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TENSOR PRODUCTS OF TOPOLOGICAL RINGOIDS

by Andrée and Charles EHRESMANN

INTRODUCTION.

A topological ringoid A is an *Ab*-category (category enriched in the category of abelian groups) *A* equipped with a topology such that the underlying category be a topological category (in the sense category internal to *Top*) and that the addition be also continuous. Topological ringoids arise in several problems of Differential Geometry: for instance the category of 1-jets from a differentiable manifold into itself «is» a topological ringoid; other topological ringoids are naturally associated to vector bundles.

If A and A' are topological ringoids and if σ is a «stable» set of subsets of A, we construct a topological ringoid $A' \otimes_\sigma A$ whose underlying *Ab*-category is the tensor product $A' \otimes A$ (it is known [10] that *Ab-Cat* admits a canonical monoidal closed structure). The continuous additive functors from $A' \otimes_\sigma A$ to a topological ringoid A" are in 1-1 correspondence with the continuous additive functors from A' to the topological ringoid $\mathrm{Hom}_\sigma(A, A'')$ of continuous additive functors from A to A", equipped with the σ-open topology. This answers a question unsolved in [17].

One of the main results gives weak enough conditions on the sets σ and σ' for the existence of an «associativity» morphism or equivalence $(- \otimes_{\sigma'} A') \otimes_\sigma A \to - \otimes_{\sigma' \otimes \sigma} (A' \otimes_{\sigma'} A)$. As a by-product, monoidal closed structures are defined on the category *RdT* of topological ringoids, on the subcategory of Hausdorff ringoids and on the category *T Ab-Cat* (where *TAb* is the category of topological abelian groups).

Several authors [11,12,16] have given general existence Theorems for monoidal closed structures on a category. But these «global» structures are rather scarce on categories related to Topology. So there is a need for «partial» tensor products, more adapted to a prescribed geometrical or topological situation; such problems were the motivation for this paper.

1. TENSOR PRODUCTS OF TOPOLOGIES.

The category *Top* of topological spaces is not cartesian closed. To remedy this hindrance several solutions have been proposed:

1° to extend *Top* into a cartesian closed category, e.g. the category of Choquet pseudo-topologies [7], the category of limit spaces [2,8], the category of Spanier quasi-topologies [21];

2° to restrict *Top*, e.g. by considering the category of Kelley spaces [13] which is cartesian closed but in which the product is different from the product in *Top*.

On *Top* itself, there are monoidal closed structures, associated to tensor product topologies defined on the product set. This is done in [1], from which we gather here some results used in the sequel.

A. σ-open topologies on functional spaces.

Let (E, T) be a topological space and σ a set of subsets Σ of E satisfying the axiom:

(a) Each point of E belongs to at least one $\Sigma \in \sigma$.

If (E', T') is a topological space, we denote by $C_\sigma(T, T')$ the set $C(T, T')$ of continuous maps $f: T \to T'$ from T to T', equipped with the *σ-open topology*, which is generated by all the sets

$$<\Sigma, U'> = \{ f: T \to T' \mid f(\Sigma) \subset U' \},$$

where $\Sigma \in \sigma$ and U' is open in T.

REMARK. In [1], $C_\sigma(T, T')$ is denoted by $C_\sigma(T', T)$; we come back here to the more usual notation.

There exists ([1], page 12) a functor $C_\sigma(T, -): Top \to Top$ associating to $g: T' \to T''$ the continuous map

$$C_\sigma(T, g): C_\sigma(T, T') \to C_\sigma(T, T'')$$

which sends $f: T \to T'$ to $g \circ f: T \xrightarrow{f} T' \xrightarrow{g} T''$.

B. σ-product of topologies ([1], page 23).

With the same hypotheses, we define on the product set $E' \times E$ a topology, called the *σ-product of (T', T)*, and denoted by $T' \times_\sigma T$ (instead of $T'\otimes_\sigma T$ in [1]). It is the finest topology $\hat{T}$ on $E'\times E$ such that:

1° For each x' in E' we have the continuous map

$$(x',-): T \to \hat{T}: x \mapsto (x',x).$$

2° For each $\Sigma\in\sigma$, the insertion from $E'\times\Sigma$ to $E'\times E$ is continuous from $T'\times(T/\Sigma)$ into $\hat{T}$ (where T/Σ is the topology induced by T on Σ).

The open sets of $T'\times_\sigma T$ are the subsets W of $E'\times E$ containing, for each point (x',x) of W:

1° a set $\{x'\}\times U$, where U is a neighborhood of x in T,

2° for each $\Sigma\in\sigma$ a set $V'\times V$, where V is a neighborhood of x in T/Σ and V' a neighborhood of x' in T'.

$T'\times_\sigma T$ has the following «universal property»: If (E'', T'') is a topological space, a map $f: E'\times E \to E''$ is continuous from $T'\times_\sigma T$ to T'' iff it satisfies the two conditions:

1° For each x' in E', we have the continuous map

$$f(x',-): T \to T'': x \mapsto f(x',x).$$

2° For each $\Sigma\in\sigma$, the restriction $f/_{E'\times\Sigma}: T'\times(T/\Sigma)\to T''$ is continuous.

In particular, $T'\times_\sigma T$ is finer than the product topology $T'\times T$, so that it is Hausdorff if so are T and T'.

EXAMPLES. 1° If σ is the set s of all the subsets with one element of E, then $T'\times_s T$ is the so-called *asterisk topology*, considered by several authors [5,6, 20], and which renders continuous the «separately continuous» maps. We get the same topology if we take for σ the set of all finite subsets of E.

2° If $E\in\sigma$, then $T'\times_\sigma T = T'\times T$.

3° If σ is the set c of all (Hausdorff) compact subspaces of T, we obtain the c-product $T'\times_c T$. When T is locally compact, we have:

$$T'\times_c T = T'\times T.$$

REMARK. In [22] other topologies are defined on $E'\times E$ by specifying not only a set σ of subsets of E but also a set σ' of subsets of E'.

C. c-stable sets.

Let (E, T) be a topological space and σ a set of subsets of E. We say σ is *c-stable* (*c(T)*-stable in [1], page 14) if it satisfies the axiom (a) above and:

(b) for each $\Sigma \in \sigma$, the topology T/Σ is compact and each $x \in \Sigma$ admits a basis of neighborhoods in T/Σ formed by elements of σ.

For example, s and c are c-stable.

THEOREM 1 *([1], page 25-27). If σ is c-stable, the functor $C_\sigma(T,-)$ from Top to Top admits as a left adjoint the functor $-\times_\sigma T$: Top $\to$ Top, associating $g \times Id_T : T' \times_\sigma T \to T'' \times_\sigma T$ to $g: T' \to T''$.*

In other words, there exists a canonical equivalence

$$C(T', C_\sigma(T,-)) \to C(T' \times_\sigma T, -)$$

between functors from *Top* to *Set*. More precisely:

THEOREM 2 *([1], page 30). Suppose σ is c-stable and σ' is a c-stable set of subsets of the topological space (E', T'). Then*

$$\sigma' \times \sigma = \{ \Sigma' \times \Sigma \mid \Sigma' \in \sigma', \Sigma \in \sigma \}$$

is c-stable in $(E' \times E, T' \times_\sigma T)$ and the canonical equivalence above lifts into an equivalence

$$C_{\sigma'}(T', C_\sigma(T,-)) \to C_{\sigma' \times \sigma}(T' \times_\sigma T, -)$$

between functors from Top to Top.

Theorems 1 and 2 imply the following «associativity» result:

THEOREM 3 *([1], page 32). With the assumptions of Theorem 2 there exists a canonical equivalence between functors from Top to Top:*

$$(-\times_{\sigma'} T') \times_\sigma T \to -\times_{\sigma' \times \sigma}(T' \times_\sigma T).$$

COROLLARY. *There exist homeomorphisms:*

$$(T'' \times_s T') \times_s T \to T'' \times_s (T' \times_s T) \text{ and } (T'' \times_c T') \times_c T \to T'' \times_c (T' \times_c T)$$

defined by $((x'', x'), x) \mapsto (x'', (x', x))$ for any topological spaces (E, T), (E', T') and (E'', T'').

D. Monoidal closed structures on Top **and its subcategories.**

Given a topological space (E, T) and a c-stable set σ on it, we have constructed functors $- \times_\sigma T$ and $C_\sigma(T, -)$ from Top to Top. Is it possible to «glue together» such functors to obtain a monoidal closed structure on Top or on subcategories of Top?

Suppose given a full subcategory S of Top containing at least a one-point topological space, and a map $\sigma(-)$ associating to each object (E, T) of S a c-stable set $\sigma(T)$ of subsets of E such that

(c) For each $f: T \to T'$ in S, we have $f(\Sigma) \epsilon \sigma(T')$ for any $\Sigma \epsilon \sigma(T)$.

EXAMPLES. 1° The map s associating to each topological space the set of its one-point subsets satisfies (c) with respect to Top.

2° The map c associating to any topological space the set of its compact subsets satisfies (c) with respect to the subcategory $HTop$ of Hausdorff spaces, but not with respect to Top itself.

THEOREM 4. *If* $T' \times_{\sigma(T)} T$ *and* $C_{\sigma(T)}(T, T')$ *are in* S *for any objects* T *and* T' *of* S, *then* S *admits a non associative (in general) monoidal closed structure whose tensor product* $\times_{\sigma(-)}$ *extends the functors* $- \times_{\sigma(T)} T: S \to S$ *and whose internal Hom functor* $C_{\sigma(-)}$ *extends the functors*

$$C_{\sigma(T)}(T, -): S \to S.$$

The tensor product always admits as a unit the one-point topology.

COROLLARY 1. Top *is a symmetric monoidal closed category* Top_s *when equipped with the tensor product* $\times_s$ *and the internal Hom* C_s.

COROLLARY 2. $HTop$ *becomes a monoidal closed category:*

1° $HTop_s$ *when equipped with* $- \times_s -$ *and* $C_s(-,-)$;

2° $HTop_c$ *when equipped with* $- \times_c -$ *and* $C_c(-,-)$.

The tensor product $- \times_c -$ on $HTop$ is not symmetric, while $- \times_s -$ is.

Let S satisfy the assumptions of Theorem 4 and let S' be a full coreflective subcategory of S containing a one-point topological space.

COROLLARY 3. *If* $T' \times_{\sigma(T)} T$ *is in* S' *when* T *and* T' *are in* S', *then* S' *is a non associative monoidal closed category for the restriction of the ten-*

sor product $\times_{\sigma(-)}$ *and the internal Hom:* $S' \times S'^* \xrightarrow[C_{\sigma(-)}(-,.)]{} S \xrightarrow[k]{} S'$ *where* k *is the coreflector.*

As an application of this last corollary, we consider the full subcategory Ke of $HTop$ whose objects are the Kelley spaces (also called compactly generated spaces) (see [13,15]).

THEOREM 5. *Ke is a cartesian closed category and the product of* (T', T) *in Ke is identical with* $T' \times_c T$.

PROOF. It is well-known that Ke is a coreflective subcategory of $HTop$, the coreflector being the Kelleyfication functor $K: HTop \to Ke$. If we prove that $T' \times_c T$ is a Kelley space for any Kelley spaces (E, T) and (E', T'), it will result from Corollaries 2 and 3 that Ke is a monoidal closed category for the tensor product $- \times_c -$ and the internal Hom: $K \circ C_c$. In fact, we shall prove that $T' \times_c T$ is identical with the product $T' \circ T$ of (T', T) in Ke, so that Ke is cartesian closed (see also [13]).

- Indeed, a subspace W of $T' \times_c T$ is open iff:

$$W_{x'} = W \cap (\{x'\} \times E) \quad \text{and} \quad W_B = W \cap (E' \times B)$$

are open in the topology induced by the product topology $T' \times T$, for each point x' of E' and each compact B of T. Now, $\{x'\} \times T$ and $T' \times B$ are Kelley spaces [13] so that $W_{x'}$ and W_B are open iff their intersection with each compact of $\{x'\} \times T$ and of $T' \times B$ are open. Hence W is open in the topology $T' \times_c T$ iff its intersection with any $B' \times B$, where B' is a compact of T', is open. But this is exactly the definition of the open sets for the Kelley product $T' \circ T$. So $T' \times_c T = T' \circ T$.

2. TENSOR PRODUCTS OF TOPOLOGICAL RINGOIDS.

A. Monoidal closed structure on *$Ab\,Cat$*.

The category Ab of abelian groups has a well-known monoidal closed structure. The tensor product $G' \otimes G$ of the abelian groups G' and G is their tensor product as Z-modules.

From general results [10], it follows that the category $Ab\,Cat$ of Ab-

categories admits a monoidal closed structure which we recollect briefly for later use.

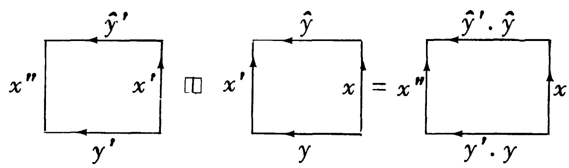
Ab-categories (i.e. categories enriched in *Ab*) are variously named; to keep the idea of «rings with several objects» [19] with a shorter name, we call them *ringoids* (annoïdes in French [3]) and we reserve the often used name «additive categories» for those ringoids admitting finite products (as in [3]). An *Ab*-category may be defined in several ways, the simplest one being probably the data A of a category $A^{\cdot}$ and of a lifting of its Hom functor $A^{*}\times A^{\cdot}\to Set$ into a functor

$$A(-,-): A^{*}\times A^{\cdot}\to Ab.$$

We denote by A_0 the set of objects of A, i.e. of $A^{\cdot}$, by A^{+} the groupoid coproduct (in *Cat*) of the abelian groups $A(e,e')$, for any objects e and e' of A, and by $0_{ee'}$ the zero of $A(e,e')$. The couple $(A^{\cdot}, A^{+})$ entirely determines the ringoid A.

We denote by *Rd* (shorter than *Ab-Cat*) the category of ringoids.

To the ringoid A is associated [3] the *horizontal ringoid* $\boxminus\!\!\!\!\!\;\mid A$ *of commutative squares of* $A^{\cdot}$, whose multiplication is:

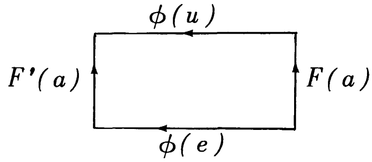


and the *vertical ringoid* $\boxminus A$; their couple $\square A$ is called *the double ringoid of squares of* A.

If A and A' are ringoids, we denote by $Hom(A,A')$ the ringoid of additive functors from A to A'. The morphisms of this ringoid, i.e. the natural transformations between additive functors from A to A', are identified [3] with additive functors from A to $\boxminus A$, by identifying

$$\phi: F \Longrightarrow F': A \to A'$$

with the additive functor $\Phi: A\to \boxminus A'$ which sends $a: e\to u$ in A onto the commutative square



This defines the «internal» *Hom* of the closed category *Rd*.

The tensor product in *Rd* associates to the ringoids A and A' the ringoid $A' \otimes A$ whose set of objects is $A'_0 \times A_0$, the abelian group from the object (e', e) to (u', u) being the tensor product group

$$A'(e', u') \otimes A(e, u).$$

The canonical bi-additive functor $J: (A', A) \to A' \otimes A$ is defined by

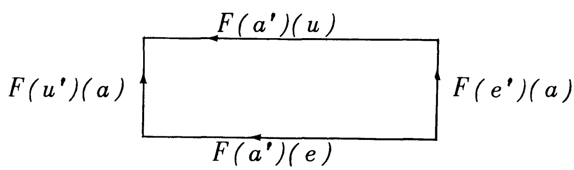
$J(a', a) = a' \otimes a$ for any morphisms a' of A' and a of A.

The image $J(A' \times A)$ «additively generates» the ringoid $A' \otimes A$.

The additive functors from $A' \otimes A$ to a ringoid A'' are in 1-1 correspondence with the bi-additive functors from (A', A) to A'', and also with the additive functors from A' to $Hom(A, A'')$. The canonical isomorphism

$$Hom(A', Hom(A, A'')) \to Hom(A' \otimes A, A'')$$

maps $F: A' \to Hom(A, A'')$ onto the additive functor sending $a' \otimes a$ onto the diagonal of the square $F(a')(a) =$



for $a: e \to u$ in A and $a': e' \to u'$ in A'.

B. Topological ringoids.

Ringoids may also be considered as sketched structures [4] : indeed there exists a projective cone-bearing category, the *sketch of ringoids*, whose realizations into *Set* «are» the ringoids [18] . The realizations of this sketch into *Top* are called topological ringoids.

A *topological ringoid* **A** is a couple (A, T) of a ringoid A and of a topology T on the set of morphisms of A, such that:

1° $(A^{\cdot}, T)$ is a topological category (in the sense: category internal to Top, i.e. the domain, codomain and composition maps are continuous [8]); let T_0 be the topology induced by T on A_0.

2° (A^+, T) is a topological groupoid (hence the addition and the opposite map are continuous); let T_0^+ be the topology induced by T on the set A_0^+ of objects of A^+, which is the set of 0-morphisms of A.

3° The continuous map $0_{ee'} \mapsto (e, e')$ from T_0^+ to $T_0 \times T_0$ is a homeomorphism.

These conditions imply that $A(e, e')$ becomes a topological group for the topology $T(e, e')$ induced by T.

EXAMPLES. 1° A topological (unitary) ring is a topological ringoid, with only one object.

2° If M is a differentiable manifold, the topological category $J^1(M)$ of 1-jets from M to M underlies a topological ringoid [9].

3° To a vector bundle is associated the topological ringoid of homomorphisms from fibre to fibre.

4° If E is a set, we have the ringoid A of couples of elements of E whose set of objects is E, the group $A(e, e')$ being reduced to its zero (e, e') for any pair of objects. If T is a topology on E, then $(A, T \times T)$ is a topological ringoid, called the *topological ringoid of pairs of* T.

General results on sketched structures (see also [18]) assert that the category of topological ringoids, denoted by RdT, admits both projective and inductive limits. The faithful functors from RdT to Rd and to Top preserve projective limits, and the first one is an initial-structure functor [23] (topological functor in the terminology of Herrlich [14], which is contradictory with ours). RdT is the category of 1-morphisms of a 2-category.

Let $\mathbf{A} = (A, T)$ be a topological ringoid. If we equip the ringoids of squares of A with the topology $\square T$ induced by the product topology T^4, we get two topological ringoids $◫\mathbf{A}$ and $⊟\mathbf{A}$, whose couple is the *topological double ringoid of squares of* $\mathbf{A}$.

Let $\mathsf{A}' = (A', T')$ be a topological ringoid; we denote by $\mathrm{Hom}(\mathsf{A}, \mathsf{A}')$ the subringoid of $Hom(A, A')$ of continuous additive functors from A to A'. Let σ be a c-stable set of subsets of A. Identifying a morphism $\bar{\mathrm{F}}$ of $\mathrm{Hom}(\mathsf{A},\mathsf{A}')$, i.e. a continuous additive natural transformation, with the corresponding continuous additive functor $\bar{\mathrm{F}}: \mathsf{A} \to \boxminus\mathsf{A}'$, we equip $\mathrm{Hom}(\mathsf{A},\mathsf{A}')$ with the topology induced by $C_\sigma(T, \square T')$ and get the topological ringoid [17] $\mathrm{Hom}_\sigma(\mathsf{A},\mathsf{A}')$. We have the endofunctor $\mathrm{Hom}_\sigma(\mathsf{A}, -)$ of RdT such that

$$\mathrm{Hom}_\sigma(\mathsf{A}, \mathrm{F}'): \mathrm{Hom}_\sigma(\mathsf{A}, \mathsf{A}') \to \mathrm{Hom}_\sigma(\mathsf{A}, \mathsf{A}''): \bar{\mathrm{F}} \mapsto \mathrm{F}' \circ \bar{\mathrm{F}},$$

if $\mathrm{F}': \mathsf{A}' \to \mathsf{A}''$, where $\circ$ is the total law of the 2-category on RdT.

C. Tensor products of topological rings.

Let $\mathsf{A} = (A, T)$ and $\mathsf{A}' = (A', T')$ be topological ringoids and σ be a set of subsets of A whose union is A.

If $\mathsf{A}'' = (A'', T'')$ is a topological ringoid, we say that

$$\mathrm{F}: (\mathsf{A}', \mathsf{A})_\sigma \to \mathsf{A}''$$

is a *σ-continuous bi-additive functor* if it is a bi-additive functor from (A', A) to A'' which is continuous from $T' \times_\sigma T$ to T''.

THEOREM 1. *1° There exists a finest topology $\hat{T}$ on the ringoid $A' \otimes A$, such that $(A' \otimes A, \hat{T})$ be a topological ringoid, denoted by $\mathsf{A}' \otimes_\sigma \mathsf{A}$, and*

$$\mathrm{J}: (\mathsf{A}', \mathsf{A})_\sigma \to \mathsf{A}' \otimes_\sigma \mathsf{A}: (a', a) \mapsto a' \otimes a$$

a σ-continuous bi-additive functor.

2° The σ-continuous bi-additive functors from $(\mathsf{A}', \mathsf{A})_\sigma$ to A'' are in 1-1 correspondence with the continuous additive functors from $\mathsf{A}' \otimes_\sigma \mathsf{A}$ to A'', for each topological ringoid A''.

PROOF. Let L be the class of all σ-continuous bi-additive functors

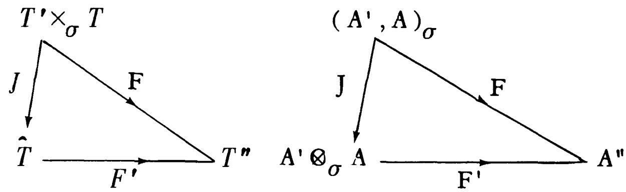
$$\mathrm{F}: (\mathsf{A}', \mathsf{A})_\sigma \to \mathsf{A}'' = (A'', T'').$$

Each F in L determines the additive functor

$$F': A' \otimes A \to A'': a' \otimes a \mapsto \mathrm{F}(a', a).$$

Let $\hat{T}$ be the initial topology associated to the family $(F', T'')_{\mathrm{F} \in L}$ (i.e. the coarser topology on $A' \otimes A$ such that $F': \hat{T} \to T''$ be continuous for any

F in L. The forgetful functor $RdT \to Rd$ being an initial-structure functor, and the functor $RdT \to Top$ preserving initial-structures, $(A' \otimes A, \hat{T})$ is a topological ringoid, $A' \otimes_\sigma A$, which is the initial topological ringoid associated to the family $(F', A'')_{F \in L}$. So by construction, each F in L determines the continuous additive functor F': $A' \otimes_\sigma A \to A''$.



- Let $J : (A', A) \to A' \otimes A$ be the canonical bi-additive functor. Each F in L being continuous from $T' \times_\sigma T$ to T'' and factorizing through J, the universal property of the initial topology implies that $J : T' \times_\sigma T \to \hat{T}$ is continuous; it follows that $\hat{T}$ is the finest ringoid topology such that

$$J : (A', A)_\sigma \to A' \otimes_\sigma A$$

be a continuous bi-additive functor J. ∎

COROLLARY 1. *With the notations of Theorem 1, the topology* $\hat{T}_0$ *induced on* $\hat{A}_0 = A'_0 \times A_0$ *by* $A' \otimes_\sigma A$ *is finer than the topology* $\hat{T}'_0$ *induced by* $T' \times T$ *and coarser than that* $\hat{T}''_0$ *induced by* $T' \times_\sigma T$. *Hence if* T_0 *and* T'_0 *are Hausdorff (resp. discrete) topologies, so is* $\hat{T}_0$.

PROOF. $J : T' \times_\sigma T \to \hat{T}$ being continuous, its restriction to $\hat{A}_0$ which is the identity on $\hat{A}_0$ is continuous from $\hat{T}''_0$ to $\hat{T}_0$. On the other hand, let B be the topological ringoid of pairs of $\hat{T}'_0$ (Example 4 above). There exists a bi-additive functor $G : (A', A) \to B$ which maps

$$(a', a) \quad \text{onto} \quad ((u', u), (e', e)),$$

if $a : e \to u$ in A and $a' : e' \to u'$ in A'. It is continuous from $T' \times T$ to $\hat{T}'_0 \times \hat{T}'_0$ (since the maps domain and codomain are continuous in A' and in A), and a fortiori σ-continuous. Hence G factors through a continuous additive functor $G' : A' \otimes_\sigma A \to B$; the identity of $\hat{A}_0$ being the restriction of G' to $\hat{A}_0$, it is continuous from $\hat{T}_0$ to $\hat{T}'_0$. Finally, $\hat{T}''_0 \to \hat{T}_0 \to \hat{T}'_0$. ∎

EXAMPLE. If A and A' are topological rings, so is $A' \otimes_\sigma A$.

THEOREM 2 *(Unitarity). Let* Z *be the ring of integers, with the discrete topology. Then*

$$Z \otimes_\sigma A \sim A \sim A \otimes_s Z .$$

PROOF. We shall construct a σ-continuous bi-additive functor

$$H : (Z, A)_\sigma \to A$$

and prove that each σ-continuous bi-additive functor from $(Z, A)_\sigma$ factors through it. From the universal property of $Z \otimes_\sigma A$, it will follow that A is isomorphic to this tensor product. Indeed, there exists a bi-additive functor

$$H : (Z, A) \to A : (z, a) \mapsto z\, a .$$

Since Z is discrete, the topology $Z \times_\sigma T$ is the coproduct of the topologies $(\{z\} \times T)_{z \in Z}$. The addition on A being continuous, each map

$$H(z, -) : T \to T : a \mapsto z\, a$$

is continuous, so that $H : Z \times_\sigma T \to T$ is continuous.

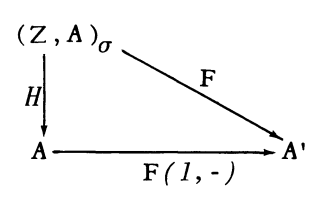
- Let $F : (Z, A)_\sigma \to A'$ be a σ-continuous bi-additive functor. In particular, $F(1, -) : A \to A'$ is a continuous additive functor. The composite

$$(Z, A)_\sigma \xrightarrow{H} A \xrightarrow{F(1,-)} A'$$

maps (z, a) onto

$$F(1, z\, a) = z\, F(1, a) = F(z, a)$$

(we use the bi-additivity of F), hence it is identical with F, and F factors through A .



- A similar method proves that A is isomorphic with $A \otimes_s Z$. ∎

If $F' : A' \to A''$ is a continuous additive functor, the map sending (a', a) onto $F'(a') \otimes a$ defines a σ-continuous bi-additive functor

$$(A', A)_\sigma \xrightarrow{F' \times \mathrm{Id}} (A'', A)_\sigma \xrightarrow{J'} A'' \otimes_\sigma A ,$$

so that it factors through an additive functor

$$F' \otimes_\sigma A : A' \otimes_\sigma A \to A'' \otimes_\sigma A .$$

This determines an endofunctor $- \otimes_\sigma A$ of RdT.

D. Some canonical isomorphisms.

THEOREM 3. *If* $A = (A, T)$ *is a topological ringoid and* σ *a c-stable set of subsets of* A, *then the functor* $- \otimes_\sigma A$ *is a right adjoint of the functor*

$$\mathrm{Hom}_\sigma (A, -) : RdT \to RdT .$$

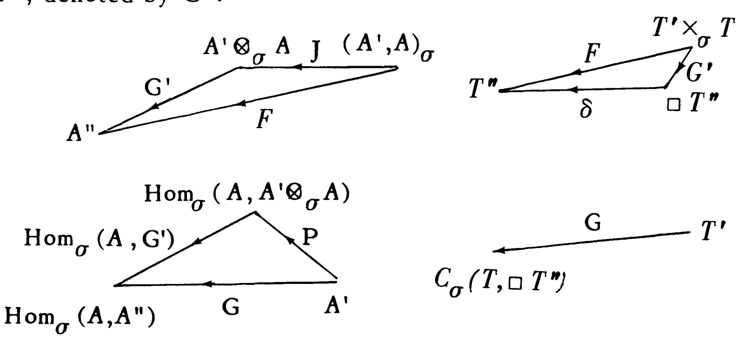
PROOF. We denote by $J : (A', A)_\sigma \to A' \otimes_\sigma A$ the canonical projection. Let

$$G : A' \to \mathrm{Hom}_\sigma (A, A''),$$

where $A' = (A', T')$ and $A'' = (A'', T'')$, be a continuous additive functor. Then G determines an additive functor from A' to $Hom(A, A'')$, hence a unique additive functor $G' : A' \otimes A \to A''$ (universal property of the tensor product). The composite

$$F : (A', A) \xrightarrow{J} A' \otimes A \xrightarrow{G'} A'' : (a', a) \mapsto G'(a' \otimes a)$$

defines a bi-additive functor. If we show that F is σ-continuous, it follows from Theorem 1 that G' defines a continuous additive functor from $A' \otimes_\sigma A$, to A'', denoted by G'.



- Indeed, by construction of $\mathrm{Hom}_\sigma (A, A'')$, we have the continuous map

$$G : T' \to C_\sigma (T, \square T'') .$$

As σ is c-stable, this implies that the map $(a', a) \mapsto G(a')(a)$ is continuous from $T' \times_\sigma T$ to $\square T''$. The diagonal map $\delta : \square T'' \to T''$ is continuous so the map

$$(a', a) \mapsto \delta\, G(a')(a) = G'(a' \otimes a)$$

is also continuous from $T' \times_\sigma T$ to T'' ; this map is F . Hence F is σ-continuous. We have constructed a canonical bijection

$$\mathrm{Hom}(A', \mathrm{Hom}_\sigma(A, A''))_0 \to \mathrm{Hom}(A' \otimes_\sigma A, A'')_0 : G \mapsto G',$$

whose inverse maps $H : A' \otimes_\sigma A \to A''$ onto

$$A' \xrightarrow{P} \mathrm{Hom}_\sigma(A, A' \otimes_\sigma A) \xrightarrow{\mathrm{Hom}_\sigma(A,H)} \mathrm{Hom}_\sigma(A, A''),$$

where P is the «liberty morphism» defined by

$$P(a') : A \to A' \otimes_\sigma A : a \mapsto a' \otimes a. \quad ■$$

Now we lift the canonical isomorphisms into topological ones. Suppose σ' is a c-stable set of subsets of A' . For each topological ringoid A'' the σ-continuous bi-additive functors $F : (A', A)_\sigma \to A''$ are objects of the ringoid $\mathrm{Hom}((A', A)_\sigma, A'')$, whose morphisms from F to G are identified with the σ-continuous bi-additive functors $\bar{F} : (A', A)_\sigma \to \boxminus A''$ such that

$$\bar{F}(a', a) = G(a', a) \quad \begin{array}{ccc} & \bar{F}(\beta a', \beta a) & \\ \uparrow & \longleftarrow & \uparrow F(a', a) \\ & \longleftarrow & \\ & \bar{F}(\alpha a', \alpha a) & \end{array}$$

(α and β being the domain and codomain maps). By this identification we equip $\mathrm{Hom}((A', A)_\sigma, A'')$ with the topology induced by $C_{\sigma' \times \sigma}(T' \times_\sigma T, \square T'')$. As $\sigma' \times \sigma$ is c-stable (Section 1), so is constructed a topological ringoid denoted by $\mathrm{Hom}_{\sigma'}((A', A)_\sigma, A'')$.

We consider the set $\sigma' \otimes \sigma$ of subsets of $A' \otimes A$ formed by the sets

$$\Sigma' \otimes \Sigma = J(\Sigma' \times \Sigma), \text{ where } \Sigma' \in \sigma', \Sigma \in \sigma,$$

and by the one-point sets $\{y\}$, where y is not in the image of the canonical projection $J : (A', A)_\sigma \to A' \otimes_\sigma A$.

THEOREM 4. *If σ and σ' are c-stable, the 1-1 correspondence η_0 between the σ-continuous bi-additive functors from $(A', A)_\sigma$ to A'' and the continuous additive functors from $A' \otimes_\sigma A$ to A'' extends into an isomorphism*

$$\eta : \mathrm{Hom}_{\sigma'}((A',A)_\sigma , A'') \to \mathrm{Hom}_{\sigma'\otimes\sigma}(A' \otimes_\sigma A, A'').$$

PROOF. 1° There is clearly a ringoid isomorphism η . We have to show that it is an homeomorphism from the topology

$$S \quad \text{induced by } C_{\sigma'\times\sigma}(T'\times_\sigma T, \square T'')$$

to the topology

$$S' \quad \text{induced by } C_{\sigma'\otimes\sigma}(\hat{T}, \square T'').$$

This will imply that $\mathrm{Hom}(A'\otimes_\sigma A, A'')$ equipped with S' is a topological ringoid, yet denoted by $\mathrm{Hom}_{\sigma'\otimes\sigma}(A'\otimes_\sigma A, A'')$, and that η is a topological isomorphism. (Remark that the existence of this topological ringoid is not obvious a priori, since $\sigma'\otimes\sigma$ is not always c-stable, and the construction of $\mathrm{Hom}_\sigma(A,-)$ uses the preservation of pullbacks by $C_\sigma(T,-)$.)

2° $$\eta^{-1} : S' \to S : \bar{F}' \mapsto \bar{F}' \circ J$$

is continuous. Indeed, it is sufficient to see that the image by η of each elementary open set of S,

$$<\Sigma'\times\Sigma, U> = \{ \bar{F} \mid \bar{F}(\Sigma'\times\Sigma) \subset U \},$$

where U open in $\square T''$ and $\Sigma' \in \sigma'$, $\Sigma \in \sigma$, is open in S'. This is true, since:

$$\eta(<\Sigma'\times\Sigma, U>) = \{ \bar{F}' \mid \bar{F}' J(\Sigma'\times\Sigma) \subset U \} = <\Sigma'\otimes\Sigma, U> .$$

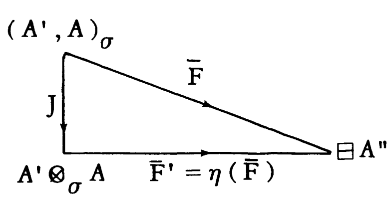
3° $\eta : S \to S'$ is continuous. Indeed, the elementary open sets of S' are of the form

$$<\Sigma'\otimes\Sigma, U> \quad \text{or} \quad <\{y\}, U> \text{ with } y \notin J(A'\times A).$$

It suffices to show that the image by η^{-1} of these sets are open sets in S. From Part 2:

$$\eta^{-1}(<\Sigma'\otimes\Sigma, U>) = <\Sigma'\times\Sigma, U>$$

is open in S. We are going to show that $\eta^{-1}(<\{y\}, U>)$ is a neighborhood of each of its elements $\bar{F}$. As $J(A'\times A)$ additively generates $A'\otimes A$, there



exist $x_1, \dots, x_n \in A' \times A$ such that

$$y = \mathrm{J}(x_1) + \dots + \mathrm{J}(x_n).$$

$\bar{\mathrm{F}} \in \eta^{-1}(<\{y\}, U>)$ implies $\eta(\bar{\mathrm{F}})(y) \in U$. We have $\eta(\bar{\mathrm{F}}) \circ \mathrm{J} = \bar{\mathrm{F}}$, so that:

$$\eta(\bar{\mathrm{F}})(y) = \eta(\bar{\mathrm{F}})(\mathrm{J}(x_1) + \dots + \mathrm{J}(x_n)) = \bar{\mathrm{F}}(x_1) + \dots + \bar{\mathrm{F}}(x_n) \in U.$$

Since the addition of $\boxminus \mathrm{A}''$ is continuous, there exist open neighborhoods U_i of $\bar{\mathrm{F}}(x_i)$ in $\square T''$, $i = 1, \dots, n$, such that $U_1 + \dots + U_n \subset U$. Each x_i is contained in a $\tilde{\Sigma}_i \in \sigma' \times \sigma$. Since $\sigma' \times \sigma$ is c-stable and $\bar{\mathrm{F}}^{-1}(U_i)$ is an open neighborhood of x_i, there exist

$$\tilde{\Sigma}'_i \in \sigma' \times \sigma \quad \text{such that } x_i \in \tilde{\Sigma}'_i \subset \bar{\mathrm{F}}^{-1}(U_i) \cap \tilde{\Sigma}_i.$$

Therefore the set $\bigcap_{i=1}^{n} <\tilde{\Sigma}'_i, U_i>$ is an open neighborhood V of $\bar{\mathrm{F}}$ in S. It is included in $\eta^{-1}<\{y\}, U>$, because $\bar{\mathrm{G}} \in V$ implies

$$\bar{\mathrm{G}}(x_i) \in \bar{\mathrm{G}}(\tilde{\Sigma}'_i) \subset U_i,$$

and so

$$\eta(\bar{\mathrm{G}})(y) = \bar{\mathrm{G}}(x_1) + \dots + \bar{\mathrm{G}}(x_n) \in U_1 + \dots + U_n \subset U. \quad \blacksquare$$

A set σ of subsets of A is called *rc-stable for* A if it is c-stable and if the images of each $\Sigma \in \sigma$ by the maps domain α and codomain β of A are in σ. For example such is the case if $\sigma = s$, or if $\sigma = c$ and T is a Hausdorff space.

If $\mathrm{A}'' = (A'', T'')$ and $\mathrm{B} = (B, S)$ are topological ringoids, we say that $\mathrm{F}: ((\mathrm{A}'', \mathrm{A}')_{\sigma'}, \mathrm{A})_{\sigma} \to \mathrm{B}$ is a (σ', σ)-*continuous tri-additive functor*, if F is a tri-additive functor, continuous from $(T'' \times_{\sigma'} T') \times_{\sigma} T$ to S.

THEOREM 5. *Let* σ *be rc-stable for* A *and* σ' *be rc-stable for* A'; *then:*

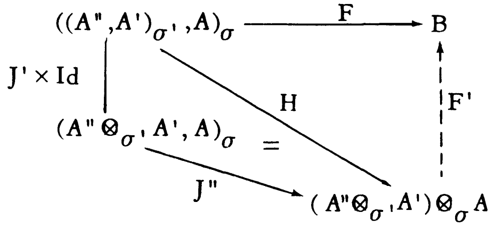
1° *Each* (σ', σ)-*continuous tri-additive functor factors through the tensor product* $(\mathrm{A}'' \otimes_{\sigma'} \mathrm{A}') \otimes_{\sigma} \mathrm{A}$.

2° *There exists a continuous additive «associativity» functor:*

$$\gamma : (\mathrm{A}'' \otimes_{\sigma'} \mathrm{A}') \otimes_{\sigma} \mathrm{A} \to \mathrm{A}'' \otimes_{\sigma' \otimes \sigma} (\mathrm{A}' \otimes_{\sigma} \mathrm{A}) :$$
$$(a'' \otimes a') \otimes a \mapsto a'' \otimes (a' \otimes a),$$

which is an isomorphism if $\sigma' \otimes \sigma$ *is c-stable.*

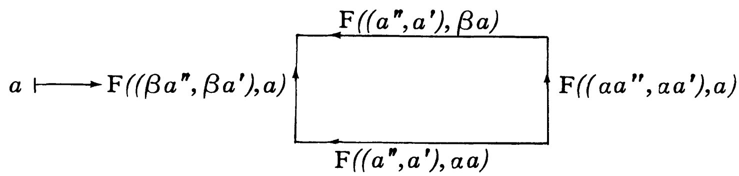
PROOF. 1° Let $F: ((A'', A')_{\sigma'}, A)_\sigma \to B = (B, S)$ be a (σ', σ)-continuous tri-additive functor. We want to show the existence of the broken line in the diagram (*):



in which J' and J" are the canonical projections; the composite H:

$$((a'', a'), a) \mapsto (a'' \otimes a') \otimes a$$

is a (σ', σ)-continuous tri-additive functor. Since F is tri-additive, it determines the bi-additive functor $G: (A'', A') \to Hom(A, B)$, which maps (a'', a') onto the additive functor $G(a'', a'): A \to \boxminus B$:



Suppose proven that $G: (A'', A')_{\sigma'} \to Hom_\sigma(A, B)$ is σ'-continuous. Then it factors through a continuous additive functor

$$G': A'' \otimes_{\sigma'} A' \to Hom_\sigma(A, B),$$

to which is associated by Theorem 3 the continuous additive functor

$$F': (A'' \otimes_{\sigma'} A') \otimes_\sigma A \to B : (a'' \otimes a') \otimes a \mapsto F((a'', a'), a).$$

- Hence it suffices to prove that $G: T'' \times_{\sigma'} T' \to C_\sigma(T, \square S)$ is continuous. Indeed, σ' being stable by α, the map

$$Id \times \alpha: T'' \times_{\sigma'} T' \to T'' \times_{\sigma'} T'$$

is continuous. As $- \times_{\sigma'} T$ and $- \times_\sigma T$ are endofunctors of Top, we have the continuous map

$$f_\alpha: (T'' \times_{\sigma'} T') \times_\sigma T \xrightarrow{(\alpha \times \alpha) \times Id} (T'' \times_{\sigma'} T') \times_\sigma T \xrightarrow{F} S$$

$$((a'', a'), a) \mapsto F((\alpha a'', \alpha a'), a).$$

Using the stability of σ by a , we find that

$$g_a : (T''\times_{\sigma'}T')\times_\sigma T \xrightarrow{Id\times a} (T''\times_{\sigma'}T')\times_\sigma T \xrightarrow{F} S :$$
$$((a'', a'), a) \mapsto F((a'', a'), aa)$$

is continuous. Let f_β and g_β be the similar maps with respect to β . These maps determine the continuous map

$$[f_\beta, g_\beta, g_a, f_a] : (T''\times_{\sigma'}T')\times_\sigma T \to \Box S \hookrightarrow S^4 :$$
$$((a'', a'), a) \mapsto G(a'', a')(a),$$

from which follows the continuity of $G: T''\times_{\sigma'}T' \to C_\sigma(T, \Box S)$.

2° We have the following diagram :

$$\begin{array}{ccc}
((A'',A')_{\sigma'}, A)_\sigma & \xrightarrow{\mu} & (A'', (A', A)_\sigma)_{\sigma'\times\sigma} \\
 & \searrow{\scriptstyle H'} & \downarrow{\scriptstyle Id\times J} \\
\downarrow{\scriptstyle H} & & (A'' ; A'\otimes_\sigma A)_{\sigma'\otimes\sigma} \\
 & & \downarrow{\scriptstyle \hat{J}} \\
(A''\otimes_{\sigma'}A')\otimes_\sigma A & \xrightarrow{\gamma} & A''\otimes_{\sigma'\otimes\sigma}(A'\otimes_\sigma A)
\end{array}$$

in which μ is the homeomorphism (cf. Section 1)

$$\mu : (T''\times_{\sigma'}T')\times_\sigma T \to T''\times_{\sigma'\times\sigma}(T'\times_\sigma T)$$

and J and $\hat{J}$ are the canonical projections ; by definition, J maps $\sigma'\times\sigma$ into $\sigma'\otimes\sigma$, so that

$$Id\times J : T''\times_{\sigma'\times\sigma}(T'\times_\sigma T) \to T''\times_{\sigma'\otimes\sigma}\hat{T}$$

is continuous, where $\hat{T}$ is the topology of $A'\otimes_\sigma A$. Therefore H' :

$$((a'', a'), a) \mapsto a''\otimes(a'\otimes a)$$

is a (σ',σ)-continuous tri-additive functor, and Part 1 implies that it factors through H to give the continuous additive functor γ .

3° Suppose that $\sigma'\otimes\sigma$ is c-stable. To prove that γ is an isomorphism, it suffices to prove that each (σ',σ)-continuous tri-additive functor F as above also factors through H' . Indeed, by a method similar to that used in Part 1 we associate to F the continuous additive functor

$$K : A'' \to \mathrm{Hom}_{\sigma'}((A', A)_\sigma, B)$$

such that $K(a''): T' \times_\sigma T \to \square S$ maps (a', a) onto the square $G(a'', a')(a)$ drawn in Part 1. As $\sigma' \otimes \sigma$ is supposed to be c-stable, Theorem 3 associates to the continuous additive functor

$$A'' \xrightarrow{K} \mathrm{Hom}_{\sigma'}((A', A)_\sigma, B) \xrightarrow{\eta} \mathrm{Hom}_{\sigma' \otimes \sigma}(A' \otimes_\sigma A, B)$$

(where η is defined in Theorem 5) a continuous additive functor

$$F'' : A'' \otimes_{\sigma' \otimes \sigma}(A' \otimes_\sigma A) \to B : a'' \otimes (a' \otimes a) \mapsto F((a'', a'), a)$$

whose composite with H' is F. ■

COROLLARY 1. *There exists an associativity isomorphism*

$$\gamma : (A'' \otimes_s A') \otimes_s A \to A'' \otimes_s (A' \otimes_s A).$$

PROOF. This follows from Theorem 5 applied in the case $\sigma = s$ and $\sigma' = s$, in which $\sigma' \otimes \sigma = s$ is c-stable. In this case there is a simple proof of Part 1 (and similarly of Part 3). Indeed, given the diagram (*) above, F defines a bi-additive functor

$$L : (A'' \otimes A', A) \to B : (a'' \otimes a', a) \mapsto F((a'', a'), a).$$

L is s-continuous, since the (s, s)-continuity of F implies the continuity of the maps:

for each $a \in A$, $L(-, a) = F(-, a) : T'' \times_s T' \to S$,
for each $x \in A'' \times A'$, $L(J'(x), -) = F(x, -) : T \to S$,
for each $y \in A'' \otimes A'$, $L(y, -) : T \to S$, since there exist $x_i \in A'' \times A'$ with $y = J'(x_1) + \ldots + J'(x_n)$, and $L(y, -) = F(x_1, -) + \ldots + F(x_n, -)$.

Hence L factors through H. ■

COROLLARY 2. *If σ and σ' are rc-stable, and if $\sigma' \otimes \sigma$ is c-stable, there exist isomorphisms*

$$\omega : \mathrm{Hom}_{\sigma'}(A', \mathrm{Hom}_\sigma(A, A'')) \to \mathrm{Hom}_{\sigma' \otimes \sigma}(A' \otimes_\sigma A, A''),$$

$$\omega' : \mathrm{Hom}_{\sigma'}(A', \mathrm{Hom}_\sigma(A, A'')) \to \mathrm{Hom}_{\sigma'}((A', A)_\sigma, A''),$$

E.g. they exist if $\sigma = s$ and $\sigma' = s$.

PROOF. ω is constructed from the identity of $\mathrm{Hom}_{\sigma'}(A', \mathrm{Hom}_\sigma(A, A''))$, by repeated use of the adjunction and «associativity» maps. Then ω' is the composite $\eta^{-1} \circ \omega$ (cf. Theorem 4). As $\sigma' \otimes \sigma$ is c-stable, ω^{-1} is deduced in a similar way from the identity of $\mathrm{Hom}_{\sigma' \otimes \sigma}(A' \otimes_\sigma A, A'')$. ■

From Theorem 3 and Corollary 1 of Theorem 5, we obtain:

THEOREM 6. *RdT admits a symmetric monoidal closed structure whose tensor product* $\otimes_S$ *extends the functors* $- \otimes_S A : RdT \to RdT$ *and whose internal Hom extends the functors* $\mathrm{Hom}_S(-, A)$.

3. HAUSDORFF RINGOIDS AND *Top*-RINGOIDS.

We study here two subcategories of *RdT*, a reflective one and a coreflective one.

A. Hausdorff ringoids.

A *Hausdorff ringoid* is defined as a topological ringoid A whose topology T is a Hausdorff topology.

We denote by *RdH* the full subcategory of *RdT* whose objects are the Hausdorff ringoids. It is complete and cocomplete, and the forgetful functors toward *Rd* and *Top* preserve projective limits.

General existence theorems prove that *RdH* is a reflective subcategory of *RdT*. Let $A = (A, T)$ be a topological ringoid and $P : A \to \tilde{A}$ the reflection morphism; its restriction $P_0 : A_0 \to \tilde{A}_0$ is onto: otherwise the restriction $P' : A \to \tilde{A}'$ of P to the full subringoid of $\tilde{A}$ such that $\tilde{A}'_0 = P(A_0)$ could not factor through P though $\tilde{A}'$ be a Hausdorff ringoid.

THEOREM 1. *If* $A = (A, T)$ *is a topological ringoid such that* T_0 *be a Hausdorff topology, then:*

1° $P : A \to \tilde{A} = (\tilde{A}, \tilde{T})$ *is onto and* $P_0 : T_0 \to \tilde{T}_0$ *is a homeomorphism.*

2° If σ *is a c-stable set of subsets of* A, *for each Hausdorff ringoid* A' *there is an isomorphism*

$$\zeta : \mathrm{Hom}_{\tilde{\sigma}}(\tilde{A}, A') \to \mathrm{Hom}_\sigma(A, A') : \bar{F} \mapsto \bar{F} \circ P,$$

where $\tilde{\sigma} = \{ P(\Sigma) \mid \Sigma \in \sigma \}$.

PROOF. 1° Let B be the topological ringoid of pairs of T_0 (Example 4-2). Its topological space of objects is T_0. The continuous additive functor:

$$G : A \to B : a \mapsto (\beta a, \alpha a)$$

admits a factorization

$$G : A \xrightarrow{P} \tilde{A} \xrightarrow{G'} B,$$

(since B is Hausdorff), and its restriction to the objects

$$G_0 : T_0 \xrightarrow{P_0} \tilde{T}_0 \xrightarrow{G'_0} T_0$$

is an identity; hence the onto map $P_0 : T_0 \to \tilde{T}_0$ is an homeomorphism (and P will be chosen so that P_0 be an identity). It follows that P(A) is a Hausdorff subringoid of $\tilde{A}$, hence $P(A) = \tilde{A}$.

2° The canonical 1-1 correspondence ζ deduced from the universal property of the reflection is an isomorphism, since it maps the set of elementary open sets

$$\langle P(\Sigma), U \rangle, \text{ where } \Sigma \in \sigma \text{ and } U \text{ open in } \boxminus A',$$

of $\mathrm{Hom}_{\tilde{\sigma}}(\tilde{A}, A')$ onto the set of elementary open sets of $\mathrm{Hom}_\sigma(A, A')$:

$$\langle \Sigma, U \rangle = \zeta(\langle P(\Sigma), U \rangle). \blacksquare$$

Let $A = (A, T)$ be a Hausdorff ringoid. Then $\boxminus A$ is also a Hausdorff ringoid. If σ is c-stable on A, the σ-open topology $C_\sigma(T, S)$ is a Hausdorff topology if S is a Hausdorff topology. It follows that, for each Hausdorff ringoid A', $\mathrm{Hom}_\sigma(A, A')$ is a Hausdorff ringoid; hence the functor $\mathrm{Hom}_\sigma(A, -)$ admits as a restriction an endofunctor of *RdH*.

On the other hand let σ be a set of subsets of A whose union is A, and let A' be a Hausdorff ringoid. The tensor product $A' \otimes_\sigma A$ is not necessarily a Hausdorff ringoid, but the set of its objects has a Hausdorff topology (Corollary 1 Theorem 1-2). We denote by $A' \tilde{\otimes}_\sigma A$ the Hausdorff ringoid associated with $A' \otimes_\sigma A$, and call it the *Hausdorff σ-tensor product of* A' and A. Theorem 1 asserts that the reflection morphism

$$P : A' \otimes_\sigma A \to A' \tilde{\otimes}_\sigma A$$

is onto and that its restriction to the objects is a homeomorphism.

$A'\tilde{\otimes}_\sigma A$ solves the universal problem to render continuous additive the σ-continuous bi-additive functors from $(A', A)_\sigma$ to Hausdorff ringoids. We denote by $-\tilde{\otimes}_\sigma A$ the composite functor (where ρ is the reflector):

$$RdH \hookrightarrow RdT \xrightarrow{-\otimes_\sigma A} RdT \xrightarrow{\rho} RdH.$$

From Theorem 3-2 and transitivity of adjunctions, we get:

THEOREM 2. *If σ is c-stable, the functor $-\tilde{\otimes}_\sigma A$ is a left adjoint of the functor* $\mathrm{Hom}_\sigma(A, -): RdH \to RdH$.

Let σ' be a c-stable set of subsets of A'. We denote by $\sigma'\tilde{\otimes}\sigma$ the set formed by the $P(\Sigma'\otimes\Sigma)$, where $\Sigma \in \sigma$ and $\Sigma' \in \sigma'$.

THEOREM 3. *Theorems 2, 4 and 5 of Section 2 are yet valid if we replace in them $\otimes$ by $\tilde{\otimes}$ and topological ringoid by Hausdorff ringoid.*

PROOF. From Theorems 4-2 and 1, we deduce the isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{\sigma'}((A', A)_\sigma, A'') & \xrightarrow{\eta} & \mathrm{Hom}_{\sigma'\otimes\sigma}(A'\otimes_\sigma A, A'') \\ & \searrow & \downarrow \zeta^{-1} \\ & & \mathrm{Hom}_{\sigma'\tilde{\otimes}\sigma}(A'\tilde{\otimes}_\sigma A, A''). \end{array}$$

The other results are proved as in Section 2. ■

COROLLARY. *1° RdH admits a symmetric monoidal closed structure whose tensor product $\tilde{\otimes}_s$ extends the functors $-\tilde{\otimes}_s A$ and whose internal Hom is a restriction of* Hom_s.

2° RdH admits a semi-associative monoidal closed structure whose tensor product $\tilde{\otimes}_c$ extends the functors $-\tilde{\otimes}_c A$ and whose internal Hom extends the functors $\mathrm{Hom}_c(A, -): RdH \to RdH$.

B. ***Top*-ringoids.**

A *Top-ringoid* is the data consisting of a ringoid A and of a topological group $\mathrm{A}(e, e')$ on $A(e, e')$ for each couple (e, e') of objects of A, such that, for each triple (e, e', e'') of objects, the composition map:

$$\mathrm{A}(e, e') \times \mathrm{A}(e', e'') \to \mathrm{A}(e, e''): (a, b) \mapsto b.a$$

be continuous.

To each topological ringoid $\mathrm{A} = (A, T)$ is associated the *Top*-ringoid obtained by taking A and on each group $A(e, e')$ the topology induced by T; this *Top*-ringoid entirely determines A if the topology induced by T on the set A_0 of objects is discrete.

Conversely, if $(A, \mathrm{A}(e, e'))$ is a *Top*-ringoid and if we equip A with the topology S coproduct of the topologies $\mathrm{A}(e, e')$, we obtain a topological ringoid in which the topological space S_0 of objects is discrete. Hence we identify the *Top*-ringoids with the topological ringoids whose topological space of objects is discrete.

We denote by *T-Rd* the full subcategory of *Rd T* whose objects are the *Top*-ringoids. It is a coreflective subcategory, the coreflection of A being the *Top*-ringoid associated above to A and the coreflection morphism being defined by the identity of A.

Let A be a *Top*-ringoid and σ a set of subsets of A whose union is A.

THEOREM 4. *1°* $\mathrm{A}' \otimes_\sigma \mathrm{A}$ *is a Top-ringoid, for each Top-ringoid* A'.

2° If σ is c-stable, the functor $- \otimes_\sigma \mathrm{A} : T\text{-}Rd \to T\text{-}Rd$ *admits as a right adjoint the functor*

$$\mathrm{H}_\sigma(\mathrm{A}, -) : T\text{-}Rd \hookrightarrow Rd\,T \xrightarrow{\mathrm{Hom}_\sigma(\mathrm{A},-)} Rd\,T \xrightarrow{\nu} T\text{-}Rd,$$

where ν is the coreflector.

PROOF. Corollary 1, Theorem 1-2 asserts that the topological space of objects of $\mathrm{A}' \otimes_\sigma \mathrm{A}$ is discrete, so that $\mathrm{A}' \otimes_\sigma \mathrm{A}$ is a *Top*-ringoid. The second assertion comes from the transitivity of adjunctions. ■

COROLLARY. *T-Rd is a symmetric monoidal closed category for the tensor product restriction of* $\otimes_S$ *and for an internal Hom extending the functors* $\mathrm{H}_S(\mathrm{A}, -)$.

REMARK. The topological ringoids $\mathrm{Hom}_\sigma(\mathrm{A}, \mathrm{A}')$ are not *Top*-ringoids (in general) since even the simplest of them $\square\,\mathrm{A}$ is a *Top*-ringoid iff the topology of A is discrete.

Similar results for *H Top*-ringoids are deduced from A.

C. Examples.

1° The category of topological rings *TR* is a full subcategory of the category *T-Rd* of *Top*-ringoids. If A is a topological ring and σ a set of subsets of A whose union is A, the functor $-\otimes_\sigma A$ admits as a restriction an endofunctor of *TR*. In particular, *TR* admits a symmetric monoidal (not closed) structure whose tensor product is a restriction of $\otimes_s$, and also a semi-associative monoidal structure for the tensor product $-\otimes_\pi-$ obtained by taking on each *A* the set π of all its subsets.

2° A topological abelian group B may be identified with the *Top*-ringoid $\hat{B}$ admitting only two objects *u* and *u'* and such that $\hat{B}(u, u') = B$ and $\hat{B}(u, u)$ and $\hat{B}(u', u')$ are discrete groups with two elements.

Let σ be a set of subsets of B whose union is B. If B' is a topological abelian group, by a method similar to that of Theorem 1-2 it is constructed a topological abelian group, denoted by $B' \otimes_\sigma B$, such that each σ-continuous bi-homomorphism from (B', B) to a topological abelian group B" factors through $B' \otimes_\sigma B$ into a continuous homomorphism toward B".

So is defined an endofunctor $-\otimes_\sigma B$ on the category *TAb* of topological abelian groups.

If σ is c-stable, $-\otimes_\sigma B$ *admits a right adjoint* $\mathrm{Hom}_\sigma(B, -)$ *such that* $\mathrm{Hom}_\sigma(B, B")$ *be the group of continuous homomorphisms from* B *to* B", *equipped with the topology induced by the σ-open topology* $C_\sigma(B, B")$, *for each topological abelian group* B".

It follows that *TAb* admits a symmetric monoidal closed structure with tensor product $-\otimes_s-$ and the internal Hom functor $\mathrm{Hom}_s(-,-)$.

It also admits *a symmetric semi-associative monoidal (not closed)* structure $(TAb)_\pi$ for the tensor product $-\otimes_\pi-$, where π associates to B the set of all its subsets. A bi-homomorphism from (B', B) is π-continuous iff it is continuous for the product topology $B' \times B$ and it then factors through $B' \otimes_\pi B$. Hence, the *Top*-ringoids may be identified with the $(TAb)_\pi$-categories (categories enriched in $(TAb)_\pi$).

4. RINGOIDS IN A CATEGORY.

A realization A of the sketch of ringoids in a category X is called a *ringoid in(ternal to)* X . Let *Rd*X be the category of ringoids in X and suppose X equipped with an intial-structure functor $\chi : \mathrm{X} \to Set$.

Then the methods and results of Section 2 may be generalized. More precisely, let A be a ringoid in X ; it is entirely determined by the couple (A, X), where A is the ringoid defined by the realization $\chi \circ \mathrm{A}$ and where $X \in \mathrm{X}_0$ is the «object of morphisms» (see [18]).

1° If $- \& X$ is an endofunctor of X such that

$$\mathrm{X} \xrightarrow{-\& X} \mathrm{X} \xrightarrow{\chi} Set = \mathrm{X} \xrightarrow{\chi} Set \xrightarrow{-\times \chi(X)} Set$$

we construct as in Theorem 1-2 an endofunctor $-\& \mathrm{A}$ of *Rd*X such that the ringoid underlying $\mathrm{A}' \& \mathrm{A}$ be $A' \otimes A$.

2° To A is associated the double ringoid $\square \mathrm{A}$ in X , over $\square A$.

3° Let $M(X, -)$ be an endofunctor of X preserving pullbacks. If A' is a ringoid in X , the realization $M(X,-) \circ \mathrm{A}'$ is a ringoid $M(X, \mathrm{A}')$ in X . Its object of morphisms is $M(X, X')$. We'll suppose moreover that

$$\mathrm{X} \xrightarrow{M(X,-)} \mathrm{X} \xrightarrow{\chi} Set = Hom_{\mathrm{X}}(X, -).$$

In this case, $M(X, \boxplus \mathrm{A}')$ admits a subringoid $M(\mathrm{A}, \mathrm{A}')$ in X over the ringoid of morphisms from A to A' (whose morphisms are the $\mathrm{F} : \mathrm{A} \to \boxminus \mathrm{A}'$).

4° If $M(X, -)$ is a right adjoint of $-\& X$, then $-\& \mathrm{A}$ admits a right adjoint $M(\mathrm{A}, -)$. If $(\mathrm{X}, \&, M(-,-))$ is a monoidal closed category, the functors $-\& \mathrm{A}$ and $M(\mathrm{A}, -)$ extend to give a monoidal closed structure on *Rd*X .

For instance, the ringoids in the cartesian closed category *Ke* (see Section 1) of Kelley spaces form a monoidal closed category. (Remark that a Kelley ringoid is not necessarily a topological ringoid, pullbacks in *Ke* differing from pullbacks in *Top* .) The ringoids in the categories of limit-spaces, or of pseudo-topologies, or of Spanier quasi-topologies,... form also monoidal closed categories.

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