CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Closed categories and Banach spaces

Cahiers de topologie et géométrie différentielle catégoriques, tome 17, nº 4 (1976), p. 335-342

<http://www.numdam.org/item?id=CTGDC_1976__17_4_335_0>

CAHIERS DE TOPOLOGIE *Vol. XVII-4 (1976)*
ET GEOMETRIE DIFFERENTIELLE

CLOSED CATEGORIES AND BANACH SPACES

*by Michael BARR**

INTRODUCTION

In previous papers we have described a duality for vector spaces over a discrete field [1] and for a category which is a natural extension of the category of Banach spaces [2]. We have also described a closed category containing «most» of the spaces of [1] (see [3]). In this paper we extend the results of [3] to the context of Banach spaces, that is, we find a largish full subcategory of the category $\mathfrak{B}$ of balls considered in [2] that, when equipped with an internal *hom* which is essentially compact convergence, becomes a closed monoidal category in which every object is reflexive.

As we did in [3], we depart from the notation of [1] and [2] and let (A, B) and A^* denote the internal *hom* and the dual space topologized by uniform convergence on compact subballs. As in [2], we let $\{x \mid \ldots\}$ denote the set of x such that... .

1. Preliminaries.

Recall from [2] that a *ball* is the unit ball of a Banach space equipped with a second, coarser locally convex topology in which the original norm is lower semi-continuous. We showed there that the topology and the norm were determined by the continuous seminorms which were bounded by the norm. The topology is determined in the usual way and the norm as the *sup* of the seminorms. A ball was called *discrete* when the second topology

* I would like to thank the Canada Council and the National Research Council of Canada for supporting this research, and the Forschungsinstitut für Mathematik, E. T. H., Zürich, for providing a congenial environment.

was that of the norm and it was shown that a seminorm p on an arbitrary B leads to a discrete ball B_p and a projection $\pi_p : B \to B_p$ such that p is π_p followed by the norm function on B_p. These facts will be used without further reference.

Most of this paper is concerned with adapting the results of [3] to the present circumstance. A reference of the form «cf. [3], $x.y$» means that proof given there works here without essential change.

PROPOSITION 1.1. *Let B be a fixed ball. The functor $(B,-)$ commutes with projective limits and has an adjoint $-\otimes B$.*

PROOF. Cf. [3], 1.1.

For similar reasons as in [3], the internal *hom* constructed above does not directly give a closed monoidal category. The same dodge used there works here too. A subset A of the ball B is called *totally bounded* if for every 0-neighborhood M of B there are elements

$$a_1, \dots, a_n \in A \quad \text{such that} \quad A \subset \cup a_i + M.$$

This is easily seen to be equivalent to the uniform notion when A is given its canonical uniformity, in which a cover of the form

$$\{ a+M \mid a \in A \}$$

is a uniform cover. In particular A is compact iff it is complete and totally bounded.

PROPOSITION 1.2. *Let B be totally bounded and C be compact. Then, $B\otimes C$ is totally bounded.*

PROOF. Given an open set $M \subset B\otimes C$, there is a discrete ball D, a map $f: B\otimes C \to D$ and an $\epsilon > 0$ such that $M \supset f^{-1}(\epsilon D)$. There corresponds a map $g: B \to (C, D)$ and the latter is a discrete ball. Hence $g^{-1}(\frac{\epsilon}{2}(C, D))$ is open and so there are $b_1, \dots, b_n$ such that:

$$B \subset \cup(b_i + g^{-1}(\tfrac{\epsilon}{2}(C, D))) \quad \text{or} \quad g(B) \subset \cup(g(b_i) + \tfrac{\epsilon}{2}(C, D)).$$

For each $i = 1, \dots, n$, $g(b_i): C \to D$ and by reasoning similar to above, there are

$$c_{i1}, \dots, c_{i,n(i)} \in C \quad \text{such that} \quad g(b_i)(C) \subset \cup (c_{ij} + \tfrac{\epsilon}{2} D).$$

The result is that

$$f(B \otimes C) = g(B)(C) \subset \cup (g(b_i)(c_{ij}) + \epsilon D)$$

or finally that

$$B \otimes C \subset \cup (b_i \otimes c_{ij} + \epsilon M).$$

Here we use $b \otimes c$ to denote the image of b at c under the map

$$B \to (C, B \otimes C)$$

given by the adjunction.

2. ζ- and ζ^*-balls.

Following [3], we say that B is a *ζ-ball* if every closed, totally bounded subball is compact (or, equivalently, complete). The full subcategory of $\mathfrak{B}$ of ζ-balls is denoted $\zeta\mathfrak{B}$.

PROPOSITION 2.1. *The ball B is a ζ-ball iff every map to B from a dense subball of a compact ball can be extended to the whole ball.*

PROOF. The proof of [3], 2.1, extends to this case. It is even easier because we already know that a compact ball is complete.

We say that B is a ζ^*-ball if B^* is a ζ-ball. As in [3] it is clear that both discrete and compact balls are ζ-ζ^*-balls.

PROPOSITION 2.2. *Let B be a ζ-ball. Then B^* is a ζ^*-ball, i. e. B^{**} is a ζ-ball.*

PROOF. Cf. [3], 2.2.

As in [3], we let $B^{\sim}$ be the completion of B. The easiest description of it is the closure of B in ΠB_p, the product being taken over all the seminorms p of B. Since each B_p is the unit ball of a Banach space, it is complete and so is the product. Now let ζB be the intersection of all the ζ-subballs of B. For a subball $A \subset B^{\sim}$ let $\zeta_1 A$ be the union of the closures of the totally bounded subballs of B. Evidently $\zeta_1 A \supset A$. If A_1 and

A_2 are two totally bounded subballs of A, so is $A_1 \cup A_2$ and then so is their convex sum ([4], 4.3, where totally bounded is called precompact). Thus $\zeta_1 A$ is a subball. Then $\zeta_\mu A$ for a cardinal μ can be defined inductively as in [3]. Finally $\zeta_\infty A$ is their union.

PROPOSITION 2.3. $\zeta B = \zeta_\infty B$.

PROOF. Cf. [3], 2.3.

PROPOSITION 2.4. *The construction* $B \mapsto \zeta B$ *is a functor which, together with the inclusion* $B \hookrightarrow \zeta B$, *determines a left adjoint to the inclusion of* $\zeta\mathfrak{B} \to \mathfrak{B}$.

PROOF. Cf. [3], 2.4.

PROPOSITION 2.5. *Let* B *be reflexive. Then so is* ζB.

PROOF. Cf. [3], 2.5.

PROPOSITION 2.6. *Let* B *be a reflexive* ζ^**-ball. Then so is* ζB.

PROOF. Cf. [3], 2.6.

We recall from [3] that given two subsets X_1 and X_2 of a topological space X, we say that X_1 *is closed in* X_2 if $X_1 \cap X_2$ is a closed subset of X_2.

PROPOSITION 2.7. *Let* $\{B_\omega\}$ *be a family of discrete balls and* B *a subball of* ΠB_ω. *Then* B *is a* ζ*-space iff for every choice of compact subballs* $C_\omega \subset B_\omega$, B *is closed in* ΠC_ω.

PROOF. The argument goes essentially as in the proof of 2.7 of [3]. One minor change has to be made. When B_0 is a closed, totally bounded subball of B, let C_ω be the closure of the image of B_0 in B_ω. Since B_ω is complete, that closure is compact.

We note, in connection with the above, that unlike the case of vector spaces over a discrete field, a compact subball of a discrete ball need not be finite dimensional. For example, let C be the unit ball of l^∞ with the weak topology in which it is compact, and let B be the unit ball of l^1.

The map

$$C \to B \quad \text{given by} \quad (\lambda_i) \mapsto (\lambda_i / 2^i)$$

embeds C in B. Since C is compact, it is isomorphic with its image.

3. The internal *hom*.

We recall that when A and B are balls, (A, B) denotes the set of continuous maps $A \to B$ which preserve the absolutely convex structure. It is topologized as a subball of $\Pi(A_\omega, B_p)$, where A_ω runs over the compact subballs of A and p over all the seminorms of B. Each (A_ω, B_p) has the discrete (i. e. norm) topology. We know from [2], 6.5, that the seminorms on A^* are all described as *suprema* on the A_ω, so that the A_ω can be thought of as indexed by the seminorms of A^*. From these observations, the following becomes a formal exercise.

PROPOSITION 3.1. *Let A and B be reflexive balls. Then the equivalence between maps $A \to B$ and $B^* \to A^*$ underlies an isomorphism*

$$(A, B) \approx (B^*, A^*).$$

LEMMA 3.2. *Let A be a reflexive ζ^*-ball and B be a reflexive ball. Then any totally bounded subball of (A, B) is equicontinuous.*

PROOF. If

$$F \subset (A, B) \approx (B^*, A^*)$$

is totally bounded, we have $F \otimes B^* \to A^*$. Let M be open in B and choose a seminorm p such that $M \supset \pi_p^{-1}(\epsilon B_p)$. The ball B_p^* is a compact subball of B^* and $F \otimes B_p^*$ is totally bounded by 1.2. The closure of its image in A^* is compact and determines a seminorm g on $A^{**} \approx A$. Tracing through the isomorphisms, we find that for any $f \in F$,

$$|g(a)| \leqslant \epsilon \quad \text{implies that} \quad |p f(a)| \leqslant \epsilon$$

and hence

$$|g(a)| \leqslant \epsilon \quad \text{implies} \quad f(a) \in M.$$

LEMMA 3.3. *Suppose A is a reflexive ζ^*-ball and B is a reflexive ζ-ball.*

Then (A, B) is a ζ-ball.

PROOF. Let $\{A_\omega\}$ range over the compact subballs of A and $\{p\}$ over the seminorms of B. A compact subball $C_{\omega p} \subset (A_\omega, B_p)$ is equicontinuous. Corresponding to the inclusion we have a map $C_{\omega p} \otimes A_\omega \to B_p$ whose image is contained in a compact subball $B_{\omega p}$ of B_p. Under the isomorphism

$$(A_\omega, B_p) \approx (B_p^*, A_\omega^*)$$

we also find a compact subball of A_ω^* which we will call $A_{\omega p}^* \subset A_\omega^*$ which contains the image. The *supremum* on $A_{\omega p}^*$ determines a seminorm on A_ω which we will also - somewhat irregularly - call p to conform with the previous name $A_{\omega p}^*$. The result is that

$$C_{\omega p} \subset (A_\omega, B_{\omega p}) \cap (A_{\omega p}, B_p).$$

(We cannot go on to claim it is in $(A_{\omega p}, B_{\omega p})$ by analogy with [3] because, for example, $A_\omega \to A_{\omega p}$ is not onto, but that does not matter.) Now suppose $C_{\omega p}$ are given for all ω and p and $A_{\omega p}$ and $B_{\omega p}$ chosen as above. Then B is closed in $\Pi B_{\omega p}$ and so (A_ω, B) is closed in $\Pi_p(A_\omega, B_{\omega p})$. This in turn implies that $\Pi_\omega(A_\omega, B_\omega)$ is closed in $\Pi_{\omega, p}(A_\omega, B_{\omega p})$. Using 3.1 and the fact that A^* is a ζ-space, we similarly conclude that $\Pi_p(A, B_p)$ is closed in $\Pi_{\omega,p}(A_{\omega p}, B_p)$ and hence that $\Pi_\omega(A_\omega, B) \cap \Pi_p(A, B_p)$ is closed in

$$\Pi_{\omega,p}((A_\omega, B_{\omega p}) \cap (A_{\omega p}, B_p)),$$

a fortiori in $\Pi C_{\omega p}$. A map which belongs to both

$$\Pi_\omega(A_\omega, B) \text{ and } \Pi_p(A, B_p)$$

determines a commutative square

$$\begin{tikzcd}
\Sigma A_\omega \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & \Pi B_p
\end{tikzcd}$$

and with the top map onto, we get a diagonal fill-in $A \to B$. Thus no matter how compact subballs $C_{\omega p} \subset (A_\omega, B_p)$ are chosen, (A, B) is closed in $\Pi C_{\omega p}$. By 2.7, (A, B) is a ζ-ball.

4. The category $\mathfrak{R}$.

We let $\mathfrak{R}$ denote the full subcategory of $\mathfrak{B}$ whose objects are the reflexive ζ-ζ^*-balls.

PROPOSITION 4.1. *The functor* $B \mapsto \delta B = (\zeta B^*)^*$ *is right adjoint to the inclusion. For any* B, $\delta B \to B$ *is 1-1 and onto.*

PROOF. Cf. [3], 4.1.

We now define, for A, B in $\mathfrak{R}$,

$$[A, B] = \delta(A, B)$$

(cf. 3.3). It consists of the same set of maps topologized by a possibly finer topology. When $B = I$, we get

$$[A, I] = \delta A^* = A^*,$$

so the dual is unchanged.

PROPOSITION 4.2. *Let* A *and* B *be in* $\mathfrak{R}$. *Any totally bounded subball of* $[A, B]$ *is equicontinuous.*

PROOF. It is a special case of 3.2.

COROLLARY 4.3. *Let* A, B, C *be in* $\mathfrak{R}$. *Then there is a 1-1 correspondence between maps* $A \to [B, C]$ *and maps* $B \to [A, C]$.

PROOF. Cf. [3], 4.5.

PROPOSITION 4.4. *Let* A *and* B *be in* $\mathfrak{R}$. *Then* $[A, B] \cong [B^*, A^*]$ *by the natural map.*

PROOF. Apply δ to both sides in 3.1.

Now define, for A, B in $\mathfrak{R}$,

$$A \otimes B = [B, A^*]^*.$$

PROPOSITION 4.5. *Let* A, B, C *be in* $\mathfrak{R}$. *Then there is a 1-1 correspondence between maps* $A \otimes B \to C$ *and maps* $A \to [B, C]$.

PROOF. Cf. [3], 4.5

COROLLARY 4.6. *For any* A, B *in* $\Re$, $A \circledcirc B \approx B \circledcirc A$.

PROPOSITION 4.7. *Let* A *and* B *be compact balls. Then* $A \circledcirc B \approx \zeta(A \otimes B)$ *and is compact ball.*

PROOF. Cf. [3], 4.7.

PROPOSITION 4.8. *Let* A, B, C *belong to* $\Re$. *The natural composition of maps* $(B, C) \times (A, B) \to (A, C)$ *arises from a map*

$$(B, C) \otimes (A, B) \to (A, C).$$

PROOF. Cf. [3], 4.7.

PROPOSITION 4.9. *Let* A, B *and* C *belong to* $\Re$. *Then natural composition arises from a map* $[B, C] \circledcirc [A, B] \to [A, B]$.

PROOF. Cf. [3], 4.8.

THEOREM 4.10. *The category* $\Re$ *equipped with* $- \circledcirc -$ *and* $[-,-]$ *is a closed monoidal category in which every object is reflexive.*

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