CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

VÁCLAV KOUBEK

On categories into which each concrete category can be embedded. II

Cahiers de topologie et géométrie différentielle catégoriques, tome 18, nº 3 (1977), p. 249-269

<http://www.numdam.org/item?id=CTGDC_1977__18_3_249_0>

CAHIERS DE TOPOLOGIE *Vol. XVIII-3 (1977)*
ET GEOMETRIE DIFFERENTIELLE

ON CATEGORIES INTO WHICH EACH CONCRETE CATEGORY CAN BE EMBEDDED. II

by Václav KOUBEK

Given a contravariant functor F from sets to sets, the category $S(F)$ has for objects pairs (X, S), with X a set and $S \subset FX$; morphisms are mappings $f: (X, S) \to (Y, T)$ such that $Ff(T) \subset S$. The paper characterizes those functors F for which $S(F)$ is a universal category, i.e. every concrete category can be fully embedded into it. The characterization is very simple: F must be nearly faithful, i.e. there must be a cardinal α such that for arbitrary mappings $f, g: X \to Y$ we have: if $f \neq g$, then either $Ff \neq Fg$ or $card\, f(X) < \alpha$, $card\, f(Y) < \alpha$.

The paper continues the author's previous characterization of covariant functors F for which $S(F)$ (defined analogously) is binding. There are striking similarities between the two cases, yet the main result here has no analogy in the covariant case.

I

CONVENTIONS. *Set* denotes the category of sets and mappings.

The word «functor» will denote a contravariant set functor.

Let e be a decomposition of a set X. Then the canonical mapping from X to X/e will be denoted by e, therefore the class of e containing x is denoted $e(x)$.

If $f: X \to Y$ is a mapping, then $Ker f$ is the canonical decomposition of f, i.e. $Ker f = \{ f^{-1}(y) \mid y \in Im f \}$.

The cardinal α is meant as the set of all ordinals with type less than α ; α^+ denotes the cardinal successor of α.

DEFINITION. A concrete category is called *universal* if every concrete category can be embedded into it.

THEOREM 1.1. *The category $S(P^-)$ is universal.*

PROOF. See [8].

NOTE. We recall the definition of the functor P^- :

$P^-(X) = \{ Z \mid Z \subset X \}$,

if $f: X \to Y$ then for every $Z \in P^- Y$, $P^- f(Z) = f^{-1}(Z)$.

DEFINITION. A full embedding Ψ from the concrete category (K, U) to the concrete category (L, V) is called *strong* if there exists a set functor $F: Set \to Set$ such that the diagram

$$\begin{array}{ccc} K & \xrightarrow{\ \Psi\ } & L \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle V} \\ Set & \xrightarrow{\ F\ } & Set \end{array}$$

commutes.

DEFINITION. An object is *rigid* if it has no non-identical endomorphism.

Now we shall describe a «behaviour» of the functor F.

CONVENTION. Let F be a functor. Then for a cardinal α, F^α denotes the subfunctor of F such that

$$F^\alpha Y = \bigcup_{card\, Z < \alpha} \bigcup_{f \in Z^Y} Im\, F f ,$$

where Z^Y is the set of all mappings from Y to Z.

DEFINITION [4]. A cardinal $\alpha > 1$ is an *unattainable cardinal* of a functor F if $F\alpha - F^\alpha \alpha \neq \emptyset$. Then put

$$F_\alpha X = F^{\alpha^+} X - F^\alpha X.$$

The class of all unattainable cardinals of F is denoted by A_F.

THEOREM 1.2. *Let X be an infinite set such that there exists $\alpha \in A_F$, with $\alpha \leqslant card X$. Then $card\, F X \geqslant card\, 2^X$.*

PROOF. See [4].

DEFINITION. Let $f, g: X \to Y$ be mappings onto. Then f, g are *diverse* if there exists $Z \subset X$ such that either

$$f(Z) = Y \quad \text{and} \quad card\, g(Z) < card\, Y$$

or

$$g(Z) = Y \quad \text{and} \quad card\, f(Z) < card\, Y .$$

A system $\mathfrak{A}$ of mappings from X to Y is called *diverse* if arbitrary distinct mappings $f, g \in \mathfrak{A}$ are diverse.

PROPOSITION 1.3. *If* α *is an unattainable cardinal of a functor* F *and if* $f, g: X \to \alpha$ *are diverse, then*

$$Ff(F_\alpha \alpha) \cap Fg(F_\alpha \alpha) = \emptyset .$$

PROOF. See [4].

LEMMA 1.4. *Let* X *be an infinite set. Then for every infinite cardinal* α *with* $\alpha \leqslant card\, X$ *there exists a diverse system* $\mathfrak{A}$ *of mappings from* X *to* α *such that* $card\, \mathfrak{A} = card\, 2^X$.

PROOF. See [4].

DEFINITION. We say that $f: X \to Y$ *is coarser than* $g: X \to Z$ if there exists $h: Z \to Y$ such that $h \circ g = f$.

PROPOSITION 1.5. *If* $f: X \to Y$ *then* $Im\, F f = \cup Im\, F g$ *where the union is taken over all* $g: X \to \alpha$ *coarser than* f *and* $\alpha \in A_F$.

PROOF. See [7].

DEFINITION. Let F be a functor, $x \in FX$. Define

$$\mathcal{F}_F^X(x) = \{ e \mid e \text{ is a decomposition of } X ,\ x \in Im\, F e \} .$$

Further we shall write

$$\| \mathcal{F}_F^X(x) \| = min \{ card\, Im\, e \mid e \in \mathcal{F}_F^X(x) \} .$$

PROPOSITION 1.6. *Let* F *be a functor; then* $\alpha \in A_F$ *iff there exists* $x \in FX$ *such that* $\| \mathcal{F}_F^X(x) \| = \alpha$ *for* $card\, X \geqslant \alpha$. *Further* $y \in F_\alpha Y$ *iff* $\| \mathcal{F}_F^Y(y) \| = \alpha$.

PROOF. Clearly $x \notin F^\alpha \alpha$. On the other hand $x \in Im\, F f$, where $f: X \to Y$ is onto and $card\, Y = \alpha$; therefore $x \in F_\alpha X$ and $\alpha \in A_F$. The rest is evident.

COROLLARY 1.7. *If* $\| \mathcal{F}_F^X(x) \|$ *is finite, then there exists* e *with*

$$\mathcal{F}_F^X(x) = \{ e' \mid e \text{ is coarser than } e' \} .$$

PROOF. If $e \neq e'$ and

$$card\, Im\, e' = card\, Im\, e < \aleph_0 ,$$

then e and e' are diverse and by Proposition 1.3 we get Corollary 1.7.

PROPOSITION 1.8. *Let F be a functor, $f: X \to Y$. Then for every $y \in FY$ it holds*

$$\mathcal{F}_F^X(Ff(y)) \supset \{ e' \mid \text{there exists } e \in \mathcal{F}_F^Y(y),\ e \circ f \text{ is coarser than } e' \}.$$

PROOF is easy.

PROPOSITION 1.9. *Let F be a functor, $y \in FY$. If for some $\bar{e} \in \mathcal{F}_F^Y(y)$ and for some $f: X \to Y$, $\bar{e} \circ f$ is onto, then*

$$\mathcal{F}_F^X(Ff(y)) = \{ e' \mid \text{there exists } e \in \mathcal{F}_F^Y(Y),\ e \circ f \text{ is coarser than } e' \}.$$

PROOF. There exists a mapping h such that $\bar{e} \circ f \circ h = id$, then

$$F\bar{e} \circ Fh \circ Ff(y) = F\bar{e} \circ Fh \circ Ff \circ F\bar{e}(z) = F\bar{e}(z) = y ,$$

where $z \in F(Y/\bar{e})$ with $F\bar{e}(z) = y$. Now by Proposition 1.8 we get Proposition 1.9.

DEFINITION. Let F be a functor. For $x \in FX$ denote by e_x the finest decomposition which is coarser than each $e \in \mathcal{F}_F^X(x)$.

NOTE. If α is a finite cardinal and $x \in F_\alpha X$, then $e_x \in \mathcal{F}_F^X(x)$.

COROLLARY 1.10. *Let F be a functor, α a finite cardinal. If, for some $f: X \to Y$ and for some $y \in F_\alpha Y$ we have $Ff(y) \in F_\alpha X$, then*

$$e_{Ff(y)} = Ker(e_y \circ f).$$

PROOF is easy.

II

LEMMA 2.1. *The object $(6, V)$ is a rigid object of $S(P^-)$, where*

$$\begin{aligned} V = \{ &\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \\ &\{1,2\}, \{1,3\}, \{1,5\}, \{2,4\}, \{2,3,4,5\}, \{1,3,4,5\}, \\ &\{1,2,4,5\}, \{1,2,3,5\}, \{0,3,4,5\}, \{0,2,4,5\}, \{0,2,3,4\}, \\ &\{0,1,3,5\}, \{1,2,3,4,5\}, \{0,2,3,4,5\}, \{0,1,3,4,5\}, \\ &\{0,1,2,4,5\}, \{0,1,2,3,5\}, \{0,1,2,3,4\} \}. \end{aligned}$$

PROOF. Since $\emptyset \notin V$ and for every $i \in 6$, $\{i\} \notin V$ we get, if $f:(6,V) \to (6,V)$ is a morphism of $S(P^-)$, then f is a bijection. Therefore for every $\{i,j\} \in V$ we have

$$f^{-1}(\{i,j\}) = \{f^{-1}(i), f^{-1}(j)\} \in V.$$

Hence

$$card\{\{i,j\} \in V \mid j \in 6-\{i\}\} \leq card\{\{f^{-1}(i),j\} \in V \mid j \in 6-\{f^{-1}(i)\}\}$$

for every $i \in 6$ and thus we get for every $i \in 6$,

$$card\{\{i,j\} \in V \mid j \in 6-\{i\}\} = card\{\{f^{-1}(i),j\} \in V \mid j \in 6-\{f^{-1}(i)\}\}.$$

Hence

$$f^{-1}(5) = 5, \quad f^{-1}(2) = 2.$$

Now it is easy to verify that $f = id_6$.

CONVENTION. An object of $S(F)$ will be called an *F-space.*

DEFINITION. Denote by $E(P^-)$ the full subcategory of $S(P^-)$ over those (X,W) for which $Z \in W$ implies $X-Z \in W$ and $Z \neq \emptyset$.

PROPOSITION 2.2. *There exists a strong embedding of* $S(P^-)$ *into* $E(P^-)$.

PROOF. Let (X,W) be a P^--space. Define $\Psi(X,W) = (X \vee 6, W_S)$, where:

$$W_S = \{Z, X\vee 6 - Z \mid Z \in V, card Z < 3\} \cup$$
$$\cup \{\{0,1,2\} \cup Z, \{3,4,5\} \cup X-Z \mid Z \in W\};$$

for a given $f:(X_1,W_1) \to (X_2,W_2)$, $\Psi f = f \vee id_6$. Clearly Ψ is an embedding. We shall prove that it is also full. Let $f:\Psi(X_1,W_1) \to \Psi(X_2,W_2)$. First we prove $f(X_1) \subset X_2$. Assume the contrary, i.e. $f(x) = i$ for some $x \in X_1$ and $i \in 6$. Then $f^{-1}(\{i\}) \in (W_1)_S$ and therefore either

$$f(\{0,1,2\}) = \{i\}, \quad \text{or } f(\{3,4,5\}) = \{i\},$$
$$\text{or } f((X_1 \vee 6)-Z) = \{i\} \text{ for some } Z \in V, \ card Z < 3.$$

Since $\emptyset \notin (W_1)_S$ and $\{i\} \in (W_2)_S$ for every $i \in 6$, we have $6 \subset Im f$ and therefore the last case is impossible. Further there exists

$$j \in 6 \quad \text{such that } \{i,j\} \in (W_2)_S$$

and so $f^{-1}(\{i,j\}) \in (W_1)_S$. Hence either $j \in f(X_1)$ or $f(X_1) = \{i\}$. In the former case we get again either

$$f(\{0,1,2\}) = \{j\} \quad \text{or} \quad f(\{3,4,5\}) = \{j\}$$

(we use the fact that $\{j\} \in (W_2)_S$) and so $6 \subset f^{-1}(\{i,j\})$; this is a contradiction. In the latter case

$$f^{-1}(\{i,j\}) = (X_1 \vee 6) - Z \quad \text{for some } Z \in V,\ card Z < 3$$

and therefore $6 \not\subset Im f$ and it is again a contradiction. Hence

$$f(X_1) \subset X_2 \quad \text{and} \quad f(6) = 6.$$

By Lemma 2.1 we have $f/6 = id_6$ and $f/X_1 : (X_1, W_1) \to (X_2, W_2)$ is a morphism of $S(P^-)$. Thus Ψ is a strong embedding.

PROPOSITION 2.3. *There exists a full subcategory* $\mathfrak{M}$ *of* $S(P^-)$ *such that:*

1° if $(X,W) \in \mathfrak{M}$ *then* $\emptyset \notin W$, $\emptyset \neq W$ *and for every* $x \in X$ *there exists* $Z \in W$ *with* $x \in Z$ *;*

2° if $f, g : (X_1, W_1) \to (X_2, W_2)$ *and* $(X_1, W_1), (X_2, W_2) \in \mathfrak{M}$, *then there exists* $Z \in W_2$ *with* $f^{-1}(Z) \neq g^{-1}(Z)$ *;*

3° there exists a strong embedding from $S(P^-)$ *to* $\mathfrak{M}$.

PROOF. Define $\Phi : S(P^-) \to S(P^-)$ as follows : $\Phi(X,W) = (X \vee 6, W_D)$ with

$$W_D = V \cup \{\{0,1,2\} \cup Z \mid Z \in W\} \cup \{\{3,4,5\} \cup Z \mid Z \subset X\} ;$$

for a given $f : (X_1, W_1) \to (X_2, W_2)$ put $\Phi f = f \vee id_6$. Evidently Φ is an embedding. Now, we shall prove that, if

$$f : (X_1 \vee 6, (W_1)_6) \to (X_2 \vee 6, (W_2)_D),$$

then $f(X_1) \subset X_2$, $f(6) \subset 6$. For every $i \in 6$,

$$\{i\} \in (W_2)_D \quad \text{and} \quad \emptyset \notin (W_1)_D ,$$

therefore $6 \subset Im f$. We assume that for some $x \in X_1$, $f(x) = i \in 6$. Then we have $f^{-1}(\{i\}) \in (W_1)_D$ and hence either

$$f(\{0,1,2\}) = \{i\} \quad \text{or} \quad f(\{3,4,5\}) = \{i\}.$$

Further there exists $j \in 6$ such that $\{i,j\} \in (W_2)_D$, and therefore

$$f^{-1}(\{i,j\}) \in (W_1)_D .$$

We get that $f^{-1}(\{j\}) \cap X_1 \neq \emptyset$ but then either

$$f(\{0,1,2\}) = \{j\} \quad \text{or} \quad f(\{3,4,5\}) = \{j\}$$

and hence $6 \subset f^{-1}(\{i,j\})$ - a contradiction. Thus

$$f(X_1) \subset X_2, \quad f(6) \subset 6.$$

By Lemma 2.1, $f/6 = id_6$ and therefore $f/X_1 : (X_1, W_1) \to (X_2, W_2)$ is a morphism of $S(P^-)$. Put $\mathfrak{M} = \Phi(S(P^-))$. Evidently $\Phi : S(P^-) \to \mathfrak{M}$ is a strong embedding and $\mathfrak{M}$ has the required properties.

NOTE. The set functor carrying Φ (or Ψ) is $I \vee C_6$ where I is the identity functor and C_6 is the constant functor to 6.

COROLLARY 2.4. *There exists a full subcategory* $\mathfrak{G}$ *of* $E(P^-)$ *such that:*

1° *if* $(X, W) \in \mathfrak{G}$, *then* $W \neq \emptyset$;

2° *if* $f, g : (X, W) \to (Y, S)$ *and* $(X, W), (Y, S) \in \mathfrak{G}$, *then there exists* $Z \in S$ *with* $f^{-1}(Z) \neq g^{-1}(Z)$;

3° *there exists a strong embedding from* $S(P^-)$ *to* $\mathfrak{G}$.

PROOF follows from Propositions 2.2 and 2.3.

THEOREM 2.5. *If* $2 \in \mathrm{A}_F$, *then there exists a strong embedding from* $S(P^-)$ *to* $S(F)$.

PROOF. Via Proposition 2.2 it suffices to prove that there exists a strong embedding from $E(P^-)$ to $S(F)$. Define

$$\Omega(X, W) = (X, W_F),$$

where

$$W_F = \{ x \in F_2 X \mid \text{there exists } Z \in W,\ Z \in e_x \};$$

for a given $f : (X, W) \to (Y, S)$, define $\Omega f = f$. Clearly Ω is an embedding; let us prove that it is full. Let $f : (X, W_F) \to (Y, S_F)$ be a morphism of $S(F)$. Then for every $x \in S_F$ it holds:

$$\text{there exist } Z_1, Z_2 \in S \text{ such that } \{Z_1, Z_2\} = e_x$$

(see Corollary 1.7 and the definition of Ω). On the other hand for every Z in S there exists

$$x \in S_F \text{ such that } \{Z, Y - Z\} = e_x.$$

Now by Corollary 1.10, we get that

$$\{f^{-1}(Z), X - f^{-1}(Z)\} = e_{F f(x)}.$$

Thus $f^{-1}(Z)\in W$ and $f:(X,W)\to(Y,S)$ is a morphism of $S(P^-)$.

III

CONSTRUCTION 3.1. Let F be a functor such that $\alpha\in A_F$, where $\alpha>1$ is a finite cardinal. Then there exists an object (X,V) of $S(F)$ such that:

1° $card\,X=\alpha+4$;

2° $V\subset F_\alpha X$;

3° for every $x\in X$ there exist $y_1,y_2\in V$ such that $\{x\}\in e_{y_1}$, $\{x\}\notin e_{y_2}$;

4° if $x\in V$, then $F(e_x)(F_\alpha\alpha)\subset V$;

5° for $x\in X$ denote by

$$gr\,x = card\{Z \mid card\,Z>1,\ x\in Z, \text{ there exists } y\in V,\ Z\in e_y\},$$

then $gr\,x>1$ with at most one exception ;

6° (X,V) is rigid ;

7° if $\alpha\geqslant 3$ then there exists $x\in X$ such that $gr\,x=1$ and if for some $Z\subset X$, $x\in Z$, $gr\,x=1$ and $Z\in e_y$ for some $y\in V$ then $card\,Z<3$.

We shall construct these objects by induction in α. For $\alpha=2$, the object exists by Lemma 2.1 and Theorem 2.5.

We assume that for $\alpha<n$ the construction is performed and $n\in A_F$. Let G be a functor with $n-1\in A_G$. Let (X',V') be a G-space fulfilling the conditions 1-7 for $n-1$. We assume that $a\notin X'$ and put $X=X'\cup\{a\}$. We choose an arbitrary decomposition $\bar{e}$ of X in n classes such that

$$card\,\bar{e}(a)=2 \text{ and if } gr\,x=1 \text{ for some } x\in X', \text{ then } x\in e(a).$$

Put

$$V=\bigcup Fe(F_\alpha X/e)\cup F\bar{e}(F_\alpha X/\bar{e})$$

where the union is taken over all e such that $\{a\}\in e$ and the restriction of e to X' is equal to e_x for some $x\in V'$. Let $f:(X,V)\to(X,V)$ be a morphism of $S(F)$; then f is a bijection by Corollary 1.10 and Condition 3 for (X',V'). Further $gr\,a=1$ and for $x\in X-\{a\}$, $gr\,x>1$. Hence $f(a)=a$. Clearly $f/X':(X',V')\to(X',V')$ is a morphism of $S(G)$ and hence $f=id_X$. The other required properties are easy to verify.

PROPOSITION 3.2. *If* $\alpha > 2$ *is a finite cardinal and* $\alpha \in A_F$, *then there exists a F-space* (X, V) *and* $x_0 \in X$ *such that the following conditions hold:*

1° (X, V) *is rigid,* $V \subset F_\alpha X$;

2° *for every* $y_0, y_1 \in V$, $card\, e_{y_i}(E_{1-i}) \leqslant \alpha - 1$ *for* $i = 0, 1$, *where* E_i *is the class of* e_{y_i} *containing* x_0 ;

3° *for every* $x \in X$ *there exist* $y_1, y_2 \in V$ *such that* $\{x\} \in e_{y_1}$, $\{x\} \notin e_{y_2}$.

PROOF. Since $\alpha > 2$ we can choose by Construction 3.1 the F-space (X, V) fulfilling Conditions 1-7. Therefore there exists $x \in X$ with $grx = 1$. Put $x = x_0$. Clearly $((X, V), x_0)$ fulfills Conditions 1-3 from Proposition 3.2.

LEMMA 3.3. *Let* α *be an infinite cardinal. Then for every set* X *such that* $card\, X = \alpha$ *and every subsets* X_1, X_2, X_3 *of* X *such that*

$$X_1 \cap X_2 = X_2 \cap X_3 = \emptyset, \quad card\, X_1 = card\, X_2 = \alpha$$

and every mapping $f: X_2 \to X_3$ *onto, there exists a diverse system* $\mathfrak{A}$ *of mappings* $g: X \to \alpha$ *such that* $card\, \mathfrak{A} = card\, 2^X$ *and every* $g \in \mathfrak{A}$ *fulfills:*

1° *for every* $i \in \alpha$, $g^{-1}(\{i\}) \cap X_j \neq \emptyset$ *for* $j = 1, 2$;

2° *there exists no non-constant mapping* h *coarser than* g *with*

$$h(x) = h(f(x)) \text{ for every } x \in X_2 .$$

PROOF. If there exists $Z \subset X_3$ such that

$$card\, Z < \alpha \quad \text{and} \quad card\, f^{-1}(Z) = \alpha ,$$

then put $Y = X_1 - Z$. By Lemma 1.4 there exists a diverse system $\mathfrak{B}$ of mappings from Y to α with $card\, \mathfrak{B} = card\, 2^X$. Now, for every $h \in \mathfrak{B}$ we choose $g_h: X \to \alpha$ such that $g_h / Y = h$, $card\, g_h(Z) = 1 = card\, g_h(X_2 - f^{-1}(Z))$ and,

$$\text{for } i \in \alpha, \quad g_h^{-1}(\{i\}) \cap f^{-1}(Z) \neq \emptyset .$$

If there exists no $Z \subset X_3$ with this property, then we choose a decomposition $\{Z_1, Z_2\}$ of X_2 such that

$$card\, Z_1 = card\, Z_2 = card(X_1 - f(Z_1)) = \alpha .$$

By Lemma 1.4 there exists a diverse system $\mathfrak{B}$ of mappings from Z_1 to α with $card\, \mathfrak{B} = card\, 2^X$. Now, for every $h \in \mathfrak{B}$ we choose $g_h: X \to \alpha$ such that

$$g_h / Z_1 = h, \quad card\, g_h(Z_2) = card\, g_h(f(Z_1)) = 1$$

and for every $i \in \alpha$,

$$g_h^{-1}(\{i\}) \cap f(Z_2) \neq \emptyset, \quad g_h^{-1}(\{i\}) \cap X_1 \neq \emptyset .$$

Then $\mathfrak{A} = \{ g_h \mid h \in \mathfrak{B} \}$ has the required properties.

CONDITION A. An F-space (X, V) fulfills the condition A if for arbitrary subsets X_1, X_2, X_3 of X such that

$$card\, X_1 = card\, X_2 = card\, X \quad \text{and} \quad X_1 \cap X_2 = X_2 \cap X_3 = \emptyset$$

and for arbitrary mapping $f: X_2 \to X_3$ onto there exists $y \in V$ such that

a) for every $e' \in \mathcal{F}_F^X(y)$ there exists $e \in \mathcal{F}_F^X(y)$ coarser than e', such that $e(x) \cap X_i \neq \emptyset$ for every $x \in X$ and $i = 1, 2$;

b) there exists $e \in \mathcal{F}_F^X(y)$ such that for every $e' \in \mathcal{F}_F^X(y)$ we have

1° $e' \cap^* e \in \mathcal{F}_F^X(y)$ and

2° a mapping h from X is constant whenever

$$h(x) = h(f(x)) \quad \text{for every} \quad x \in X_2$$

and h is coarser than e ($e' \cap^* e$ denotes a co-intersection of e' and e).

PROPOSITION 3.4. *Let $\alpha \in A_F$ be an infinite cardinal such that there exists $x \in F_a \alpha$ with non-trivial e_x. Then there exists an F-space (X, V) and a x_0 of X such that:*

a) (X, V) is rigid;

b) $card\, X = \alpha$, $V \subset F_a X$;

c) for every $a \in X$ there exists $y_a \in V$ such that $e_{y_a}(a) \neq e_{y_a}(x_0)$;

d) (X, V) fulfills condition A.

PROOF. We choose a set X with $card\, X = \alpha$ and choose $x_0 \in X$. For every a we choose a bijection $f_a : X \to \alpha$ such that $e_x(f_a(a)) \neq e_x(f_a(x_0))$. Then

$$e_{F f_a(x)}(a) \neq e_{F f_a(x)}(x_0) .$$

Put

$$\mathfrak{B}_0 = \{ F f_a(x) \mid a \in X - \{ x_0 \} \} .$$

Now, we choose bijections

$$\Psi_1 : card\, 2^X \to \{ f: X \to X \mid f \neq id_X \},$$

$$\Psi_2 : card\, 2^X \to \{ (X_1, X_2, X_3, f) \mid card\, X_1 = card\, X_2 = \alpha, \quad X_1 \cap X_2 = X_2 \cap X_3 = \emptyset,\ f: X_2 \to X_3 \text{ is onto} \}.$$

For $i \in card\, 2^X$ denote

$$C_i = \{ y \in F_\alpha X \mid F(\Psi_1(i))(y) \neq y \}.$$

As an application of Lemma 1.4 we get that $card\, C_i = card\, 2^X$. Further for $\Psi_2(i) = (X_1, X_2, X_3, f)$ denote

$D_i = \{ y \in F_\alpha X \mid$ there exists $e \in \mathcal{F}^X_F(y)$ with
1° for every $x \in X$, $e(x) \cap X_j \neq \emptyset$ for $j = 1, 2$,
2° for every $e' \in \mathcal{F}^X_F(y)$, $e' \cap^* e \in \mathcal{F}^X_F(y)$,
3° there exists no non-constant mapping from X coarser than e, with $h(x) = h(f(x))$ for every $x \in X_2 \}$.

If we construct the system $\mathcal{A}$ from Lemma 3.3 for $(X_1, X_2, X_3, f) = \Psi_2(i)$, then for every $g \in \mathcal{A}$ we have $Fg(x) \in D_i$ and therefore $card\, D_i = card\, 2^X$. Now we shall construct, by induction on $i \in card\, 2^X$, sets $\mathcal{B}_i$, $\mathcal{C}_i$ such that:

$$card\, \mathcal{C}_i < card\, 2^X, \quad \mathcal{B}_i \subset \mathcal{C}_i \cap F_\alpha X \text{ for every } i .$$

Put $\mathcal{C}_0 = \mathcal{B}_0$. We assume that we have the sets $\mathcal{B}_i$, $\mathcal{C}_i$ for $i < j$. If j is a limit ordinal, put

$$\mathcal{B}_j = \bigcup_{i<j} \mathcal{B}_i, \quad \mathcal{C}_j = \bigcup_{i<j} \mathcal{C}_i.$$

If $j = k + 1$ then

a) we choose $x^1_k \in C_k - \mathcal{C}_k$ such that $F(\Psi_1(k))(x^1_k) \notin \mathcal{C}_k$,

b) we choose $x^2_k \in D_k - (\mathcal{C}_k \cup \{ x^1_k, F(\Psi_1(k))(x^1_k)\})$.

Put

$$\mathcal{B}_j = \mathcal{B}_k \cup \{ x^1_k, x^2_k \}, \quad \mathcal{C}_j = \mathcal{C}_k \cup \{ x^1_k, F(\Psi_1(k))(x^1_k), x^2_k \}.$$

Evidently

$$card\, \mathcal{C}_j < card\, 2^X \text{ and } \mathcal{B}_j \subset \mathcal{C}_j \cap F_\alpha X.$$

Put $V = \cup \mathcal{B}_j$ where the union is taken over all $j \in card 2^X$. The F-space (X, V) has the required properties.

LEMMA 3.5. *Let (X, V) be a rigid F-space. If $g_1 : Y_1 \to X$, $g_2 : Y_2 \to X$ are onto, then every mapping $h : Y_1 \to Y_2$ such that $Fh(Fg_2(V)) \subset Fg_1(V)$ fulfills $g_2 \circ h = g_1$.*

PROOF. Assume the contrary, i.e. $g_2 \circ h \neq g_1$. Then there exists $f : X \to Y_1$ such that

$$g_1 \circ f = id_X \quad \text{but} \quad g_2 \circ h \circ f \neq id_X .$$

Further it is clear to verify that $g_2 \circ h \circ f$ is an F-morphism of (X, V) - a contradiction.

CONSTRUCTION 3.6. Let $\mathfrak{O} = ((X, V), x_0)$ be a couple where (X, V) is an F-space, $card X > 1$ and $x_0 \in X$. For every set Y and every $Z \subset Y$ define $g_Z : U \to X$ where $U = (Y \times (X - \{x_0\})) \vee \{x_0\}$ as follows

$$g_Z(x_0) = x_0 , \quad g_Z(y, x) = x_0 \quad \text{if } y \in Y - Z,\ x \in X - \{x_0\},$$

$$g_Z(y, x) = x \quad \text{if } y \in Z,\ x \in X - \{x_0\} .$$

Define a functor $\Sigma_{\mathfrak{O}} : S(P^-) \to S(F)$:

$$\Sigma_{\mathfrak{O}}(Y, W) = (U, \bigcup_{Z \in W} F g_Z(V)),$$

and for $f : (Y_1, W_1) \to (Y_2, W_2)$ put

$$\Sigma_{\mathfrak{O}} f = (f \times id_{X - \{x_0\}}) \vee id_{\{x_0\}} .$$

Clearly $\Sigma_{\mathfrak{O}}$ is faithful and if $\Sigma_{\mathfrak{O}}$ is full, then $\Sigma_{\mathfrak{O}}$ is a strong embedding.

NOTE. If Z_1, Z_2 are distinct subsets of Y, then g_{Z_1} and g_{Z_2} are diverse.

LEMMA 3.7. *Let $\mathfrak{O} = ((X, V), x_0)$ and $Z \subset Y$. If $y \in V$ fulfills:*

let $e \in \mathfrak{F}_F^X(y)$ such that for every $e' \in \mathfrak{F}_F^X(y)$, $e' \cap^* e \in \mathfrak{F}_F^X(y)$,

then $F g_Z(y)$ fulfills :

for every $\bar{e} \in \mathfrak{F}_F^U(F g_Z(y))$, $\bar{e} \cap^* Ker(e \circ g_Z) \in \mathfrak{F}_F^U(F g_Z(y))$.

PROOF follows from Proposition 1.9.

LEMMA 3.8. *Let n be a finite unattainable cardinal of F. Let an F-space (X, V) and $x_0 \in X$ fulfill the conditions 1 - 3 from Proposition 3.2. If*

$$g : \Sigma_{\mathfrak{O}}(Y_1, W_1) \to \Sigma_{\mathfrak{O}}(Y_2, W_2)$$

is an $S(F)$*-morphism and* $\emptyset \notin W_1$, *then for every* $Z_2 \in W_2$, $Z_2 \neq \emptyset$, *there exists* $Z_1 \in W_1$ *such that* $F(g_{Z_2} \circ g)(V) \subset F g_{Z_1}(V)$.

PROOF. Assume the contrary, i. e. there exist $y_0, y_1 \in F g_{Z_2}(V)$ such that:

$$F g(y_0) \in F g_{Z_0}(V) \text{ and } F g(y_1) \in F g_{Z_1}(V) \text{ where } Z_0 \neq Z_1.$$

We can assume that there exists $v \in Z_0 - Z_1$. Put $F g(y_i) = z_i$ for $i = 0, 1$. Then

$$card\, g_{Z_0}(\{v\} \times (X - \{x_0\})) > n-1 \text{ and } \{v\} \times (X - \{x_0\}) \subset g_{Z_1}(x_0).$$

By Corollary 1.10 we get that

$$card\, e_{t_0}(g_{Z_2} \circ (\{v\} \times (X - \{x_0\}))) > n-1$$

and

$$e_{t_1}(x_0) \supset g_{Z_2} \circ g(\{v\} \times (X - \{x_0\}))$$

where $t_0, t_1 \in V$ such that $F g_{Z_2}(t_i) = y_i$ for $i = 0, 1$; but this contradicts the Condition 2 from Proposition 3.2.

LEMMA 3.9. *Let* α *be an infinite unattainable cardinal of* F. *Let* (X, V) *and* $x_0 \in X$ *fulfill the conditions a- d from Proposition 3.4. If*

$$g : \Sigma_{\mho}(Y_1, W_1) \to \Sigma_{\mho}(Y_2, W_2)$$

is an $S(F)$*-morphism and* $\emptyset \notin W_1$, *then for every* $Z_2 \in W_2$, $Z_2 \neq \emptyset$ *there exists* $Z_1 \in W_1$ *such that* $F(g_{Z_2} \circ g)(V) \subset F g_{Z_1}(V)$.

PROOF. Assume the contrary, i. e. there exist $y_0, y_1 \in F g_{Z_2}(V)$ such that:

$$F g(y_i) \in F g_{Z_i}(V) \text{ for } i = 0, 1,$$

where $Z_0 \neq Z_1$. We can assume that $v \in Z_0 - Z_1$ and $w \in Z_1$. Put

$$U_i = (Y_i \times (X - \{x_0\})) \vee \{x_0\} \text{ for } i = 1, 2.$$

By Proposition 1.8 and Lemma 3.7 there exists $e_i \in \mathcal{F}_F^{U_1}(F g(y_i))$ such that e_i is coarser than

$$Ker\, g_{Z_2} \circ g \cap^* Ker\, g_{Z_i} \text{ for } i = 0, 1.$$

Therefore we get that

$$card(g_{Z_2}\circ g(\{v\}\times(X-\{x_0\}))-g_{Z_2}\circ g(\{w\}\times(X-\{x_0\})))=$$
$$=card(g_{Z_2}\circ g(\{w\}\times(X-\{x_0\}))-g_{Z_2}\circ g(\{v\}\times(X-\{x_0\})))=\alpha\ .$$

Put

$$X_1=g_{Z_2}\circ g(\{v\}\times(X-\{x_0\}))-g_{Z_2}\circ g(\{w\}\times(X-\{x_0\}))$$
$$X_2=g_{Z_2}\circ g(\{w\}\times(X-\{x_0\}))-g_{Z_2}\circ g(\{v\}\times(X-\{x_0\})),$$
$$X_3=\{\,g_{Z_2}\circ g((v,x))\mid g_{Z_2}\circ g((w,x))\in X_2\},$$
$$f(g_{Z_2}\circ g((w,x)))=g_{Z_2}\circ g((v,x)).$$

Since $v\notin Z_1$ we have $X_3\cap X_2=\emptyset$. Clearly $X_1\cap X_2=\emptyset$ and f is onto. Therefore there exists $t\in V$ from Condition A for (X_1,X_2,X_3,f). Denote:

$$y_3=Fg_{Z_2}(t),\quad z_3=Fg(y_3),\quad z_3\in Fg_{Z_3}(V).$$

By a, Condition A we have $v,w\in Z_3$. Further there is $e_1\in\mathfrak{F}_F^{U_1}(z_3)$ coarser than $Ker\,g_{Z_3}$ and $Ker(e_0\circ g_{Z_2}\circ g)$, where $e_0\in\mathfrak{F}_F^X(t)$ from b of Condition A. Hence there exists a mapping p such that $e_1=p\circ e_0\circ g_{Z_2}\circ g$. Since e_1 is coarser than $Ker\,g_{Z_3}$ we get that

$$p\circ e_0(x)=p\circ e_0(f(x))\quad\text{for every}\quad x\in X_2$$

- and thus $p\circ e_0$ is constant and so is e_1. This contradicts

$$z_3\in Fg_{Z_3}(V)\subset Fg_{Z_3}(F_\alpha X)\subset F_\alpha U_1.$$

THEOREM 3.10. *Let $\alpha>2$ be an unattainable cardinal of F. Then there exists a strong embedding from $S(P^-)$ to $S(F)$ whenever there exists $x\in F_\alpha X$ such that e_x is non-trivial.*

PROOF. Let (X,V) be an F-space and $x_0\in X$ fulfilling the conditions 1-3 from Proposition 3.2 if α is finite, or the conditions a-d from Proposition 3.4 if α is infinite. We shall restrict the functor $\Sigma_{\mathcal{O}}$ to the category $\mathfrak{M}$, where $\mathcal{O}=((X,V),x_0)$. By Lemmas 3.8 and 3.9, if

$$g:\Sigma_{\mathcal{O}}(Y_1,W_1)\to\Sigma_{\mathcal{O}}(Y_2,W_2)$$

is an $S(F)$-morphism, then for every $Z_2\in W_2$ there exists

$$Z_1\in W_1\quad\text{such that}\quad F(g_{Z_2}\circ g)(V)\subset Fg_{Z_1}(V)$$

and then by Lemma 3.5 $g_{Z_2} \circ g = g_{Z_1}$. Since W_2 is a cover of Y_2, we get that

$$g(Y_1 \times \{a\}) \subset Y_3 \times \{a\} \text{ for every } a \in X - \{x_0\} \text{ and } g(x_0) = x_0 .$$

For every $a \in X - \{x_0\}$, define $g_a : Y_1 \to Y_2$ as follows :

$$g_a(y_1) = y_2 \text{ iff } g((y_1, a)) = (y_2, a).$$

Then $g_a : (Y_1, W_1) \to (Y_2, W_2)$ is an $S(P^-)$-morphism for every $a \in X - \{x_0\}$. Further for every $a, b \in X - \{x_0\}$ and every $Z \in W$,

$$g_a^{-1}(Z) = g_b^{-1}(Z).$$

Properties of $\mathfrak{M}$ imply that $g_a = g_b$ for every $a, b \in X - \{x_0\}$, thus $\Sigma_{\mathfrak{O}}$ is a strong embedding from $\mathfrak{M}$ to $S(F)$. By Proposition 2.3, we obtain the Theorem.

IV

DEFINITION [9]. We say that a colimit of a diagram $D : \mathfrak{D} \to \mathcal{K}$ *is absolute* if every covariant functor $F : \mathcal{K} \to \mathfrak{L}$ preserves it.

LEMMA 4.1. *Let*

$$f_i : A \to B_i, \quad g_i : B_i \to C, \quad i = 1, 2,$$

be morphisms of the category $\mathcal{K}$ *and let*

$$h_1 : B_1 \to A, \quad h_2 : C \to B_2$$

be morphisms of $\mathcal{K}$ *such that*

$$g_1 \circ f_1 = g_2 \circ f_2, \quad f_2 \circ h_1 = h_2 \circ g_1, \quad f_1 \circ h_1 = id_{B_1}, \quad g_2 \circ h_2 = id_C .$$

Then the push-out of $f_i : A \to B_i$, $i = 1, 2$, *is absolute.*

PROOF. See [10].

LEMMA 4.2. *Let* $f : X \to Y$, $g : X \to Z$ *be mappings onto such that there exists exactly one* $z \in Z$ *with* $\operatorname{card} g^{-1}(z) > 1$. *Then the push-out of* f, g *is absolute.*

PROOF. Let $h_1 : Y \to V$, $h_2 : Z \to V$ be this push-out. Choose $k_1 : Y \to X$

such that $f \circ k_1 = id_Y$ and

$$k_1(y) \in g^{-1}(\{g(z)\}) \quad \text{whenever} \quad f^{-1}(\{y\}) \cap g^{-1}(\{g(z)\}) \neq \emptyset .$$

Further we choose $k_2 : V \to Z$ such that

$$h_2 \circ k_2 = id_V \quad \text{and} \quad k_2 \circ h_2 \circ g(z) = g(z)$$

and

$$k_2(v) = g \circ k_1(h_1^{-1}(v)) \quad \text{for} \quad v \in V - \{h_2 g(z)\} .$$

It is easy to verify that the definition of k_2 is correct and $g \circ k_1 = k_2 \circ h_1$. Now, Lemma 4.2 follows from Lemma 4.1.

DEFINITION. A decomposition e is called *finite* if every class of e is finite and e has only a finite number of non-singleton classes.

COROLLARY 4.3. *Let F be a functor, $x \in FX$. If $e_x = \{X\}$, then every finite decomposition is an element of $\mathcal{F}_F^X(x)$.*

PROOF. If e is a finite decomposition, then e is a co-intersection of decompositions e_i, $i = 1, 2, \dots . n$ such that every e_i has only one non-singleton class. If $e_i \in \mathcal{F}_F^X(x)$ then by induction we get from Lemma 4.2 that $e \in \mathcal{F}_F^X(x)$. Further every decomposition e_i is a co-intersection of

$$e_i^j , \quad j = 1, 2, \dots , m ,$$

where every decomposition e_i^j has only one non-singleton class and every class of e_i^j has at most two points. Now, by induction we get from Lemma 4.2 that

$$e_i \in \mathcal{F}_F^X(x) \quad \text{whenever every} \quad e_i^j \in \mathcal{F}_F^X(x).$$

Since $e_x = \{X\}$, it is easy to verify by Lemma 4.2 that every $e_i^j \in \mathcal{F}_F^X(x)$.

We recall the definition of the union and the co-union.

DEFINITION. Let $f : Y \to X$, $g : Z \to X$ be monomorphisms. The monomorphism $h : V \to X$ is called a *union of* f, g (we shall write $f \cup g = h$) if there exist

$$f_1 : Y \to V, \ g_1 : Z \to V \quad \text{such that} \quad h \circ f_1 = f, \ h \circ g_1 = g$$

and for every $h' : V' \to X$ for which there exist

$$f_2 : Y \to V', \ g_2 : Z \to V' \quad \text{such that} \quad h' \circ f_2 = f, \ h' \circ g_2 = g$$

there exists

$$h_1 : V \to V' \quad \text{with} \quad h = h' \circ h_1 .$$

The dual notion is a *co-union* (we shall write $h = f \cup^* g$ if h is a co-union of f, g).

The covariant set functor F *preserves finite unions* if for arbitrary one-to-one mappings $f: Y \to X$, $g: Z \to X$ we have

$$Ff \cup Fg = F(f \cup g).$$

F *preserves unions with a finite set* if for arbitrary one-to-one mappings

$$f: X \to Y, \ g: Z \to Y \quad \text{with } Z \text{ finite},$$

we have $Ff \cup Fg = F(f \cup g)$.

The contravariant set functor F *dualizes finite co-unions* if for arbitrary mappings $f: X \to Y$, $g: X \to Z$ onto, we have

$$Ff \cup Fg = F(f \cup^* g) ;$$

F *dualizes co-unions with a finite decomposition* if for arbitrary mappings $f: X \to Y$, $g: X \to Z$ onto, where $Ker\, g$ is a finite decomposition, we have

$$Ff \cup Fg = F(f \cup^* g).$$

DEFINITION. A set functor F (covariant or contravariant) is said to be *nearly faithful* if there exists a cardinal α such that, for arbitrary mappings $f \neq g: X \to Y$, $Ff = Fg$ implies that

$$card\, f(X) < \alpha \quad \text{and} \quad card\, g(X) < \alpha .$$

MAIN THEOREM 4.4. *Let F be a contravariant set functor. Then $S(F)$ is a universal category if and only if F is nearly faithful.*

To prove the Main Theorem we shall first prove a detailed characterization Theorem analogous to the covariant case (see below). Notice that the (covariant) identity functor I is faithful but $S(I)$ is far from universal.

First we recall that a permutation with only one 2-cycle is called a transposition.

THEOREM 4.5. *For a contravariant functor F the following conditions are* *equivalent:*

1° $S(F)$ *is universal;*

2° *there exists a strong embedding from* $S(P^{-})$ *to* $S(F)$;

3° $S(F)$ *has more than* $\operatorname{card} 2^{F\emptyset} + \operatorname{card} 2^{F1}$ *non-isomorphic rigid spaces;*

4° *there exists a rigid F-space* (X, V) *with* $\operatorname{card} X > 1$;

5° F *does not dualize co-unions with finite decomposition;*

6° *there exists a set* X *and* $x \in FX$ *such that* e_x *is non-trivial;*

7° *there exists a set* X *and a transposition*

$$t: X \to X \quad \text{such that} \quad Ft \neq F\, id_X;$$

8° *there exists a cardinal* α *such that for every set* X *with* $\operatorname{card} X \geq \alpha$ *and every transposition* $t: X \to X$ *it holds* $Ft \neq F\, id_X$.

PROOF. We recall that $6 \Longrightarrow 2$ follows from Theorems 2.5 and 3.10. The implication $2 \Longrightarrow 1$ follows from Theorem 1.1. The implications

$$1 \Longrightarrow 3 \Longrightarrow 4$$

are evident. Further $5 \Longrightarrow 6$ follows from Corollary 4.3 and Proposition 1.5. The implication $8 \Longrightarrow 7$ is obvious and so is

$$\text{non } 8 \Longrightarrow \text{non } 4 \text{ - thus } 4 \Longrightarrow 8.$$

Therefore the theorem will be proved as soon as we show that

$$7 \Longrightarrow 6 \text{ and } 6 \Longrightarrow 5.$$

$7 \Longrightarrow 6$. Let $t: X \to X$ be a transposition such that $Ft \neq F\, id_X$, therefore there exists $x \in FX$ such that $Ft(x) \neq x$. Denote a, b distinct points of X such that

$$t(a) = b, \quad t(b) = a.$$

If $e_x = \{X\}$, then there exists $e \in \mathcal{F}_F^X(x)$ with $e(a) = \{a, b\}$ and

$$e(y) = \{y\} \quad \text{for } y \in X - \{a, b\}.$$

Then $e = e \circ t$ and thus $Ft \circ Fe = Fe$ - hence $Ft(x) = x$, because x is in $Im\, Fe$ - a contradiction.

$6 \Longrightarrow 5$. Let $x \in FX$ such that e_x is non-trivial. By Proposition 1.9 we can suppose that there exists $a \in X$ such that $\{a\} \notin e_x$. We choose $b \in X$ such that $e_x(a) \neq e_x(b)$. Let

$$e_1 = \{X - \{a\}, \{a\}\}, \quad e_2 = \{\{a, b\}\} \cup \{\{x\} \mid x \in X - \{a, b\}\}.$$

We have that $x \notin Im\, F e_1 \cup Im\, F e_2$. On the other hand e_2 is a finite decomposition and $e_1 \cup^* e_2 = id_X$.

PROOF OF MAIN THEOREM. If F is nearly faithful, then F fulfills the condition 8 of Theorem 4.5 and thus $S(F)$ is universal. If $S(F)$ is universal, then F fulfills the condition 7 of Theorem 4.5 and by [5] it is nearly faithful.

We recall the analogous results on covariant set functors. Here, instead of universality, those F are characterized for which $S(F)$ is binding. (This means that the category of graphs is fully embeddable in $S(F)$ and, assuming the non-existence of too many non-measurable cardinals, it is the same as universality, see [3].) Let us remark that via Theorem 4.5, $S(F)$ is universal iff it is binding, for contravariant F.

For a covariant set functor F, denote for $x \in FX$,

$$\mathcal{F}_F^X(x) = \{ Z \subset X \mid x \in Im\, F i,\ i : Z \to X \text{ is the inclusion}\}.$$

It is well-known (see [11]) that either $\mathcal{F}_F^X(x)$ is a filter or

$$\mathcal{F}_F^X(x) \cup \{\emptyset\} = exp\, Z = \{ Z \mid Z \subset X \}.$$

THEOREM 4.6. *For a covariant set functor F the following conditions are equivalent:*

1° $S(F)$ is binding;

2° there exists a strong embedding from the category of graphs to $S(F)$;

3° $S(F)$ has more than

$$card\, 2^{F\emptyset} + (card\, 2^{F1} \,.\, card\, 2^{2^{F1}})$$

non-isomorphic rigid spaces;

4° there exists a rigid F-space (X, V) such that $card\, X > card\, 2^{F1}$;

5° F does not preserve unions with a finite set;

6° there exists a set X and $x \in FX$ such that $\mathcal{F}_F^X(x)$ is not an ultrafilter and $\cap Z \neq \emptyset$ where the intersection is taken over all $Z \in \mathcal{F}_F^X(x)$;

7° there exists a set X, a transposition $t : X \to X$ and a mapping $p : X \to X$

such that $p(y) = y$ *iff* $t(y) \neq y$ *and there exists* $x \in FX$ *with*

$$Ft(x) \neq x \neq Fp(x);$$

8° there exists a cardinal α *such that for every set* X, $\operatorname{card} X > \alpha$ *and every transposition* $t: X \to X$ *and every mapping* $p: X \to X$ *such that* $p(y) = y$ *iff* $t(y) \neq y$, *there exists* $x \in FX$ *with* $Ft(x) \neq x$, $Fp(x) \neq x$.

COROLLARY 4.7. *In the finite set theory,* $S(F)$ *is a universal category if and only if* F *is a non-constant functor, i. e.* $S(F)$ *is universal iff* F *does not dualize co-unions.*

Again, the situation for covariant functors was described in [6].

THEOREM 4.8. *In the finite set theory,* $S(F)$ *is a universal category if and only if* F *is not naturally equivalent to* $(I \times C_M) \vee C_N$ *for some* M, N *(we recall that* C_M *is the constant functor to* M *and* I *is the identity functor), i. e.* $S(F)$ *is universal iff* F *does not preserve unions.*

EXAMPLE 4.9 (A non-constant functor which is not nearly faithful). Denote by β the usual set functor, assigning to a set X the set βX of all ultrafilters on X, and to a mapping f the mapping βf which sends an ultrafilter $\mathfrak{F}$ to the ultrafilter with base

$$\{ f(Z) \mid Z \in \mathfrak{F} \}.$$

Let $\bar{\beta}$ be the factor-functor of β with $\mathfrak{F}$, $\mathfrak{G} \in \beta X$ merged iff either $\mathfrak{F} = \mathfrak{G}$ or $\mathfrak{F}$ and $\mathfrak{G}$ are fixed (i.e. $\cap Z \neq \emptyset$, where the intersection is taken over all $Z \in \mathfrak{F}$). Then, clearly, $\bar{\beta}$ merges transpositions and so does the (non-constant) functor $F = P^{-} \circ \bar{\beta}$.

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Faculty of Mathematics and Physics
Charles University
PRAGUE
TCHECOSLOVAQUIE