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# On simply bireflective subcategories

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ET GEOMETRIE DIFFERENTIELLE

# ON SIMPLY BIREFLECTIVE SUBCATEGORIES

by Jan MENU and Aleš PULTR

A description of an everyday-life concrete category follows often the following pattern : There is given

( 1 ) a construction producing structures on sets,

( 2 ) a mechanism of choice of well-behaved mappings among all mappings between structured sets,

( 3 ) a delimitation of the desired objects among all the ones obtained by the construction from ( 1 ) (the «system of axioms» on the structure in question ).

In [5] there was proved that the approach over the categories $S(F)$ ( [3], [4], [7], etc...; $F$ is a functor $Set \to Set$, the objects of $S(F)$ are the couples $(X, r)$ with $r \subset F(X)$, the morphisms $(X, r) \to (Y, s)$ are the triples $(r, f, s)$ with $f: X \to Y$ such that $F(f)(r) \subset s$ ) is of a fairly general validity for the tasks ( 1 ) and ( 2 ).

In the present paper we are going to discuss the tamest case of ( 3 ); namely, the delimitations leading to the subcategories that are both reflective and coreflective with, moreover, both the reflection and coreflection morphisms identity carried ( following [4], we call them simply bireflective ).

Consider the example of the symmetry axiom for binary relations. The category of sets with binary relations coincides with $S(Q)$ where $Q$ sends a set $X$ to $X \times X$ and a mapping $f$ to $f \times f$. We see that its subcategory of the symmetric relations is again of the type $S(G)$, namely with

$$G(X) = \{ A \mid A \subset X,\ 1 \leqslant |A| \leqslant 2 \}, \quad G(f)(A) = f(A).$$

and, moreover, its embedding into $S(Q)$ is naturally induced by the epitransformation $Q \to G$ sending $(x, y)$ to $\{x, y\}$. This observation led us for a moment to the conjecture that one might describe the simply bireflective subcategories of $S(F)$, generally, by means of epi-transformations $\varepsilon: F \to G$. One sees easily that this conjecture is false. There are simply bireflective subcategories which are not thus induced (e.g. in $S(Q)$ the subcategory of the $(X, r)$ such that $(x, x) \in r$ whenever $(x, y) \in r$ for a $y$). On the other hand, an epi-transformation always induces an embedding onto a simply reflective, but not necessarily onto a simply bireflective subcategory. If one, however, generalizes the definition of $S(F)$ to functors $F$ with values in the category of quasidiscrete spaces (see n° 1), the situation is more satisfactory. Now, every simply bireflective subcategory is induced by an epi-transformation and one can give an explicit characterization of the epi-transformations which induce one. This is shown and discussed in n° 4 and n° 5 (the first three paragraphs are of a technical character). As an application, we give at the end of n° 5 a complete list of «systems of axioms» on $A$-nary relations leading to simply bireflective subcategories of the category of all $A$-nary relations.

## 1. Quasidiscrete spaces.

1.1. A *quasidiscrete space* (abbr., QD-space) is a topological space in which the intersection of any system of open sets is open (see [1]).

In a QD-space we denote by $\mathcal{O}_p A$ the smallest open subset containing $A$.

The category of all QD-spaces and their continuous mappings will be denoted by $QD\,Top$, its full subcategory generated by the $T_0$-spaces will be denoted by $QD\,Top_0$.

1.2. Let $(X, \tau)$ be a QD-space. Define a preorder $\leqslant$ (more exactly, $\leqslant_\tau$) on $X$ by:

$$x \leqslant y \text{ iff } \mathcal{O}_p\{x\} \subset \mathcal{O}_p\{y\}.$$

On the other hand, with a preorder $\leqslant$ on $X$ we can associate a quasidiscrete topology declaring $A \subset X$ for open iff

$$x \leqslant y \text{ and } y \in A \text{ imply } x \in A.$$

It is well-known (and very easy to check) that this construction yields an isomorphism between *QD Top* and the category of preordered sets and monotone mappings. Since obviously a QD-space is $T_0$ iff

$$\mathcal{O}_{\mathfrak{p}}\{x\} = \mathcal{O}_{\mathfrak{p}}\{y\} \text{ implies } x = y,$$

*QD Top*$_0$ corresponds under this isomorphism to the category of partially ordered sets.

1.3. Obviously, the continuity of a mapping $f: X \to Y$ is characterized by the formula: $f(\mathcal{O}_{\mathfrak{p}}\{x\}) \subset \mathcal{O}_{\mathfrak{p}}\{f(x)\}$.

1.4. CONVENTIONS. Whenever convenient, we will deal with the corresponding preorders instead of the topologies. If there is no danger of confusion, the preorders are indicated by $\leqslant$ simply without further specifications. We write

$$x \sim y \quad \text{for} \quad x \leqslant y \ \& \ y \leqslant x.$$

Instead of $\mathcal{O}_{\mathfrak{p}}\{x\}$, we write $\mathcal{O}_{\mathfrak{p}}x$.

The proofs of the following two lemmas are easy.

1.5. LEMMA. *Let $f: X \to Y$ in QD Top be such that*

$$M, N \text{ open } \& \ f^{-1}(M) = f^{-1}(N) \text{ implies } M = N.$$

*Then for every $y \in Y$ there is an $x \in X$ such that $f(x) \sim y$. Consequently, if moreover $Y$ is $T_0$, $f$ is onto.*

1.6. LEMMA. *Let $f, g: X \to Y$ in QD Top be such that*

$$f^{-1}(M) = g^{-1}(M) \text{ for every open } M.$$

*Then $f(x) \sim g(x)$ for every $x \in X$. Consequently, if moreover $Y$ is $T_0$, $f = g$.*

1.7. If $(X, \tau)$ is a QD-space, we denote by $\mathcal{O}(X, \tau)$ the lattice of all the open subsets of $(X, \tau)$. It is well-known (and very easy to see) that $\mathcal{O}(X, \tau)$ *is irreducibly generated, and the irreducible elements are exactly the sets of the form $\mathcal{O}_{\mathfrak{p}}x$ with $x \in X$.* (*)

(*) An irreducible element of a lattice $\Lambda$ is a non-zero element $a \in \Lambda$ such that $a \leqslant \bigvee b_i$ implies there is an $i$ with $a \leqslant b_i$. $\Lambda$ is said to be irreducibly generated if every $x \in \Lambda$ is a union of irreducible elements.

1.8. The following fact is also well-known (and easy to prove):

*An irreducibly generated lattice $\Lambda$ is a Boolean algebra iff its irreducible elements are disjoint. Consequently, $\mathcal{O}(X, \leqslant)$ is a Boolean algebra iff $\leqslant$ is symmetric (and, hence, an equivalence).*

1.9. $QD\,Top_0$ is a reflective subcategory of $QD\,Top$. Let us denote by $J$ the embedding $QD\,Top_0 \subset QD\,Top$, by $L$ the reflection functor and by

$$\eta : 1_{QD\,Top} \to J\,L$$

the reflection transformation. (Suitable $L$ and $\eta$ may be obtained putting first

$$L'(X) = (\{\mathcal{O}_{\mathfrak{p}}\, x \mid x \in X\}, \subset),$$

$$L'(f)(\mathcal{O}_{\mathfrak{p}}\, x) = \mathcal{O}_{\mathfrak{p}}\, f(x), \quad \eta'_X(x) = \mathcal{O}_{\mathfrak{p}}\, x,$$

$$\phi_X = \begin{cases} 1_{L(X)} & \text{for } X \notin QD\,Top_0 \\ \eta'^{-1}_X & \text{for } X \in QD\,Top_0 \end{cases}$$

and then putting $L(f) = \phi_Y \circ L'(f) \circ \phi_X^{-1}$.)

1.10. The following property of the mappings $\eta_X$ is evident:

*$M$ is open in $X$ iff $M = \eta_X^{-1}(N)$ for an open $N$ in $L(X)$.*

## 2. The categories $S(F)$ with $F : Set \to QD\,Top$.

2.1. Let $F : Set \to QD\,Top$ be a functor ($Set$ is the category of all sets and mappings). The category $S(F)$ is defined as follows: The objects are the couples $(X, r)$ with $r$ an *open* subset of $F(X)$; the morphisms from $(X, r)$ to $(Y, s)$ are the triples $(r, f, s)$ with $f : X \to Y$ such that $F(f)(r) \subset s$.

$S(F)$ will be regarded as a concrete category with the forgetful functor sending $(r, f, s)$ to $f$.

2.2. Thus defined categories $S(F)$ include the categories $S(F)$, with $F : Set \to Set$ introduced in [2] and studied in various papers. It suffices to regard a functor into $Set$ as a functor into $QD\,Top$ with discrete values.

2.3. Let $\theta : F \to G$ be a transformation. Define a functor

$$[\theta] : S(G) \to S(F)$$

putting

$$[\theta](X, r) = (X, \theta_X^{-1}(r)), \quad [\theta](r, f, s) = (\theta_X^{-1}(r), f, \theta_Y^{-1}(s)).$$

2.4. OBSERVATION. $[\theta]$ is faithful and a right adjoint.

2.5. A transformation $\theta : F \to G$ is said to be an *epi-transformation* if every $\theta_X$ is a mapping onto.

(The epi-transformations are exactly the epimorphisms in the illegitimate category $[\,Set, QD\,Top\,]$, which follows immediately by the cocontinuousness of the evaluation functor and by the fact that the epimorphisms in $QD\,Top$ are onto.)

2.6. PROPOSITION. *If* $\varepsilon$ *is an epi-transformation, then* $[\varepsilon]$ *is a full embedding.*

PROOF (quite analogous to the corresponding one for *Set*-valued functors - see [7] ). Since $\varepsilon_X$ are onto,

$$\varepsilon_X^{-1}(r) = \varepsilon_X^{-1}(s) \quad \text{implies} \quad r = s.$$

Thus, since $[\varepsilon]$ is faithful, it is one-to-one. If

$$F(f)(\varepsilon_X^{-1}(r)) \subset \varepsilon_Y^{-1}(s),$$

we have

$$G(f)(r) = G(f)\,\varepsilon_X\,\varepsilon_X^{-1}(r) = \varepsilon_Y F(f)\,\varepsilon_X^{-1}(r) \subset \varepsilon_Y\,\varepsilon_Y^{-1}(s) = s.$$

Thus $[\varepsilon]$ is also full.

2.7. REMARK. If $[\varepsilon]$ is a full embedding, $\varepsilon$ is not necessarily an epi-transformation, which is easily seen. It is necessarily an epi-transformation if the values of $G$ are in $QD\,Top_0$ (see 1.5).

2.8. Evidently, we have:

PROPOSITION. *Let* $\varepsilon : F \to G$ *be an epi-transformation. Then* $[\varepsilon]$ *is an isofunctor mapping* $S(G)$ *onto* $S(F)$ *iff the following condition holds:*

*For every* $X$, $M$ *is open in* $F(X)$ *iff* $M = \varepsilon_X^{-1}(N)$ *for an open set* $N$ *in* $G(X)$.

## 3. The transformations $\eta F$.

3.1. In the notation of 1.9, for an $F: Set \to QD\,Top$ put $F' = J\,L\,F$. We have epi-transformations $\eta F: F \to F'$.

3.2. We have obviously $F'' = F'$ and $\eta F' = 1_{F'}$.

3.3. By 1.10 and 2.8 we obtain immediately:

PROPOSITION. *$[\eta F]$ is an isofunctor of $S(F')$ onto $S(F)$.*

3.4. THEOREM. *Let $\phi: S(F) \to S(G)$ be an isofunctor such that $U'\phi = U$ for the natural forgetful functors $U$, $U'$. Then there is a natural equivalence $\kappa: G' \cong F'$ such that the diagram*

$$\begin{array}{ccc} S(F) & \xrightarrow{\quad\phi\quad} & S(G) \\ \uparrow{\scriptstyle [\eta F]} & & \uparrow{\scriptstyle [\eta G]} \\ S(F') & \xrightarrow{\quad[\kappa]\quad} & S(G') \end{array}$$

*commutes.*

PROOF. Given $H, K: Set \to QD\,Top_0$ and an isofunctor $\psi: S(H) \cong S(K)$ preserving the underlying mappings, the formula

$$\psi(X, r) = (X, \phi_X(r))$$

defines obviously lattice isomorphisms $\phi_X: \mathcal{O}(H(X)) \to \mathcal{O}(K(X))$. Since $\phi_X$ sends irreducibles to irreducibles and since $K(X)$ is a $T_0$-space, the formula

$$\mathcal{O}_P \lambda_X(x) = \phi_X(\mathcal{O}_P\, x)$$

defines homeomorphisms $\lambda_X: H(X) \cong K(X)$ (see 1.2) such that

$$\phi_X(r) = \lambda_X(r) \quad \text{for an open } r \subset H(X).$$

Let $f: X \to Y$ be a mapping, $x \in X$. Since $H(f)(\mathcal{O}_P\, x) \subset \mathcal{O}_P\, H(f)(x)$, we have

$$K(f)(\lambda_X(x)) \in K(f)(\mathcal{O}_P \lambda_X(x)) = K(f)(\phi_X(\mathcal{O}_P\, x)) \subset \phi_Y \mathcal{O}_P\, H(f)(x),$$

so that

$$\mathcal{O}_{\mathfrak{p}} K(f) \lambda_X(x) \subset \mathcal{O}_{\mathfrak{p}} \lambda_Y H(f)(x).$$

Similarly, $\mathcal{O}_{\mathfrak{p}} H(f) \lambda_X^{-1}(y) \subset \mathcal{O}_{\mathfrak{p}} \lambda_Y^{-1} K(f)(y)$, and hence

$$\mathcal{O}_{\mathfrak{p}} \lambda_Y H(f)(x) = \mathcal{O}_{\mathfrak{p}} \lambda_Y H(f) \lambda_X^{-1}(y) = \phi_Y(\mathcal{O}_{\mathfrak{p}} H(f) \lambda_X^{-1}(y)) \subset$$
$$\subset \phi_Y(\mathcal{O}_{\mathfrak{p}} \lambda_Y^{-1} K(f)(y)) = \mathcal{O}_{\mathfrak{p}} K(f)(y) = \mathcal{O}_{\mathfrak{p}} K(f) \lambda_X(x).$$

Thus

$$\mathcal{O}_{\mathfrak{p}} K(f) \lambda_X(x) = \mathcal{O}_{\mathfrak{p}} \lambda_Y H(f)(x),$$

and since the spaces are $T_0$, consequently, $K(f)\lambda_X = \lambda_Y H(f)$, so that $\lambda$ is a natural equivalence. Now, apply the just proved assertion to

$$\psi = [\eta G]^{-1} \phi [\eta F], \qquad \text{and put } \kappa = \lambda^{-1}.$$

3.5. REMARK. In particular, we see that $S(F)$ and $S(G)$ are equally carried (i.e. there is an isofunctor $\phi$ with $U'\phi = U$) iff $F' \cong G'$. In fact, if $S(F)$ and $S(G)$ are isomorphic, they are equally carried necessarily (which, in essence, may be proved characterizing internally up to isomorphism the objects $(1, \emptyset)$). Thus, if $S(F) \cong S(G)$, then $F' \cong G'$.

## 4. Simply reflective subcategories.

4.1. Let $(\mathfrak{K}, U)$ be a concrete category. A subcategory $\mathfrak{L}$ of $\mathfrak{K}$ is said to be *simply reflective* (resp. *coreflective*) in $(\mathfrak{K}, U)$ if it is reflective (resp. coreflective) and if there is a reflection (resp. coreflection) transformation.

$$\rho = (\rho_X : X \to X') \quad (\text{resp. } \rho = (\rho_X : X' \to X))$$

such that

$$\text{for every } X \in obj\,\mathfrak{K}, \quad U(\rho_X) = 1_{U(X)} \qquad (\text{cf. } [4]).$$

4.2. PROPOSITION. *Let* $\varepsilon: F \to G$ *be an epi-transformation. Then* $[\varepsilon]$ *maps* $S(G)$ *onto an isomorphic simply reflective subcategory of* $S(F)$ *such that, moreover,*

$(*)$ *if* $(X, a_i) \in \mathfrak{K}$ *for* $i \in J$, *then* $(X, \bigcup_J a_i) \in \mathfrak{K}$.

PROOF follows immediately by 2.4, 2.6 and the fact that

$$f^{-1}(\bigcup_J A_i) = \bigcup_J f^{-1}(A_i) \quad (\text{cf. also } [6]).$$

4.3. Let $\mathcal{K}$ be a subcategory of an $S(F)$. Denote by $\tilde{\mathcal{K}}$ the full subcategory of $S(F)$ generated by all the objects of the form

$$(X, \bigcup_J a_i) \quad \text{with} \quad (X, a_i) \in obj\,\mathcal{K} \text{ for every } i \in J .$$

4.4. REMARK. Since $(X, \bigcup_J a_i)$ may be expressed as a colimit of the diagram consisting of the identity carried morphisms $(X, \emptyset) \to (X, a_i)$, and since the forgetful functor of $S(F)$ preserves colimits, we see that for a simply coreflective $\mathcal{K}$, we have $\mathcal{K} = \tilde{\mathcal{K}}$. On the other hand, one sees easily that the condition $\mathcal{K} = \tilde{\mathcal{K}}$ does not imply simple coreflectivity.

4.5. Let $\mathcal{K}$ be a simply reflective subcategory of $S(F)$. Denote by

$$\rho_{(X,a)}: (X, a) \to (X, \rho a)$$

the identity carried reflection. Define a functor $F_{\mathcal{K}}: Set \to QD\,Top$ as follows: The underlying set of $F_{\mathcal{K}}(X)$ coincides with that of $F(X)$ and the topology of $F_{\mathcal{K}}(X)$ is given by the preorder:

$$u \leqslant_{\mathcal{K}} v \quad \text{iff} \quad \rho \mathcal{O}_{\mathfrak{p}} u \subset \rho \mathcal{O}_{\mathfrak{p}} v ;$$

for a mapping $f$ put $F_{\mathcal{K}}(f)(u) = F(f)(u)$. (This is correct: if we have $\rho \mathcal{O}_{\mathfrak{p}} u \subset \rho \mathcal{O}_{\mathfrak{p}} v$, then by the basic property of reflections and by 1.3:

$$\rho \mathcal{O}_{\mathfrak{p}}\, F(f)(u) \subset \rho \mathcal{O}_{\mathfrak{p}}\, F(f)(v) \text{ . )}$$

Since the preorder $\leqslant_{\mathcal{K}}$ is obviously stronger than the original one, we have an epi-transformation $\varepsilon_{\mathcal{K}}: F \to F_{\mathcal{K}}$ defined by $(\varepsilon_{\mathcal{K}})_X(u) = u$.

4.6. LEMMA. *$M$ is open in $F_{\mathcal{K}}(X)$ iff $(X, M) \in \tilde{\mathcal{K}}$.*

PROOF. Let $M$ be open in $F_{\mathcal{K}}(X)$. Then for every $u \in M$, $\rho \mathcal{O}_{\mathfrak{p}} u \subset M$, so that $M = \bigcup_{u \in M} \rho \mathcal{O}_{\mathfrak{p}} u$. Thus, $(X, M) \in \tilde{\mathcal{K}}$. On the other hand, suppose that $(X, M) \in \tilde{\mathcal{K}}$ and $u \leqslant_{\mathcal{K}} v \in M$. Then, $u$ is in $\rho \mathcal{O}_{\mathfrak{p}} v \subset M$. Thus, $M$ is open.

4.7. THEOREM. *Let $\mathcal{K}$ be a simply reflective subcategory of $S(F)$. Then $S(F_{\mathcal{K}}) = \tilde{\mathcal{K}}$ and $[\varepsilon_{\mathcal{K}}]$ is the embedding $\tilde{\mathcal{K}} \subset S(F)$.*

PROOF follows immediately by 2.6 and 4.6.

4.8. REMARK. By 4.7 and 4.2, for a simply reflective $\mathcal{K}$, the category $\tilde{\mathcal{K}}$ is also simply reflective.

4.9. Theorem 4.7 is in a way converse to Proposition 4.2, stating that every simply reflective subcategory satisfying ( * ) is an image of an epi-transformation induced functor. By 3.3 we have another epi-transformation

$$\varepsilon'_{\mathcal{K}} = \eta F_{\mathcal{K}} \circ \varepsilon_{\mathcal{K}} : F \to F'_{\mathcal{K}}$$

inducing also an isomorphism of $S(F'_{\mathcal{K}})$ and $\mathcal{K}$. We will show now that every epi-transformation $\varepsilon : F \to G$ such that $[\varepsilon]$ represents the embedding $\mathcal{K} \subset S(F)$ lies in between $\varepsilon_{\mathcal{K}}$ and $\varepsilon'_{\mathcal{K}}$. We have

THEOREM. *Let $\mathcal{K}$ be a simply reflective subcategory of $S(F)$ such that $\mathcal{K} = \tilde{\mathcal{K}}$. Let $\varepsilon : F \to G$ be an epi-transformation and let there be an isofunctor $\phi$ such that the diagram*

(1)
$$\begin{tikzcd} & S(G) \arrow[dl, "\phi"'] \arrow[dr, "{[\varepsilon]}"] & \\ \mathcal{K} \arrow[rr, hook] & & S(F) \end{tikzcd}$$

*commutes. Then there are epi-transformations*

$$F_{\mathcal{K}} \xrightarrow{\beta} G \xrightarrow{\gamma} F'_{\mathcal{K}}$$

*such that the diagram*

(2)
$$\begin{tikzcd} & & F_{\mathcal{K}} \arrow[dl, "\beta"'] \arrow[dd, "\eta F_{\mathcal{K}}"] \\ F \arrow[urr, "\varepsilon_{\mathcal{K}}"] \arrow[r, "\varepsilon"] \arrow[drr, "\varepsilon'_{\mathcal{K}}"'] & G \arrow[dr, "\gamma"] & \\ & & F'_{\mathcal{K}} \end{tikzcd}$$

*commutes.*

PROOF. By 3.4 there is a natural equivalence $\kappa : G' \to F'_{\mathcal{K}}$ such that

(3) $$\phi^{-1}[\eta F_{\mathcal{K}}] = [\eta G][\kappa].$$

Put $\gamma = \kappa \circ \eta G : G \to F'_{\mathcal{K}}$. Now, let $M$ be open in $G(X)$. Then $(X, M)$ is in $S(G)$ and hence $(X, \varepsilon_X^{-1}(M)) = \phi(X, M) \in \mathcal{K}$. Thus, by 4.6, $\varepsilon_X^{-1}(M)$

is open in $F\mathcal{K}(X)$. Consequently, we can define an epi-transformation $\beta: F\mathcal{K} \to G$ putting $\beta_X(u) = \varepsilon_X(u)$. Thus, we obtain immediately

$$\beta \circ \varepsilon_{\mathcal{K}} = \varepsilon . \tag{4}$$

Further, we obtain (using (3), (4) and (1))

$$[\varepsilon_{\mathcal{K}}]\ [\gamma \circ \beta] = [\gamma \circ \beta \circ \varepsilon_{\mathcal{K}}] = [\beta \circ \varepsilon_{\mathcal{K}}]\ [\gamma] =$$
$$= [\beta \circ \varepsilon_{\mathcal{K}}]\ \phi^{-1}\ [\eta F_{\mathcal{K}}] = [\varepsilon]\ \phi^{-1}\ [\eta F_{\mathcal{K}}] = [\varepsilon_{\mathcal{K}}]\ [\eta F_{\mathcal{K}}] .$$

Since $[\varepsilon_{\mathcal{K}}]$ is an embedding, we have, hence, $[\gamma \circ \beta] = [\eta F_{\mathcal{K}}]$. Since

$$(\gamma \circ \beta)_X,\ (\eta F_{\mathcal{K}})_X : F\mathcal{K}(X) \to F'\mathcal{K}(X)$$

and since $F'\mathcal{K}(X)$ is a $T_0$-space, we obtain by 1.6 finally $\eta F_{\mathcal{K}} = \gamma \circ \beta$. Thus, the diagram (2) commutes.

4.10. REMARK. On the other hand, if (2) in 4.9 commutes and if $\varepsilon$ is an epi-transformation, then it induces a full embedding onto $\mathcal{K}$. Really, $\beta$ is then necessarily an epi-transformation and we have $[\eta F_{\mathcal{K}}] = [\beta]\ [\gamma]$. Since $[\eta F_{\mathcal{K}}]$ is an isofunctor and $[\beta]$ a full embedding, $[\beta]$ is an isofunctor.

## 5. Simply bireflective subcategories.

In this paragraph, we are going to characterize the epi-transformations $\varepsilon$ such that $[\varepsilon]$ is a full embedding onto a simply coreflective subcategory. By 2.4 and 2.7, every $[\varepsilon]$ is a full embedding onto a simply reflective subcategory, by 4.4 and 4.7 every simply bireflective (i.e. simply reflective and simply coreflective) subcategory of an $S(F)$ is represented by an embedding $[\varepsilon]$. Thus, we will obtain a characterization of all simply bireflective subcategories of $S(F)$.

5.1. LEMMA. *Let $F, G: Set \to QD\,Top$ be functors and $\tau: F \to G$ a transformation. Let $R: S(F) \to S(G)$ be a functor such that there is a natural equivalence*

$$\kappa_{xy} : S(F)([\tau](x), y) \cong S(G)(x, R(y))$$

*such that*

$U'(\kappa_{\bullet}(\phi)) = U(\phi)$ *for* $U$, $U'$ *the natural forgetful functors.*

*Then* $(X, r') \cong R(X, r) = (X, C(r))$, *where*

$$C(r) = \cup\{\mathcal{O}_{\mathfrak{p}}\, b \mid \tau_X^{-1}(\mathcal{O}_{\mathfrak{p}}\, b) \subset r\}.$$

PROOF. Since $\tau_X^{-1}(Cr) \subset r$, $1_X$ carries a morphism $[\tau](X, Cr) \to (X, r)$ and hence it carries also a morphism $(X, Cr) \to (X, r')$. Thus, $Cr \subset r'$. On the other hand, if $b \in r'$, $1_X$ carries a morphism $(X, \mathcal{O}_{\mathfrak{p}}\, b) \to R(X, r)$. Consequently, $\tau_X^{-1}(\mathcal{O}_{\mathfrak{p}}\, b) \subset r$, and hence $b \in Cr$.

5.2. REMARK. For the $R$ from 5.1 we have $U'R = U$. The formula

$$U'R(\xi) = U(\xi) \quad \text{for objects } \xi$$

is obvious immediately. Now consider a

$$\phi = (r, f, s): (X, r) \to (Y, s).$$

We have

$$\dot{R}(\phi) = S(G)(1, R(\phi))\kappa_{\bullet}\, \kappa_{\bullet}^{-1}(1) =$$

$$= \kappa_{\bullet}(S(F)(1, \phi)(\kappa_{\bullet}^{-1}(1))) = \kappa_{\bullet}(\phi\kappa_{\bullet}^{-1}(1)).$$

Thus

$$U'R(\phi) = U'(\kappa_{\bullet}(\phi\kappa_{\bullet}^{-1}(1))) = U(\phi)U(\kappa_{\bullet}^{-1}(1)) = U(\phi).$$

5.3. THEOREM. *Let* $F, G: Set \to QD\,Top$ *be functors,* $\varepsilon: F \to G$ *an epitransformation.* $[\varepsilon]$ *is a full embedding onto a simply coreflective (and, hence, simply bireflective) subcategory iff:*

(A) *for every* $f: X \to Y$ *and every* $b \in G(X)$,

$$\mathcal{O}_{\mathfrak{p}}\, F(f)\, \varepsilon_X^{-1}(\mathcal{O}_{\mathfrak{p}}\, b) = \varepsilon_Y^{-1}(\mathcal{O}_{\mathfrak{p}}\, G(f)(b)).$$

PROOF. Let $[\varepsilon]$ be a full embedding onto a simply coreflective subcategory. Take an $f: X \to Y$ and a $b \in G(X)$. We have

$$\mathcal{O}_{\mathfrak{p}}\, F(f)\, \varepsilon_X^{-1}(\mathcal{O}_{\mathfrak{p}}\, b) \subset \varepsilon_Y^{-1}(\mathcal{O}_{\mathfrak{p}}\, G(f)(b)),$$

since, if

$$y \in F(f)(\varepsilon_X^{-1}(\mathcal{O}_{\mathfrak{p}}\, b)) \quad \text{and} \quad z \in \varepsilon_X^{-1}(\mathcal{O}_{\mathfrak{p}}\, b)$$

is such that $F(f)(z) = y$, then

$$\varepsilon_Y(y) = G(f)\,\varepsilon_X(z) \in G(f)(\mathcal{O}_{\mathfrak{p}}\, b) \subset \mathcal{O}_{\mathfrak{p}}\, G(f)(b).$$

Put $r = \mathcal{O}_{\mathfrak{p}}\, F(f)(\varepsilon_X^{-1}(\mathcal{O}_{\mathfrak{p}}\, b))$. Then $f$ carries a morphism

$$[\varepsilon](X, \mathcal{O}_{\mathfrak{p}}\, b) \to (Y, r) \quad \text{in } S(F)$$

and hence also a morphism

$$(X, \mathcal{O}_{\mathfrak{p}}\, b) \to R(Y, r) = (Y, Cr) \quad \text{in } S(G)$$

( $C$ from 5.1), so that in particular $G(f)(b) \in Cr$, i.e.

$$\varepsilon_Y^{-1}(\mathcal{O}_{\mathfrak{p}}\, G(f)(b)) \subset r = \mathcal{O}_{\mathfrak{p}}\, F(f)\,\varepsilon_X^{-1}(\mathcal{O}_{\mathfrak{p}}\, b).$$

Now, let (A) hold. Define $R: S(F) \to S(G)$ by $R(r, f, s) = (Cr, f, Cs)$. (This is correct:

$$G(f)(Cr) = G(f)(\cup\{\mathcal{O}_{\mathfrak{p}}\, b \mid \varepsilon_X^{-1}(\mathcal{O}_{\mathfrak{p}}\, b) \subset r\}) =$$
$$= \cup\{G(f)(\mathcal{O}_{\mathfrak{p}}\, b) \mid \varepsilon_X^{-1}(\mathcal{O}_{\mathfrak{p}}\, b) \subset r\} \subset \cup\{\mathcal{O}_{\mathfrak{p}}\, G(f)(b) \mid \varepsilon_X^{-1}(\mathcal{O}_{\mathfrak{p}}\, b) \subset r\}$$
$$\subset \cup\{\mathcal{O}_{\mathfrak{p}}\, G(f)(b) \mid \mathcal{O}_{\mathfrak{p}}\, F(f)\,\varepsilon_X^{-1}(\mathcal{O}_{\mathfrak{p}}\, b) \subset s\} =$$
$$= \cup\{\mathcal{O}_{\mathfrak{p}}\, G(f)(b) \mid \varepsilon_Y^{-1}(\mathcal{O}_{\mathfrak{p}}\, G(f)(b)) \subset s\} \subset Cs\,.)$$

We have $F(f)(\varepsilon_X^{-1}(r)) \subset s$ iff $\mathcal{O}_{\mathfrak{p}}\, F(f)\,\varepsilon_X^{-1}(r) \subset s$ iff

$$\cup\{\mathcal{O}_{\mathfrak{p}}\, F(f)\,\varepsilon_X^{-1}(\mathcal{O}_{\mathfrak{p}}\, b) \mid b \in r\} \subset s$$

iff

$$\cup\{\varepsilon_Y^{-1}(\mathcal{O}_{\mathfrak{p}}\, G(f)(b)) \mid b \in r\} \subset s$$

iff

$$\varepsilon_Y^{-1}(\mathcal{O}_{\mathfrak{p}}\, G(f)(r)) \subset s \quad \text{iff} \quad \mathcal{O}_{\mathfrak{p}}\, G(f)(r) \subset Cs.$$

Thus, $f$ carries a morphism $[\varepsilon](x) \to y$ iff it carries a morphism $x \to R(y)$.

5.4. REMARKS. 1° From the first part of the proof of 5.3 we see that the inclusion

$$\mathcal{O}_{\mathfrak{p}}\, F(f)\,\varepsilon_X^{-1}(\mathcal{O}_{\mathfrak{p}}\, b) \subset \varepsilon_Y^{-1}(\mathcal{O}_{\mathfrak{p}}\, G(f)(b))$$

holds for any $\varepsilon$. Thus, the condition (A) is equivalent to the reverse inclusion

$$\varepsilon_Y^{-1}(\mathcal{O}_{\mathfrak{p}}\, G(f)(b)) \subset \mathcal{O}_{\mathfrak{p}}\, F(f)\,\varepsilon_X^{-1}(\mathcal{O}_{\mathfrak{p}}\, b).$$

Rewriting this, we obtain the following condition on $\varepsilon$ equivalent to (A):

(B) *For every $f: X \to Y$, every $a \in F(Y)$ and every $b \in G(X)$ such that*

$\varepsilon_Y(a) \leqslant G(f)(b)$, *there is a* $c \in F(X)$ *such that*

$$a \leqslant F(f)(c) \quad \textit{and} \quad \varepsilon_X(c) \leqslant b.$$

2° In the case of an $\varepsilon: F \to G$ such that $F$ and $G$ have discrete values, the condition (A) reduces to:

*For every* $f: X \to Y$ *and for every* $b \in G(X)$,

$$F(f)(\varepsilon_X^{-1}(b)) = \varepsilon_Y^{-1}(G(f)(b)).$$

5.5. PROPOSITION. *Let* $\varepsilon: F \to G$ *satisfy* (A). *Then, whenever* $F(f)$ *is an open mapping,* $G(f)$ *is also an open mapping.*

PROOF. We have

$$G(f)(\mathcal{O}_{\mathfrak{p}} b) = G(f)\,\varepsilon_X \varepsilon_X^{-1}(\mathcal{O}_{\mathfrak{p}} b) = \varepsilon_Y F(f)\,\varepsilon_X^{-1}(\mathcal{O}_{\mathfrak{p}} b).$$

If $F(f)$ is an open mapping, we continue

$$\ldots = \varepsilon_Y \mathcal{O}_{\mathfrak{p}} F(f)\,\varepsilon_X^{-1}(\mathcal{O}_{\mathfrak{p}} b) = \varepsilon_Y\,\varepsilon_Y^{-1} \mathcal{O}_{\mathfrak{p}} G(f)(b) = \mathcal{O}_{\mathfrak{p}} G(f)(b).$$

5.6. *Simply bireflective subcategories of the category of sets with A-nary relations*: Let $A$ be a set. An $A$-nary relation on a set $X$ is a subset of $X^A$; if $r$ (resp. $s$) is an $A$-nary relation on $X$ (resp. $Y$), a mapping $f: X \to Y$ is an $rs$-homomorphism if

$$\alpha \in r \text{ implies } f \circ \alpha \in s.$$

Thus, the category of sets with $A$-nary relations and their homomorphisms coincides with $S(Q_A)$ where

$$Q_A(X) = X^A, \quad Q_A(f)(\alpha) = f \circ \alpha,$$

the topology on $Q_A(X)$ being discrete. By 4.7 and 5.4, we will obtain a complete list of bireflective subcategories of $S(Q_A)$ if we list all the epi-transformations $\varepsilon: Q_A \to G$ satisfying (A) and such that the underlying mappings of $\varepsilon$ are the identities.

Consider such an epi-transformation. Since $Q_A(X)$ are discrete, we have all $Q_A(f)$ open and, hence, by 5.6, all

$$G(f): G(X) = (X^A, \leqslant) \to G(Y) = (Y^A, \leqslant)$$

are open. Thus, in particular, if $\alpha \leqslant \beta$ in $G(X)$, then, since we have $\beta = G(\beta)(1_A)$, there is a $\phi \leqslant 1_A$ in $G(A)$ such that

$$\alpha = G(\beta)(\phi) = \beta \circ \phi .$$

Conversely, of course, if $\phi \leqslant 1_A$, we have necessarily

$$\beta \circ \phi = G(\beta)(\phi) \leqslant G(\beta)(1) = \beta .$$

Thus, in particular, if $\phi' \leqslant 1_A$ and $\psi \leqslant 1_A$, we have

$$\phi \circ \psi \leqslant \phi \leqslant 1_A .$$

Hence, there is a submonoid $M$ of $A^A$ such that

(1) $\alpha \leqslant \beta$ in $G(X)$ iff $\exists \phi \in M,\ \alpha = \beta \circ \phi$ .

On the other hand, let there be given a submonoid $M$ of $A^A$ and let us put $G(X) = (X^A, \leqslant)$ with $\leqslant$ defined by the formula (1), $G(f)(\alpha) = f \circ \alpha$ . (Obviously, thus defined $\leqslant$ is transitive, and

$$\alpha \leqslant \beta \text{ implies } G(f)(\alpha) \leqslant G(f)(\beta) .)$$

Define $\varepsilon: Q_A \to G$ putting $\varepsilon_X(\alpha) = \alpha$. If $\varepsilon_Y(\alpha) \leqslant G(f)(\beta)$, i.e. if $\alpha \leqslant f \circ \beta$, we have $\alpha = f \circ \beta \circ \phi$ for a $\phi \in M$. Put $\gamma = \beta \circ \phi$. Then

$$\alpha = G(f)(\gamma) \text{ and } \varepsilon_X(\gamma) = \gamma \leqslant \beta .$$

Hence, the condition (B) is satisfied.

Thus, we conclude that *the simply bireflective subcategories* $\mathcal{K}$ *of* $S(Q_A)$ *are exactly those obtained as follows: A submonoid* $M$ *of* $A^A$ *is given, and an object* $(X, r)$ *of* $S(Q_A)$ *is in* $\mathcal{K}$ *iff*

$$\alpha \circ \phi \in r, \textit{ for every } \alpha \in r \textit{ and } \phi \in M.$$

Moreover, we see easily that, among them, the ones representable by an $\varepsilon: Q_A \to G$ with discrete $G(X)$ are exactly those where $M$ is a group. (By 1.8, $\phi \in M$, i.e. $\phi \leqslant 1$, implies $1 \leqslant \phi$ ; hence there exists a

$$\psi \in M \text{ with } 1 = \phi \circ \psi .)$$

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