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Dualities of concrete categories

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ET GEOMETRIE DIFFERENTIELLE

DUALITIES OF CONCRETE CATEGORIES

by Hans-E. PORST

ABSTRACT

We consider categories with a set of separators, i.e. with a faithful underlying functor to some power of $\underline{S}$, the category of sets. In case these categories are premonadic, we give a necessary and sufficient condition, which guaranties the existence of a duality with another category of this kind. Furthermore, we show - following an idea of Linton - how to construct this duality. There are further results on the abelianess and the rank of the involved categories if these are even algebraic.

Among the special cases of such dualities are those of Pontrjagin, Stone and Oberst.

INTRODUCTION

Many authors have been concerned with the examination of dualities by means of categorical methods during the last years. Though most of these use similar methods (monads and comparison functors), the starting points and the directions of examination are quite different: categorical description of known dualities [10], categorical generalization of a special duality [3], description of the dual of a varietal category under certain conditions [9], tripleability of the dual of certain abelian categories [8].

In this paper we start to answer the following more general questions (where the answers will include many of the results cited above): What are exactly the conditions such that there *exists* a duality between suitable nice categories? How a duality may be *constructed*, if it should exist? Why «all» dualities have to be constructed in the same way? (By duality we always think of a duality «in both directions».)

In doing so, we first have to elect a class $\mathcal{K}$ of categories which should be large enough to contain the categories which are involved in well known dualities (e.g. the dualities of Pontrjagin, Stone, Oberst), and which on the other side in some sense is a class of «nice» categories. Here one might think of:

- suitable classes of abelian categories (as in [8]) - but then one would exclude Stone duality for example, and furthermore there is no canonical concept to construct equivalences between abelian categories ;

- locally presentable categories - but these do only admit trivial dualities [5] ;

- categories of sketched structures [1] ; this concept seems to be too general in this context;

- monadic (varietal) categories - but this class is too small (e.g. the category of Stone spaces is not monadic).

Convenient classes of categories arise if we consider certain classes of subcategories of the latter, as will be pointed out in n° 2 and n° 3.

The first of the questions above will be answered by means of internal characterizations of the involved categories, whereas we answer the second one generalizing the construction of [9] .

Though starting from an axiomatic point of view in developing this theory of dualities we do not omit the constructions being essential in the context of this problem, where these constructions, involving nothing else but the underlying functors of the categories in question, are done from the «point de vue algébriste» in the sense of Ehresmann's introduction to [4] .

Among the special cases of dualities arising from this theory are those of Pontrjagin (discrete and compact abelian groups), Gelfand (compact spaces) and the Oberst duality for Grothendieck categories. Furthermore it follows from this theory that there are no suitable dualities for a lot of nice categories as for example the category of groups or the category of (commutative) rings (with unit).

0. Notations and definitions, basic facts.

For any set I we denote by $\underline{S}^I$ the I-fold power of the category $\underline{S}$ of sets and mappings; $p_i : \underline{S}^I \to \underline{S}$ are the projection functors ($i \in I$). A pair $(\underline{A}, U)$ with a faithful functor $U: \underline{A} \to \underline{S}^I$ is called an *I-concrete category*; if the functors $p_i U$ are representable we call $(\underline{A}, U)$ *representable*. If $(\underline{A}, U)$ is a cocomplete I-concrete category, U has an adjoint iff all $p_i U$ ($i \in I$) have adjoints, that is iff $(\underline{A}, U)$ is representable; in this case the set of representing objects of the $p_i U$ is a set of separators of $\underline{A}$. If a category $\underline{A}$ has a set of separators I we denote by $\Upsilon: \underline{A} \to \underline{S}^I$ the functor determined by $p_i \Upsilon = \underline{A}(i, \cdot)$ for all $i \in I$; thus $(\underline{A}, \Upsilon)$ becomes a representable I-concrete category.

If $(\underline{A}, U)$ and $(\underline{B}, V)$ are I-concrete categories, $F: (\underline{A}, U) \to (\underline{B}, V)$ denotes a functor $F: \underline{A} \to \underline{B}$ over $\underline{S}^I$. If such a functor has an adjoint G and the involved categories are representable, the separators A_i of $\underline{A}$ and B_i of $\underline{B}$ correspond via G ($GB_i = A_i$).

In slight generalization of [6] an I-concrete category $(\underline{A}, U)$ is called *algebraic* iff $\underline{A}$ has coequalizers, U has an adjoint and U preserves and reflects regular epimorphisms. I-algebraic categories have the same properties with respect to limits, colimits and factorizations as in the special case $|I| = 1$.

If $(\underline{A}, U)$ is an I-concrete category and admits an adjoint situation $F \dashv U$, the induced monad on $\underline{S}^I$ is denoted by $\langle U \rangle$, and the corresponding I-concrete Eilenberg-Moore category by $(\langle \underline{U} \rangle, ||_U)$. Then C_U denotes the semantical comparison functor. We will omit the subscript U if no misunderstanding is possible.

With respect to a class C of morphisms of some category $\underline{B}$ we will use the self-understanding notions of *C-projective* objects and *C-separating sets* if $\underline{B}$ has coproducts. $E_{\underline{B}}$ or simply E ($rE_{\underline{B}}$ or rE) will denote the class of (regular) $\underline{B}$-epimorphisms, $M_{\underline{B}}$ (rM) the class of (regular) $\underline{B}$-monomorphisms.

Let $(\underline{A}, U)$ and $(\underline{B}^{op}, V')$ be I-concrete categories and let $(\underline{B}, V)$ and $(\underline{A}^{op}, U')$ be J-concrete. If there are functors

$$S : (\underline{A}^{op}, U') \to (\underline{B}, V) \quad \text{and} \quad T : (\underline{B}^{op}, V') \to (\underline{A}, U)$$

such that

$$1_{\underline{A}} \cong T S^{op} : (\underline{A}, U) \to (\underline{A}, U) \quad \text{and} \quad 1_{\underline{B}} \cong S T^{op} : (\underline{B}, V) \to (\underline{B}, V),$$

we call (S, T) *a duality*, denoted by

$$S : (\underline{A}, U) \rightleftarrows (\underline{B}, V) : T .$$

1. Dualities of representable I-concrete categories.

Of course it is of no interest to establish a duality-theory for a class of I-concrete categories, but it can be shown that even for these classes the form of the duality functors and duality transformations can be described uniquely.

PROPOSITION 1. *Let $\underline{A}$ and $\underline{B}$ be categories and $I \subset ob\underline{A}$, $J \subset ob\underline{B}$, be sets of non-isomorphic objects such that $(\underline{A}, \Upsilon)$ is representable I-concrete and $(\underline{B}, \Upsilon)$ is representable J-concrete. Furthermore let $S : \underline{A}^{op} \to \underline{B}$ and $T : \underline{B}^{op} \to \underline{A}$ be functors such that*

$$1_{\underline{A}} \cong T S^{op} \quad \text{and} \quad 1_{\underline{B}} \cong S T^{op}.$$

Then the following assertions hold with respect to the sets

$$TJ = \{ Tj \mid j \in J \} \subset ob\underline{A} \ (\cong J)$$

and

$$SI = \{ Si \mid i \in I \} \subset ob\underline{B} \ (\cong I).$$

(i) TJ (resp. SI) is a coseparating set of $\underline{A}$ (resp. $\underline{B}$) of ΥS^{-1}- (resp. ΥT^{-1}-)injectives.

(ii) $\Upsilon S = \widetilde{TJ}$ and $\Upsilon T = \widetilde{SI}$.

(iii) $p_i \Upsilon(Tj) = p_j \Upsilon(Si)$.

The meaning of these assertions might be clearer by the following remarks: (i) and (ii) say that

$$S : (\underline{A}, \Upsilon) \rightleftarrows (\underline{B}, \Upsilon) : T$$

is a duality by means of the J- (resp. I-)concrete categories $(\underline{A}^{op}, \widetilde{TJ})$ and $(\underline{B}^{op}, \widetilde{SI})$, where (ii) alone says that duality functors are to be constructed as morphism-functors. (iii) says, in the special case $|I| = |J| = 1$, that the objects which define the duality functors have the same underlying set.

PROOF. Considering T^{op} as an adjoint of S, one gets (ii) and (iii). (i) follows from the properties of free objects over a singleton set.

The following corollary we write down only in the case

$$I = \{ A_1 \}, \quad J = \{ B_1 \} \text{ and } U = \tilde{I}, \quad V = \tilde{J}.$$

COROLLARY. *If under the conditions of Proposition 1 the category $\underline{B}$ has products, then there is a natural monomorphism $m: S \to S A_1^{U\text{-}}$ such that each $V m_A$ corresponds to an injective map $V S A \to \underline{S}(U A, U T B_1)$.*

PROOF. Let m_A be the composition of the canonical monomorphism

$$S A \to S A_1^{\underline{B}(SA, SA_1)}$$

and the isomorphism induced by the bijection $\underline{B}(S A, S A_1) \cong U A$.

We can also describe the duality transformation on this level. For the sake of lucidity we again do this in the case $|I| = |J| = 1$:

PROPOSITION 2. *Under the conditions of Proposition 1 the unit λ of the adjunction $T \dashv S$ serves as the duality transformation, i. e. it holds the formula*

$$U(V \lambda_B(x))(f) = f \circ x,$$

where $B \in ob \underline{B}$, $f \in \underline{B}(B, S A_1)$, $x \in \underline{B}(B_1, B)$.

PROOF. To let this formula make sense we first have to identify

$$V S T B = \underline{A}(T B, T B_1) \quad \text{and} \quad \underline{B}(B_1, S T B)$$

via the adjunction. Thus we have immediately $V \lambda_B(x) = T x$ and from this the assertion.

Just for the sake of later specifications we write down the following obvious assertion on the existence of dualities.

PROPOSITION 3. *The representable I-concrete category $(\underline{A}, U)$ admits a duality with a representable J-concrete category iff $\underline{A}$ has a coseparating set with cardinality $|J|$.*

2. Dualities of I-premonadic categories.

In order to construct a duality the formal assertion of Proposition 3 is not very helpful; for example it is of no interest to know that the concrete category of sets $(\underline{S}, 1_{\underline{S}})$ admits a duality with $(\underline{S}^{op}, \underline{S}(\cdot, \{1, 2\}))$; yet some importance is arising when one can interprete this concrete category as the concrete category of complete atomic Boolean algebras. So we should try to answer the question how to find an I-concrete category within a category of algebras again. If we restrict ourselves to a cocomplete representable I-concrete category $(\underline{A}, U)$, there comes out a connection between $(\underline{A}, U)$ and the Eilenberg-Moore category $(<\underline{U}>, |\;|_U)$ by means of the comparison functor C_U which is faithful and has an adjoint. But in general it is not possible to identify $(\underline{A}, U)$ with a subcategory of $(<\underline{U}>, |\;|_U)$ via C, because C might identify non-isomorphic $\underline{A}$-objects, as the example of topological spaces shows. So we should restrict ourselves to *I-premonadic categories* (e.g. [14]), which are characterized by the following equivalences.

PROPOSITION 4. *Let $(\underline{A}, U)$ be an I-concrete category such that U has an adjoint. Then the following assertions are equivalent:*

(i) $(\underline{A}, U)$ is equivalent with a full reflexive subcategory of

$$(<\underline{U}>, |\;|_U).$$

(ii) $\underline{A}$ is cocomplete and $C_U : \underline{A} \to <\underline{U}>$ is full and faithful (and has an adjoint).

(iii) $\underline{A}$ is cocomplete and U reflects regular epimorphisms.

(iv) $\underline{A}$ is cocomplete and has an rE-separating set I such that $U = \tilde{I}$.

PROOF. (i) and (ii) are obviously equivalent.

For (ii) $\Longleftrightarrow$ (iii) and (ii) $\Longrightarrow$ (iv), see [14], 10.1.

(iv) $\Longrightarrow$ (iii) also follows from [14], 10.1, and the fact that the canonical epimorphism from the coproduct of the separators to an object A is the counit of the adjunction defined by U.

As a corollary we have, in analogy to Proposition 3:

PROPOSITION 5. *The I-premonadic category* $(\underline{A}, U)$ *admits a duality with a J-premonadic category iff* $\underline{A}$ *has an rM-coseparating set* J.

Furthermore we are able to construct this duality as follows. Let us start with an I-premonadic category $(\underline{A}, \tilde{I})$ and an rM-coseparating set J of objects of $\underline{A}$. Thus we get the J-premonadic category $(\underline{A}^{op}, \tilde{J})$ and the comparison functor $C_{\tilde{J}}$ serves as an equivalence (after restriction on its image) of the latter with a full reflexive subcategory of $(\langle \tilde{J} \rangle, |\,|_{\tilde{J}})$ which is J-premonadic too, and will be denoted by $(\underline{CA^{op}}, |\,|_{\tilde{J}})$.

$$CI := C_{\tilde{J}} I := \{ Ci \mid i \in I \}$$

is an rM-coseparating set in this category, such that we can start the same procedure again. Thus we get an I-premonadic category $((\underline{CA^{op}})^{op}, \widetilde{CI})$ which is equivalent with a full reflexive subcategory

$$(\underline{C(CA^{op})^{op}}, |\,|_{\widetilde{CI}}) \quad \text{of} \quad (\langle \underline{\widetilde{CI}} \rangle, |\,|_{\widetilde{CI}})$$

by means of $C_{\widetilde{CI}}$. We claim this category to be equivalent to $(\underline{A}, \tilde{I})$: of course

$$C_{\widetilde{CI}} C_{\tilde{J}}^{op} : \underline{A} \to (\underline{CA^{op}})^{op}$$

is an equivalence, and it is an equivalence of the I-concrete categories in question, too, respecting the representing objects of the underlying functors. These results legitimate the following notion:

$$(\underline{A}, \tilde{I})_J^{op} := (\underline{CA^{op}}, |\,|_{\tilde{J}}).$$

We may summarize our results, up to a little calculation, as follows:

THEOREM. *Let* $(\underline{A}, \tilde{I})$ *be an I-premonadic category and* $J \subset ob\,\underline{A}$ *an rM-coseparating set. The following assertions hold:*

(i) There is an equivalence $(\underline{A}, \tilde{I}) \xrightarrow{\sim} ((\underline{A}, \tilde{I})_J^{op})_{CI}^{op}$.

(ii) There is (neglecting the equivalence of (i)) the duality:

$$C_{\tilde{J}} : (\underline{A}, \tilde{I}) \rightleftarrows (\underline{A}, \tilde{I})_J^{op} : C_{\widetilde{CI}}.$$

3. Dualities of I-algebraic categories.

In this paragraph we are concerned with the problem to find conditions on an I-premonadic category such that it admits a duality not only with a J-premonadic one, but with a J-algebraic category. In this context we will be able to give a more precise description of the category sought after. First we have the more or less folklore

PROPOSITION 6. *Let $(\underline{A}, U)$ be an I-concrete category such that U has an adjoint. Then the following assertions are equivalent:*

(i) $(\underline{A}, U)$ is equivalent with a full regular-epi-reflexive subcategory of $(<\underline{U}>, |\,|_U)$.

(ii) $\underline{A}$ is cocomplete and $C_U : \underline{A} \to <\underline{U}>$ serves as an embedding of a full reflexive subcategory, which is closed under products and subobjects.

(iii) $\underline{A}$ is an I-algebraic category.

(iv) $\underline{A}$ is cocomplete and has an rE-separating set I of rE-projectives such that $U = \tilde{I}$.

Again we have as a corollary:

PROPOSITION 7. *The I-premonadic category $(\underline{A}, U)$ admits a duality with a J-algebraic category iff $\underline{A}$ has an rM-coseparating set of rM-injectives with cardinality $|J|$.*

The duality might be constructed as in the above theorem.

The following more precise description is essentially due to Linton [9] :

PROPOSITION 8. $(\underline{B}, V) = (\underline{A}, U)_J^{op}$ *is the full subcategory of $(<\underline{\tilde{I}}>, |\,|_{\tilde{J}})$ which is (regularly) cogenerated by CI.* (*)

PROOF. $\underline{CA}^{op}$ consists exactly of the (regular) subobjects of products of elements of CI, because $\underline{CA}^{op}$ is closed under subobjects and C preserves products and equalizers.

The regularity condition on the (co-)separators in this context is

(*) $\underline{B}$ might also be described in terms of a localization functor in the sense of [8].

fulfilled trivially if the categories in question are balanced or even abelian as it is the case in many examples. The key lemma on dualities of balanced algebraic categories is the following slight generalization of a result of [9].

LEMMA. *Let* $\underline{A}$ *be a locally small complete category with an* rM*-coseparating set such that extremal epimorphisms are regular. Then the following assertions are equivalent:*

(i) $M_{\underline{A}} = rM_{\underline{A}}$.

(ii) $E_{\underline{A}} = rE_{\underline{A}}$.

(iii) $\underline{A}$ *is balanced.*

PROOF. For the non-trivial parts use the (extremal-epi, mono)-factorization (resp. the dominion factorization) as in [9].

From this lemma and the preceding facts, then one has immediately:

PROPOSITION 9. *Let* $(\underline{A}, U)$ *be an* I*-algebraic category admitting a duality with a* J*-algebraic category. Then* $\underline{A}$ *is abelian iff* $\underline{A}$ *is balanced and if* $|p_i\, U F(\emptyset)| = 1$ *(where* $(\emptyset)$ *is the initial object of* $\underline{S}^I$*) for all* $i \in I$.

There are dualities of balanced algebraic categories which are not abelian (e.g. Stone duality). Therefore the following consequence of the preceding facts is of some interest:

PROPOSITION 10. *If* $(\underline{A}, U)$ *is an* I*-algebraic category, then the following conditions are equivalent:*

(i) U *preserves epimorphisms and* $\underline{A}$ *has a coseparating set* J *of injectives.*

(ii) $\underline{A}$ *is balanced and has a coseparating set* J *of injectives.*

(iii) $\underline{A}^{op}$ *is* J*-algebraic with respect to an epimorphism preserving functor* V.

Furthermore it should be mentioned in this context that an I-algebraic category which is abelian even is an Eilenberg-Moore category [8]. Therefore in constructing a duality for an abelian I-algebraic category $(\underline{A}, U)$ with a coseparating set J of injectives we have not only

$$(\underline{A}, U)^{op}_J \subset (\langle \tilde{\underline{I}} \rangle, |\,|_{\tilde{J}}),$$

but even

$$(\underline{A}, U)_J^{op} \cong (<\underline{\tilde{T}}>, |\,|_{\tilde{J}}).$$

This situation for instance is given in case of Pontrjagin and Roos-Oberst dualities; especially the last part of the Negrepontis proof of Pontrjagin duality might be omitted.

4. Duality and rank.

The well known examples of dualities show that in case one of the categories in question has a rank in some sense (e.g. the category of abelian groups) the other one hasn't (e.g. the category of compact abelian groups). Gabriel and Ulmer did prove this in general for locally presentable categories. Here we consider categories with a more general notion of rank, following [6].

DEFINITION. An I-concrete category $(\underline{A}, U)$ *has a rank* $\leqslant k$ (k any infinite regular cardinal number) iff, for any k-direct limit (l_j, L) in $\underline{A}$, then (Ul_j, UL) is an epi-sink (i.e. if $gUl_j = fUl_j$ for all j, then $g = f$).

It is well known that this definition is equivalent to the usual notion of rank in case of Eilenberg-Moore categories, but that it is a weaker notion in case of I-algebraic categories [6].

PROPOSITION 11 [12]. *An I-algebraic category $(\underline{A}, U)$ has a rank iff the Eilenberg-Moore category $(<\underline{U}>, |\,|_U)$ has a rank.*

From this we may get the Gabriel-Ulmer result, also for I-algebraic categories:

PROPOSITION 12. *If $(\underline{A}, U)$ is an I-algebraic category with rank and if $(\underline{A}^{op}, V)$ is J-algebraic with rank, then each $\underline{A}$-object has exactly one endomorphism.*

PROOF. Carry out the Gabriel-Ulmer construction in the categories

$$(<\underline{U}>, |\,|_U), \text{ resp. } (<\underline{V}>, |\,|_V),$$

which both are locally presentable by Proposition 11, and «reflect» it to $\underline{A}$

using the fact that the unit of the reflection is a regular epimorphism.

From this we get:

THEOREM. *If* $(\underline{A}, U)$ *is an* I*-algebraic category with rank such that* $\underline{A}^{op}$ *is* J*-algebraic with rank with respect to some functor* V*, then there is a subset* $I' \subset I$ *such that* $\underline{A} \cong \mathcal{P}(I')$.

PROOF. If (n_i) is an arbitrary $\underline{S}^I$-object, then from Proposition 12 follows $|\underline{S}^I((n_i), UF(n_i))| = 1$ and from this

$$|p_j UF(n_i)| \leqslant 1 \quad \text{and} \quad |p_j UF(n_i)| = 1 \text{ if } n_j \neq \emptyset$$

and of course

$$|p_j UF(n_i)| = |p_j UF(m_i)| \quad \text{if } n_j = m_j.$$

Take

$$I' = \{ i \in I \mid p_i UF(\emptyset) \neq \emptyset \}$$

and

$$T(F(n_i)) = \{ j \in I' \mid p_j UF(n_i) \neq \emptyset \}.$$

Since each $\underline{A}$-object is a regular quotient of a free one, one may extend T to all $ob\,\underline{A}$ so that

$$TA = TA' \text{ iff } A \cong A'.$$

T then serves as an equivalence $\underline{A} \cong \mathcal{P}(I')$.

5. Examples.

a) Pontrjagin duality: both involved categories are abelian Eilenberg-Moore categories over $\underline{S}$.

b) Stone duality: both involved categories are balanced but not abelian; the Stone spaces do only form an algebraic category.

c) Oberst-Roos duality: the Grothendieck category is I-premonadic ($|I| > 1$ in general); its dual is algebraic and thus, being abelian, it is an Eilenberg-Moore category. The Oberst construction is canonical as the formulas of Proposition 1 show.

d) Tannaka duality: the used coseparating set has cardinality $\aleph_0$.

e) Let $\underline{V_F}$ be the category of F-vectorspaces for some field F. Using F as an injective coseparator of $\underline{V_F}$, one gets a duality with an abelian Eilenberg-Moore category which is not a category of modules because it has no rank.

f) The category of left exact abelian group-valued functors on a small abelian category admits a duality with an abelian Eilenberg-Moore category.

g) There are no suitable dualities available for the categories of monoids, semigroups, groups, (commutative) rings (with unit), because these categories don't have coseparating sets [2].

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