are two maps $C' \rightrightarrows \Sigma(C_\omega \times C)$ with the same composite in $(\Sigma C_\omega)\times C$. A finite number of indices, say $\omega = 1,\ldots,n$, suffice so that the maps from C' factor

$$\begin{array}{ccc} C' \rightrightarrows \sum_{\omega=1}^{n}(C_\omega \times C) & \longrightarrow & \Sigma(C_\omega \times C) \\ \| & & \downarrow \\ (\sum_{\omega=1}^{n} C_\omega)\times C & \longrightarrow & (\Sigma C_\omega)\times C \end{array}$$

and with the injection into the sum a monomorphism, it follows the two maps are equal. An analysis of the argument used in (5.4) shows that this suffices to show $A \otimes C = A \times C$ for $C \in \mathfrak{C}$ which can be fed right back into this argument to derive the hypothesis for arbitrary A. This can be summarized as

(5.7) THEOREM. *Given the hypotheses of Section 1 with the tensor on* $\mathfrak{C}$

the cartesian product, then the resultant category $\pi\mathfrak{A}$ *is cartesian closed provided the hypothesis of (5.3) is satisfied.*

6. EXAMPLES.

(6.1) The first example leads to the category of compactly generated spaces. Let $\mathfrak{A}$ be the category of Hausdorff spaces and $\mathfrak{C}$ the full subcategory of compact ones. The tensor is the cartesian product and the hom

$$(-,-): \mathfrak{C}^{op} \times \mathfrak{A} \to \mathfrak{A}$$

equips the function space with the uniform convergence topology. That the hypotheses of Section 1 are satisfied is standard (Kelley [1955]), and that these of (5.6) are is trivial. The construction gives the category of compactly generated spaces with internal hom given by the compact/open topology.

(6.2) With $\mathfrak{A}$ as above, any full subcategory of compact spaces may be taken as $\mathfrak{C}$ provided it is closed under finite products. In fact, even that restriction may be dropped provided we can show that finite products of objects in $\mathfrak{C}$ are $\mathfrak{C}$-generated. For then the classes of spaces generated by $\mathfrak{C}$ and these generated by its finite product closure are the same. Since the full subcategory generated by the one is cartesian closed - the hypothesis of (5.3) being satisfied in any case - it follows that the full subcategory generated by $\mathfrak{C}$ is also.

(6.3) For example, take for $\mathfrak{C}$ the category consisting of the space

$$C = \{ 1, 1/2, 1/3, \dots, 0 \}$$

with the usual topology and its endomorphisms. We call the $\mathfrak{C}$-generated spaces *sequentially generated*. It is clear that first countable spaces - in particular all C^n - are sequentially generated and so are their quotients (see Kelley, problem 3-R where it is shown that the euclidean plane with one axis shrunk to a point is a quotient of a first countable space that isn't first countable) . Conversely, any sequentially generated space is a quotient of a union of copies of C and hence a quotient of a first countable space.

Thus sequentially generated means a quotient of a first countable space.

(6.4) For another example let $\mathfrak{C}$ be the category consisting of the unit interval J and its endomorphisms. Let X be any space such that each point has a countable decreasing basis of pathwise connected sets. Call it pathwise first countable. Then the topology is determined by the convergent sequences. Given such a sequence $\{x_n\} \to x$, let $\{U_n\}$ be a neighborhood base at x. By refining the sequence, if necessary, we may suppose that $x_n \epsilon U_n$. Then map $J \to X$ by letting the interval $[1/n+1, 1/n]$ go to a path between x_{n+1} and x_n which lies entirely inside U_n. This describes a continuous map $J \to X$ and it is clear that such maps determine the topology. Since each J^n is pathwise first countable the $\mathfrak{C}$-generated spaces - usually called simplicially generated - form a cartesian closed category. They are the same as the pathwise first countable spaces.

These topological examples have also been considered, from slightly different points of view - by Day [1972] and Wyler [1973] - who obtain substantially the same results.

(6.5) We can also play this game with the category of separated uniform spaces. A direct limit in the category of uniform spaces of compact Hausdorff spaces has the fine uniformity. For any function which is continuous is continuous, hence uniformly continuous, on every compact subspace, and hence uniformly continuous on the whole space. The same argument shows it is compactly generated in the category of completely regular spaces. The converse is also true so the resultant category consists of the completely regular spaces which are compactly generated in *that category.*

(6.6) We may vary the examples of (6.2) - (6.4) by considering pointed Hausdorff spaces for $\mathfrak{A}$. The tensor product is now the smash product (the cartesian product with the naturally embedded sum shrunk to a point) and the internal hom of C to X is the subspace of base point preserving maps with the same compact/open topology. The fact that the map

$$C_1 \times C_2 \to C_1 \otimes C_2$$

consists in the identification of a single compact set to a point allows the proof that the *inverse* image of a compact set is compact. From that observation, the equivalence

$$(C_1 \otimes C_2, A) = (C_1, (C_2, A))$$

follows readily from the analogous result for non-pointed spaces.

This example may also be varied by considering different possibilities for $\mathfrak{C}$ to get pointed sequentially generated spaces, pointed simplicially generated spaces, etc...

(6.7) By an MT (for mixed topology) is meant a real or complex topological space equipped with an auxiliary norm. Morphisms are required to be linear, continuous in the topology and norm reducing. Various conditions may be imposed on the spaces but the one most useful here is the supposition that the norm and the topology are those induced by mappings to Banach spaces. This means the space is a subspace, both in norm and topology, of a product of Banach spaces. If complete, it is a closed subspace. We let $\mathfrak{A}$ be the category of such complete MT spaces. Let $\mathfrak{B}$ be the full subcategory of the Banach spaces (i.e. the topology is that of the norm) and $\mathfrak{C}$ the full subcategory of those whose unit ball is compact. It is known (see Semadeni [1960] or Barr [1976]) that $\mathfrak{B}$ and $\mathfrak{C}$ are dual by functors

$$B \longmapsto B^*, \qquad C \longmapsto C^*$$

(describable as the space of linear functionals topologized, respectively, with the weak and the norm topologies). If $B \in \mathfrak{B}$, $C \in \mathfrak{C}$, (C, B) is defined to be the space of morphisms $C \to B$ normed by the sup norm. It is a Banach space. If $A \in \mathfrak{A}$, let $A \subset \Pi B_\omega$, then (C, A) is the space of morphisms given the topology and norm induced by $(C, A) \subset \Pi(C, B_\omega)$. Finally, for C_1, C_2 in $\mathfrak{C}$, we define

$$C_1 \otimes C_2 = (C_1, C_2^*)^*,$$

which lies in $\mathfrak{C}$. The proof that the conditions of section 1 are satisfied (which uses completeness of the spaces, by the way) can be worked out from Barr [1976a].

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