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NATURAL FACTORIZATIONS AND THE KAN EXTENSION OF COHOMOLOGY THEORIES

by John L. MACDONALD

1. Introduction.

The question of when a Kan extension of a cohomology theory is again a cohomology theory has been investigated by Deleanu and Hilton in $[2,3]$. The *cohomology theories* which they consider each consist of a sequence $h = \{ h^n \}$ of contravariant functors from an «admissible» category J_0 of topological spaces to the category Ab of abelian groups, together with a sequence

$$\sigma = \{ \sigma^n \mid \sigma^n : h^n \to h^{n+1} \Sigma \}$$

of natural transformations where Σ is the suspension functor, all subject to the conditions:

(1) If $f \overset{\sim}{-} g$ then $h^n(f) = h^n(g)$ (Homotopy axiom).

(2) $\sigma^n : h^n \overset{\sim}{\sim} h^{n+1} \Sigma$. (Suspension axiom).

(3) If $X' \to X \to X''$ is a mapping cone sequence in J_0, then

$$h^n(X') \leftarrow h^n(X) \leftarrow h^n(X'')$$

is exact (Exactness axiom).

An *admissible* category J_0 of topological spaces is a full subcategory of the category of all pointed topological spaces which is non-empty, closed under construction of mapping cones and contains complete homotopy types.

The examples of such theories are known to include ordinary cohomology theory, stable cohomotopy theory, real and complex K-theory plus a number of other examples described in $[4]$.

In this paper we introduce the notion of natural factorizations in order to describe an abstract or archetypal cohomology axiom of which the three

preceding axioms may be regarded as examples. The aim of this description is to obtain cofinality conditions which will ensure that the Kan extensions of a functor satisfying an archetypal axiom again satisfies that axiom.

We then translate the abstract cofinality condition on the Kan extension of the archetype into the language of the three axiom examples, with an added twist for the exactness axiom. These conditions result in theorems for extending cohomology theories which include those of Deleanu and Hilton.

Furthermore we work in abstract categories $\mathcal{P}$ and $\mathcal{C}$ instead of the categories J_0 and Ab, respectively, in order to analyze better the conditions which are essential in the constructions. In this way we can see, for example, why the homotopy and suspension axioms extend for quite general $\mathcal{P}$ and $\mathcal{C}$ and why the exactness axiom needs some restrictions on $\mathcal{C}$.

We next introduce some background information which will be needed in the later sections. Accordingly, we let $K : \mathcal{P} \to \mathcal{T}$ be a functor, and let $b : \mathcal{P} \to \mathcal{C}$ be a contravariant functor. The *left contravariant Kan extension of b along K* is a functor $b_1 : \mathcal{T} \to \mathcal{C}$ defined on objects X in $\mathcal{C}$ by

$$b_1 X = \varinjlim ((X \downarrow K) \xrightarrow{D_X} \mathcal{P} \xrightarrow{b} \mathcal{C})$$

with the obvious definition on morphisms derived from the universal property of $\varinjlim$. We observe that an object of the *comma category* $(X \downarrow K)$ is a $\mathcal{T}$-morphism $f : X \to K Y$ and that if $g : X \to K Y'$ is another $(X \downarrow K)$-object, then a morphism $u : f \to g$ of $(X \downarrow K)$ is a morphism

$$u : Y \to Y' \text{ of } \mathcal{P} \text{ with } K u \circ f = g.$$

The functor D_X projects

$$f : X \to K Y \quad \text{to} \quad Y \quad \text{and} \quad u : f \to g \quad \text{to} \quad u : Y \to Y'.$$

The comma category $(K \downarrow X)$ is defined similarly (see Mac Lane [5], p. 46).

A functor L is said to be *initial* if condition (a) of the following theorem holds.

THEOREM 1. *Let* $L : I' \to I$ *be a functor. Then the following are equivalent:*

(a) For each object i of I, the category $(L \downarrow i)$ is non-empty and connected.

(b) For every contravariant functor $F: I \to \mathcal{C}$ with $\mathcal{C}$ cocomplete, the canonical map $\varinjlim FL \to \varinjlim F$ is an isomorphism.

This theorem is proved simply by adapting the finality results of Mac Lane in [5], pages 213-14, to our setting.

2. The archetypal Cohomology axiom.

Let $h: \mathcal{P} \to \mathcal{C}$ be a contravariant functor and let $F: \mathcal{P} \to \mathcal{Q}$ be any (covariant) functor. The functor h is said to *satisfy the cohomology axiom of type (F, σ)* if there is a diagram

(2.1)

$$
\begin{array}{ccc}
 & \mathcal{P} & \\
 & {\Big\downarrow} \;\;\; \searrow^{F} & \\
h & & \mathcal{Q} \\
 & \overset{\sigma}{\Longrightarrow} \;\; \nearrow_{\bar{h}} & \\
 & \mathcal{C} &
\end{array}
$$

such that $\sigma: h \to \bar{h} F$ is a natural equivalence. If σ is a natural transformation (but not necessarily an equivalence) then we call σ a *natural F-factorization of h*. We remark that the Kan extension of h along F, if it exists, determines an F-factorization of h which is universal in the sense of [5], pages 232-33.

Suppose that

(2.2)

$$
\begin{array}{ccc}
\mathcal{P} & \overset{K}{\longrightarrow} & \mathcal{T} \\
F {\Big\downarrow} & & {\Big\downarrow} G \\
\mathcal{Q} & \underset{L}{\longrightarrow} & \mathcal{U}
\end{array}
$$

is a commutative diagram of categories and functors. Given $X \in |\mathcal{T}|$ we let $M_X : (X {\downarrow} K) \to (GX {\downarrow} L)$ be the functor defined by

$$M_X (f: X \to K Y) = Gf: GX \to GKY = LFY ,$$

for $f: X \to K Y$ an object of $(X {\downarrow} K)$ with the obvious definition on morphisms.

THEOREM 2. *Suppose that* $h: \mathcal{P} \to \mathcal{C}$ *is a functor with natural F-factorization* $\sigma: h \to \overline{h} F$ *where F is a functor appearing in diagram* (2.2). *Then the Kan extension* $h_1: \mathcal{T} \to \mathcal{C}$ *of h along K has a natural G-factorization*

(2.3)

$$
\begin{array}{c}
\mathcal{T} \\
h_1 \downarrow \quad \sigma_1 \searrow^{G} \mathcal{U} \\
\mathcal{C} \quad \nearrow (\overline{h})_1
\end{array}
$$

for $(\overline{h})_1: \mathcal{U} \to \mathcal{C}$ *the Kan extension of* $\overline{h}$ *along L and* σ_1 *the natural transformation given by the explicit formula*

$$(\sigma_1)_X = M_X^* \circ \varinjlim \sigma D_X:$$

$$h_1 X = \varinjlim h D_X \xrightarrow{\ \varinjlim \sigma D_X\ } \varinjlim \overline{h} F D_X =$$

$$= \varinjlim \overline{h} D_{GX} M_X \xrightarrow{\ M_X^*\ } \varinjlim \overline{h} D_{GX} = (\overline{h})_1 G X.$$

PROOF. The left square of the diagram

(2.4)

$$
\begin{array}{ccccc}
X\!\downarrow\!K & \xrightarrow{D_X} & \mathcal{P} & \xrightarrow{h} & \mathcal{C} \\
M_X \downarrow & & F \downarrow & \sigma \Big/\!\!\Big/ & \downarrow \\
GX\!\downarrow\!L & \xrightarrow[D_{GX}]{} & \mathcal{Q} & \xrightarrow[\overline{h}]{} & \mathcal{C}
\end{array}
$$

commutes and the right square represents the given natural factorization of h. Thus

$$\sigma D_X: h D_X \to \overline{h} F D_X = \overline{h} D_{GX} M_X.$$

Furthermore M_X induces the obvious morphism

$$M_X^*: \varinjlim \overline{h} D_{GX} M_X \to \varinjlim \overline{h} D_{GX}.$$

The result now follows immediately from the explicit formula for $(\sigma_1)_X$ plus an easy verification of naturality.

We remark that σ_1 is the unique natural transformation determined

by the fact that h_1 arises from a universal K-factorization of h .

If $h: \mathcal{P} \to \mathcal{C}$ has a natural factorization $(\bar{h} \, F, \sigma)$, then the associated canonical G-extension of h along K is the functor $(\bar{h})_1 \, G: \mathcal{J} \to \mathcal{C}$ described above. By Theorem 2 there is a natural transformation σ_1 from the Kan extension of h along K to the canonical extension. The next theorem shows when σ_1 is an equivalence.

THEOREM 3. *Suppose that* $h: \mathcal{P} \to \mathcal{C}$ *is a functor which satisfies the cohomology axiom of type* (F, σ) *for F a functor appearing in the commutative diagram* (2.2). *Then the Kan extension* $h_1: \mathcal{J} \to \mathcal{C}$ *satisfies the cohomology axiom of type* (G, σ_1) *provided that* $M_X: (X{\downarrow}K) \to (GX{\downarrow}L)$ *is initial for each* $X \in |\mathcal{J}|$ *and that* σ_1 *is the natural transformation from the Kan extension to the canonical G-extension described in Theorem 2.*

PROOF. By Theorem 2 there is a natural transformation

$$\sigma_1 : h_1 \to (\bar{h})_1 G \quad \text{with} \quad (\sigma_1)_X = M_X^* \circ \varinjlim \sigma D_X.$$

If M_X is initial, then by Theorem 1 it follows that M_X^* is an equivalence and if h satisfies the axiom of type (F, σ) , then σ and hence $\varinjlim \sigma D_X$ are equivalences. Thus σ_1 is a natural equivalence, as required.

We next consider how the crucial initiality condition for

$$M_X : (X{\downarrow}K) \to (GX{\downarrow}L)$$

can be formulated in terms of objects and morphisms drawn from the commuting diagram

(2.5)

$$
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\quad K \quad} & \mathcal{J} \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\
\mathcal{Q} & \xrightarrow[\quad L \quad]{} & \mathcal{U}
\end{array}
$$

of categories and functors. The purpose of this is to provide a procedure for testing in particular examples whether M_X is initial.

From Section 1 we recall that M_X is initial if and only if for each object k of $(GX{\downarrow}L)$ the comma category $(M_X{\downarrow}k)$ is non-empty and con-

nected. Thus it is obvious that M_X is initial if for each object k of $(GX{\downarrow}L)$ it follows that

(A) the category $(M_X{\downarrow}k)$ is non-empty, and

(B) any two objects u_1 and u_2 of $(M_X{\downarrow}k)$ can be connected by a pair of morphisms of the form $u_1 \to u \leftarrow u_2$.

If (A) and (B) hold, then we say that M_X *satisfies the initiality condition of type* $\to . \leftarrow$:

Condition (A) can be interpreted as follows:

A. Given any morphism $k : GX \to LZ$ of $\mathfrak{U}$ there exists a morphism $g : X \to KY$ in $\mathfrak{I}$ and a morphism $u : FY \to Z$ in $\mathfrak{Q}$ such that $u : M_X g \to k$ in $(GX{\downarrow}L)$, i. e. such that

(2.6)

$$
\begin{array}{ccc}
GX & \xrightarrow{\ \ Gg\ \ } & GKY = LFY \\
 & k \searrow & \downarrow Lu \\
 & & LZ
\end{array}
$$

commutes in $\mathfrak{U}$.

Condition (B) has the following interpretation:

B. For each diagram

(2.7)

$$
\begin{array}{ccc}
GX & \xrightarrow{\ Gg_1\ } & GKX_1 = LFX_1 \\
Gg_2 \downarrow & \ \ \searrow k & \downarrow Lu_1 \\
GKX_2 = LFX_2 & \xrightarrow[Lu_2]{} & LZ
\end{array}
$$

commuting in $\mathfrak{U}$ there is a commutative diagram

(2.8)

$$
\begin{array}{ccc}
GX & \xrightarrow{\ \ Gg_1\ \ } & GKX_1 = LFX_1 \\
Gg_2 \downarrow & GKv_1 \swarrow & \downarrow Lu_1 \\
 & GKY = LFY & \\
GKv_2 \nearrow & Lu \searrow & \\
GKX_2 = LFX_2 & \xrightarrow[Lu_2]{} & LZ
\end{array}
$$

in $\mathfrak{U}$ with

$$K v_1 \circ g_1 = K v_2 \circ g_2 \quad \text{and} \quad u \circ F v_2 = u_2 \quad \text{and} \quad u \circ F v_1 = u_1.$$

If M_X satisfies the initiality condition of type $\rightarrow . \leftarrow$, then we also say that $\mathfrak{U}$ *has weak local pushouts relative to* M_X. We note that the following sections reveal that Hilton's notion of *local right adjunctability* in [2] is an instance of such a local pushout.

In a similar fashion the same functor $M_X : (X \downarrow K) \rightarrow (G X \downarrow L)$ is said to satisfy the *initiality condition of type* $\leftarrow . \rightarrow$, or $\mathfrak{U}$ is said to have *weak local pullbacks relative to* M_X (cf. [2]), if (A) holds and if in addition any two objects u_1 and u_2 of $(M_X \downarrow k)$ can be connected by a pair of morphisms of the form $u_1 \leftarrow u \rightarrow u_2$. This condition, of course, can be interpreted in terms of $\mathfrak{U}$-morphisms using diagram (2.7) and a «weak local pullback» diagram in place of the «weak local pushout» diagram (2.8).

A similar interpretation in terms of $\mathfrak{U}$-morphisms exists for other initiality conditions (e. g. of type

$$\longrightarrow . \longleftarrow . \longrightarrow . \longleftarrow \quad)$$

or even for the most general initiality condition, namely that any two objects u_1 and u_2 of $(M_X \downarrow k)$ can be connected by a finite sequence of $(M_X \downarrow k)$-morphisms.

We remark that the initiality conditions ensure that the cohomology axiom of type (G, σ_1) is satisfied by any existing Kan extension h_1 along K of *any* functor $h : \mathcal{P} \rightarrow \mathcal{C}$ satisfying an axiom of type (F, σ), that is, h, $\mathcal{C}$ and σ may vary while holding $\mathcal{P}, F, G$ fixed. One might also investigate whether there are conditions weaker than initiality and depending in some way on h which would ensure that h_1 satisfies (G, σ).

3. The Homotopy axiom.

Let there be given a commutative diagram of categories and functors where $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{T}}$ are quotient categories of $\mathcal{P}$ and $\mathcal{T}$, respectively, with canonical functor $[\ ]$:

$$\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\quad K \quad} & \mathcal{T} \\
{\scriptstyle [\,]}\downarrow & & \downarrow{\scriptstyle [\,]} \\
\widetilde{\mathcal{P}} & \xrightarrow[\quad \widetilde{K} \quad]{} & \widetilde{\mathcal{T}}
\end{array}$$

(3.1)

We define quotient category in the sense of MacLane [5]. For example, $\mathcal{T}$ could be the category of topological spaces and $\widetilde{\mathcal{T}}$ the homotopy category.

A functor $h: \mathcal{P} \to \mathcal{C}$ satisfies the *homotopy axiom* if h factors through the canonical functor $[\,] : \mathcal{P} \to \widetilde{\mathcal{P}}$. We write $h = \widetilde{h}\,[\,]$ and note that this is the same as saying that h satisfies the cohomology axiom of type $(\,[\,]\,,1\,)$. Given an object X of $\mathcal{T}$ let $L_X : (X{\downarrow}K) \to (X{\downarrow}\widetilde{K})$ be the functor defined by:

$$L_X(\,f: X \to KY\,) = [f] : X \to \widetilde{K}Y$$

with the obvious definition on morphisms.

THEOREM 4. *If* $h: \mathcal{P} \to \mathcal{C}$ *satisfies the homotopy axiom, then so does its Kan extension* $h_1 : \mathcal{T} \to \mathcal{C}$ *along* K *provided that* $L_X : (X{\downarrow}K) \to (X{\downarrow}\widetilde{K})$ *is initial for each* $X \in |\mathcal{T}|$.

PROOF. Follows immediately from Theorem 3 by letting $(F,\sigma) = (\,[\,]\,,1\,)$, since then h_1 satisfies axiom of type (G,σ_1), with

$$G = [\,] \quad \text{and} \quad (\sigma_1)_X = L_X^* \, .$$

COROLLARY 5. *If* $h: \mathcal{P} \to \mathcal{C}$ *satisfies the homotopy axiom, then so does its Kan extension* $h_1 : \mathcal{T} \to \mathcal{C}$ *along* K *provided that* $\widetilde{\mathcal{T}}$ *has weak local pullbacks (or weak local pushouts) relative to* L_X.

PROOF. Immediate since these are initiality conditions for L_X by the preceding section.

4. The Suspension axiom.

Let

(4.1)

$$\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\quad K \quad} & \mathcal{T} \\
{\scriptstyle \Sigma}\downarrow & & \downarrow{\scriptstyle \Sigma} \\
\mathcal{P} & \xrightarrow[\quad K \quad]{} & \mathcal{T}
\end{array}$$

be a given commutative diagram of categories and functors.

A functor $h^n : \mathcal{P} \to \mathcal{C}$ satisfies the *suspension axiom* if h^n factors through the functor $\Sigma : \mathcal{P} \to \mathcal{P}$ up to a natural equivalence $\sigma : h^n \to h^{n+1} \Sigma$ for some functor $h^{n+1} : \mathcal{P} \to \mathcal{C}$. This is the same as saying that h^n satisfies the cohomology axiom of type (Σ, σ). Given an object X of $\mathcal{T}$ let

$$M_X : (X \downarrow K) \to (\Sigma X \downarrow K)$$

be the functor defined by

$$M_X (f : X \to K Y) = \Sigma f : \Sigma X \to \Sigma K Y = K \Sigma Y$$

with the obvious definition on morphisms.

THEOREM 6. *If $h^n : \mathcal{P} \to \mathcal{C}$ satisfies the suspension axiom, then so does its Kan extension $h^n_1 : \mathcal{T} \to \mathcal{C}$ along K provided that $M_X : (X \downarrow K) \to (\Sigma X \downarrow K)$ is initial for each $X \in |\mathcal{T}|$.*

PROOF. Let

$$(F, \sigma) = (\Sigma, \sigma) \quad \text{for} \quad \sigma : h^n \to h^{n+1} \Sigma$$

and apply Theorem 3. Then h^n_1 satisfies the cohomology axiom of type

$$(G, \sigma_1) \quad \text{where} \quad G = \Sigma : \mathcal{T} \to \mathcal{T} \quad \text{and} \quad \sigma_1 : h^n_1 \to h^{n+1}_1 \Sigma$$

for h^{n+1}_1 the Kan extension of h^{n+1} along K.

COROLLARY 7. *If $h^n : \mathcal{P} \to \mathcal{C}$ satisfies the suspension axiom, then so does its Kan extension $h^n_1 : \mathcal{T} \to \mathcal{C}$ along K provided that $\mathcal{T}$ has weak local pushouts (or weak local pullbacks) relative to M_X.*

PROOF. This is immediate because existence of weak local pushouts implies M_X is initial by results of Section 1.

Suppose that $h^n : \mathcal{P} \to \mathcal{C}$ satisfies both the homotopy and suspension axioms, then by Theorem 4 and Theorem 6 so does the Kan extension

$$h^n_1 : \mathcal{T} \to \mathcal{C} \quad \text{along} \quad K$$

provided that

$$M_X : (X \downarrow K) \to (\Sigma X \downarrow K) \quad \text{and} \quad L_X : (X \downarrow K) \to (X \downarrow \tilde{K})$$

are initial for all X. We show next how to get the same result from weaker conditions.

In the following theorem let

$$\widehat{M}_X : (X \downarrow \widetilde{K}) \to (\Sigma X \downarrow \widetilde{K}) \quad \text{and} \quad N_X : (X \downarrow K) \to (\Sigma X \downarrow \widetilde{K})$$

be the functors defined on objects by

$$\widehat{M}_X ([f] : X \to \widetilde{K} Y) = [\Sigma f] : \Sigma X \to \Sigma \widetilde{K} Y = \widetilde{K} \Sigma Y$$

and

$$N_X (f : X \to K Y) = [\Sigma f]$$

respectively, with the obvious definition on morphisms. Clearly $N_X = \widehat{M}_X L_X$. We also assume that commutative diagrams (3.1) and (4.1) are given and that Σ commutes with $[\ ]$ and $\widetilde{K}$.

THEOREM 8. *Let* $h^n : \mathcal{P} \to \mathcal{C}$ *be a functor satisfying the homotopy axiom with* $h^n = \widetilde{h}^n [\ ]$ *and suppose* $\widetilde{h}^n : \widetilde{\mathcal{P}} \to \mathcal{C}$ *satisfies the suspension axiom with given natural equivalence* $\sigma : \widetilde{h}^n \to \widetilde{h}^{n+1} \Sigma$. *Then the following state-ments hold:*

1. The canonical $\Sigma [\ ]$*-extension* $\widehat{\widetilde{h}}_1^{n+1} \Sigma [\ ] : \mathcal{J} \to \mathcal{C}$ *of* h^n *along* K *as well as* h^n *itself both satisfy the homotopy and suspension axioms, where* $\widehat{\widetilde{h}}_1^{n+1}$ *is the Kan extension of* $\widetilde{h}^{n+1} : \widetilde{\mathcal{P}} \to \mathcal{C}$ *along* $\widetilde{K}$.

2. The canonical $[\ ]$*-extension* $\widehat{\widetilde{h}}_1^n [\ ] : \mathcal{J} \to \mathcal{C}$ *of* h^n *along* K *satisfies the homotopy and suspension axioms provided* $\widehat{M}_X : (X \downarrow \widetilde{K}) \to (\Sigma X \downarrow \widetilde{K})$ *is initial for all* X. *The functor* $\widehat{\widetilde{h}}_1^n$ *is by definition the Kan extension of*

$$\widetilde{h}^n : \widetilde{\mathcal{P}} \to \mathcal{C} \quad \text{along} \quad \widetilde{K}.$$

3. The Kan extension $h_1^n : \mathcal{J} \to \mathcal{C}$ *of* h^n *along* K *satisfies the homotopy and suspension axioms provided* $N_X : (X \downarrow K) \to (\Sigma X \downarrow \widetilde{K})$ *is initial for all* X. *In particular this holds if* L_X *and* $\widehat{M}_X$ *are initial for all* X *in* $|\mathcal{J}|$.

PROOF. The first statement is obvious since

$$\sigma [\ ] : h^n = \widetilde{h}^n [\ ] \to \widetilde{h}^{n+1} \Sigma [\ ] = \widetilde{h}^{n+1} [\ ] \Sigma$$

is an equivalence. For the second statement we assume that

$$\widehat{M}_X : (X \downarrow \widetilde{K}) \to (\Sigma X \downarrow \widetilde{K})$$

is initial and note that by hypothesis $\tilde{b}^n \colon \tilde{\mathcal{P}} \to \mathcal{C}$ satisfies the suspension axiom with specified equivalence $\sigma \colon \tilde{b}^n \to \tilde{b}^{n+1}\Sigma$. Then by Theorem 6 the Kan extension $\tilde{b}^n_1$ along $\tilde{K}$ satisfies the suspension axiom with equivalence σ_1 from $\tilde{b}^n_1$ to the canonical Σ-extension $\tilde{b}^{n+1}_1\Sigma$. Thus $\sigma_1[\,]$ is an equivalence from $\tilde{b}^n_1[\,]$ to the canonical $\Sigma[\,]$-extension

$$\tilde{b}^{n+1}_1\Sigma[\,] = \tilde{b}^{n+1}_1[\,]\Sigma.$$

For the third statement we apply Theorem 3 to the commutative diagram

(4.2)

$$
\begin{array}{ccc}
\mathcal{P} & \overset{K}{\longrightarrow} & \mathcal{J} \\
\Sigma[\,] \downarrow & & \downarrow \Sigma[\,] \\
\tilde{\mathcal{P}} & \underset{\tilde{K}}{\longrightarrow} & \tilde{\mathcal{J}}
\end{array}
$$

where $b^n \colon \mathcal{P} \to \mathcal{C}$ satisfies the cohomology axiom of type $(\Sigma[\,], \sigma[\,])$, where

$$\sigma[\,] \colon b^n = \tilde{b}^n[\,] \to \tilde{b}^{n+1}\Sigma[\,].$$

It then follows that the induced natural transformation $(\sigma[\,])_1$ from b^n_1 to the canonical $\Sigma[\,]$-extension $\tilde{b}^{n+1}_1\Sigma[\,]$ is an equivalence if

$$N_X \colon (X \!\downarrow\! K) \to (\Sigma X \!\downarrow\! \tilde{K})$$

is initial.

We next interpret the preceding results in the topological context considered by Hilton in [2]. Accordingly we let $K \colon J_0 \to J_1$ be the inclusion of non-empty full subcategories of the category of topological spaces, each closed under the suspension functor Σ. Let

$$[\,] \colon J_0 \to \tilde{J}_0 \quad \text{or} \quad [\,] \colon J_1 \to \tilde{J}_1$$

represent the canonical functor to the homotopy categories. Then we have the results of the preceding theorems for $J_0 = \mathcal{P}$ and $J_1 = \mathcal{J}$.

In particular the statement that $\Sigma \colon \tilde{J}_1 \to \tilde{J}_1$ is *locally right $\tilde{J}_0$-adjunctable* as defined in [2] is equivalent to the statement from section 2 of this paper that $\tilde{J}_1$ has *weak local pushouts relative to*

$$\hat{M}_X : (X \downarrow \hat{K}) \to (\Sigma X \downarrow \hat{K})$$

where in this case the reference diagram (2.5) becomes $\Sigma \hat{K} = \hat{K} \Sigma$. This is just the initiality condition of type $\longrightarrow . \longleftarrow$ for $\hat{M}_X$. We may now state the following corollary to Theorem 8.

COROLLARY 9. *Let* $b^n : J_0 \to \mathcal{C}$ *be a functor satisfying the homotopy and suspension axioms for* $\mathcal{C}$ *arbitrary and*

$$b^n = \tilde{b}^n \, [\,] \xrightarrow{\quad \sigma \, [\,] \quad} \tilde{b}^{n+1} \Sigma \, [\,] = \tilde{b}^{n+1} \, [\,] \, \Sigma$$

a given natural equivalence. If $\Sigma : \tilde{J}_1 \to \tilde{J}_1$ *is locally right* $\tilde{J}_0$*-adjunctable, then the canonical* $[\,]$*-extension* $\tilde{b}_1^n [\,] : J_1 \to \mathcal{C}$ *satisfies the homotopy and suspension axioms.*

PROOF. This is immediate from statement 2 of Theorem 8 since the adjunctability condition implies $\hat{M}_X$ initial.

In a similar fashion the statement that J_0 *has weak local pullbacks* relative to $\tilde{J}_1$ as defined in [2] is equivalent to the statement derived from section 2 that $\tilde{J}_1$ has weak local pullbacks relative to

$$L_X : (X \downarrow K) \to (X \downarrow \hat{K})$$

where in this case diagram (2.5) becomes $[\,] K = \hat{K} \, [\,]$. This is just the initiality condition of type $\longleftarrow . \longrightarrow$ for L_X.

COROLLARY 10. *Suppose that* $b^n : J_0 \to \mathcal{C}$ *is a functor satisfying the homotopy and suspension axioms as in Corollary 9. Then the Kan extension:*

$$b_1^n : J_1 \to \mathcal{C} \quad \text{of} \quad b^n \quad \text{along} \quad K : J_0 \to J_1$$

satisfies the homotopy and suspension axioms provided that $\tilde{J}_0$ *has weak local pullbacks relative to* $\tilde{J}_1$ *and that* $\Sigma : \tilde{J}_1 \to \tilde{J}_1$ *is locally right* $\tilde{J}_0$*-adjunctable.*

PROOF. The hypotheses given imply that L_X, $\hat{M}_X$ and hence $N_X = \hat{M}_X L_X$ are initial for all $X \in |J_1|$. The result is now immediate from Theorem 8.

We remark that the last statement of Theorem 8 interpreted in terms of J_0 and J_1 provides the result of Corollary 10 under more general hypotheses. We record this explicitly in the following corollary.

COROLLARY 11. *The following conditions are equivalent and imply that the validity of the homotopy and suspension axioms for the Kan extension* $b_1^n : J_1 \to \mathcal{C}$ *along* K *of any functor* $b^n : J_0 \to \mathcal{C}$ *follows from the validity of these axioms for* b^n :

1. The functor

$$N_X = \hat{M}_X L_X : (X \downarrow K) \to (X \downarrow \tilde{K}) \to (\Sigma X \downarrow \tilde{K})$$

is initial for each $X \in |J_1|$.

2. The categories $(N_X \downarrow k)$ *are non-empty and connected for each object* $k : \Sigma X \to \tilde{K} Z$ *of* $(\Sigma X \downarrow \tilde{K})$ *and each* $X \in |J_1|$.

3. For each $X \in |J_1|$ *and each object* $k : \Sigma X \to \tilde{K} Z$ *of* $(\Sigma X \downarrow \tilde{K})$, *there exists an object*

$$u : N_X g \to k \quad of \quad (N_X \downarrow k)$$

for some object g *of* $(X \downarrow K)$, *and for any pair* $(u, \bar{u})$ *of* $(N_X \downarrow k)$-*objects there exists a commutative diagram*
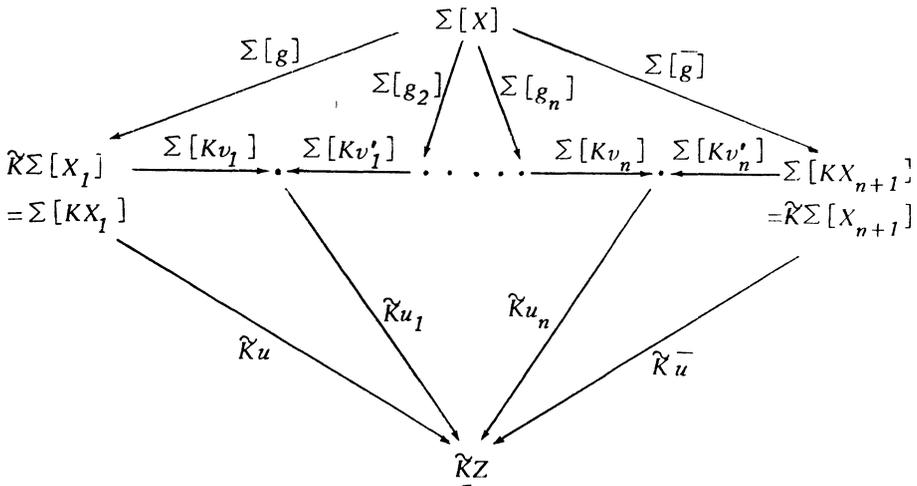
in $\tilde{J}_1$ *with* n *finite and*

$$K v_i \circ g_i = K v_i' \circ g_{i+1} \quad in \quad J_1 \quad for \quad i = 1, \ldots, n,$$

and

$$u = u_1 \circ \Sigma [v_1], \quad u_i \circ \Sigma v_i' = u_{i+1} \circ \Sigma v_{i+1} \quad for \quad i = 1, \ldots, n-1,$$

and $u_n \circ \Sigma v_n' = \bar{u}$ *in* $\tilde{J}_0$.

PROOF. We have simply written down an explicit interpretation of the comma

category morphisms involved, as suggested near the end of Section 2.

5. The Exactness axiom.

Let $h_\iota : \mathcal{P}_* \to \mathcal{C}$ be a contravariant functor which satisfies the co-homology axiom of type $(1_{\mathcal{P}_*}, \sigma)$. This means that there is a diagram

(5.1)

$$
\begin{array}{c}
\mathcal{P}_* \\
\\
h_\iota \quad \sigma \quad \mathcal{P}_* \\
\\
\quad \quad h_K \\
\mathcal{C}
\end{array}
$$

with $\sigma : h_\iota \to h_K$ a natural equivalence.

Given a commutative diagram

(5.2)

$$
\begin{array}{ccc}
\mathcal{P}_* & \xrightarrow{K_*} & \mathcal{J}_* \\
\| & & \| \\
\mathcal{P}_* & \xrightarrow{K_*} & \mathcal{J}_*
\end{array}
$$

it follows from Theorem 3 that the Kan extension $(h_\iota)_1$ of h_ι along K_* satisfies the cohomology axiom of type $(1_{\mathcal{J}_*}, \sigma_1)$ where the natural equivalence σ_1 of

(5.3)

$$
\begin{array}{c}
\mathcal{J}_* \\
\\
(h_\iota)_1 \quad \sigma_1 \quad \mathcal{J}_* \\
\\
\quad \quad (h_K)_1 \\
\mathcal{C}
\end{array}
$$

is given by

$$
(\sigma_1)_{\underline{X}} = \varinjlim \sigma D_{\underline{X}} : (h_\iota)_1 \underline{X} = \varinjlim h_\iota D_{\underline{X}} \to \varinjlim h_K D_{\underline{X}} = (h_K)_1 \underline{X}
$$

for each object $\underline{X}$ of $\mathcal{T}_*$.

We relate the preceding to exactness in the following way. Let $\mathcal{P}$ and $\mathcal{T}$ be categories possessing a weak cokernel for each map, that is, we require universality without requiring uniqueness, as in [5], page 231. For example, in the pointed homotopy category, the weak cokernel of a map $f: X \to Y$ is the usual map $Y \to M_f$ for M_f the mapping cone of f.

Let $K: \mathcal{P} \to \mathcal{T}$ be a functor preserving weak cokernels and let $K_*: \mathcal{P}_* \to \mathcal{T}_*$ be the functor induced by K where $\mathcal{P}_*$ and $\mathcal{T}_*$ are categories whose objects are weak cokernel sequences in $\mathcal{P}$ and in $\mathcal{T}$, respectively, with the obvious definition for morphisms.

Next we let $\mathcal{C}$ be any category in which each map has a kernel and an epic-monic factorization unique up to equivalence. The category $Comp\,\mathcal{C}$ is the one whose objects are short complexes

$$C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \quad \text{in } \mathcal{C}, \text{ i.e. } \beta\alpha = 0,$$

and whose morphisms are chain mappings. Then we have the «image to kernel» natural monomorphism

$$\mu: Im \to Ker: Comp\,\mathcal{C} \to \mathcal{C}$$

where Im and Ker are the functors assigning $Im\,\alpha$ (= the monic part of α) and $Ker\,\beta$, respectively, to the object

$$C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \quad \text{of } Comp\,\mathcal{C}.$$

If $h: \mathcal{P} \to \mathcal{C}$ is a contravariant functor inducing a functor

$$h_*: \mathcal{P}_* \to Comp\,\mathcal{C},$$

then we have a natural monomorphism $\sigma: h_I \to h_K$ where by definition

$$h_I = Im \circ h_*, \quad h_K = Ker \circ h_* \text{ and } \sigma = \mu \circ 1_{h_*}.$$

The functor $h: \mathcal{P} \to \mathcal{C}$ is *exact* if h takes each weak cokernel sequence

$$Y' \xrightarrow{\alpha} Y \xrightarrow{\beta} Y''$$

to an exact sequence

$$h\,Y'' \xleftarrow{h\alpha} h\,Y \xleftarrow{h\beta} h\,Y'' \quad \text{in } \mathcal{C}.$$

PROPOSITION 12. *The functor* $h : \mathcal{P} \to \mathcal{C}$ *is exact if and only if the functor* $h_l : \mathcal{P}_* \to \mathcal{C}$ *satisfies the cohomology axiom of type* $(1_{\mathcal{P}_*}, \sigma)$.

COROLLARY 13. *If* $h : \mathcal{P} \to \mathcal{C}$ *is exact, then the Kan extension*

$$(h_l)_1 : \mathcal{T}_* \to \mathcal{C} \quad of \quad h_l \quad along \quad K_* : \mathcal{P}_* \to \mathcal{T}_*$$

satisfies the cohomology axiom of type $(1_{\mathcal{T}_*}, (\mu \circ 1_{h_*})_1)$.

PROOF. This is immediate from the first paragraphs of the section.

Thus, in the sense of the corollary, exactness for $h : \mathcal{P} \to \mathcal{C}$ is always a Kan extendable property.

However to say that the Kan extension

$$h_1 : \mathcal{T} \to \mathcal{C} \quad of \quad h \quad along \quad K : \mathcal{P} \to \mathcal{T}$$

is exact is a little different. By Prop. 12, $h_1 : \mathcal{T} \to \mathcal{C}$ is exact if and only if the functor $(h_1)_l : \mathcal{T}_* \to \mathcal{C}$ satisfies the cohomology axiom of type

$$(1_{\mathcal{T}_*}, \mu \circ 1_{(h_1)_*}).$$

The next Theorem shows how exactness for $h_1 : \mathcal{T} \to \mathcal{C}$ follows from that of $h : \mathcal{P} \to \mathcal{C}$ provided that a certain canonical limit interchange map is an epimorphism.

THEOREM 14. *Suppose that* $h : \mathcal{P} \to \mathcal{C}$ *is an exact functor. Then there is a commutative diagram*

$$(5.4) \qquad
\begin{array}{ccc}
(h_K)_1 & \xrightarrow{\;\;\eta_K\;\;} & (h_1)_{\nu} \\
{\scriptstyle(\mu \circ 1_{h_*})_1}\Big\uparrow & & \Big\uparrow{\scriptstyle \mu \circ 1_{(h_1)_*}} \\
(h_l)_1 & \xrightarrow[\;\;\eta_l\;\;]{} & (h_1)_l
\end{array}$$

of natural transformations with $(\mu \circ 1_{h_*})_1$ *a natural equivalence and with* $\mu \circ 1_{(h_1)_*}$ *a natural monomorphism. Furthermore the Kan extension* $h_1 : \mathcal{T} \to \mathcal{C}$ *along* K *is exact if the canonical limit interchange map* η_K *is an epimorphism.*

PROOF. By Corollary 13 the left vertical arrow is an equivalence. The right

vertical map is just the monomorphism from image to kernel. We next define η_K. Given a contravariant functor $h : \mathcal{P} \to \mathcal{C}$ we let $h_1 : \mathcal{J} \to \mathcal{C}$ be the Kan extension of h along $K : \mathcal{P} \to \mathcal{J}$ defined by

$$h_1 X = \varinjlim (X{\downarrow}K \longrightarrow \mathcal{P} \xrightarrow{\ h\ } \mathcal{C})$$

and let $\delta : h \to h_1 K$ be the associated universal K-factorization. In the same way there is a universal K_*-factorization

(5.5)

$$\begin{array}{ccc}
\mathcal{P}_* & \xrightarrow{\quad K_* \quad} & \mathcal{J}_* \\
{\scriptstyle h_*}\downarrow & {\scriptstyle \varepsilon_K} & \diagdown \\
Comp\,\mathcal{C} & & (Ker \circ h_*)_1 \\
{\scriptstyle Ker}\downarrow & \diagup & \\
\mathcal{C} & &
\end{array}$$

We then let

$$\eta_K : (h_K)_1 = (Ker \circ h_*)_1 \to Ker \circ (h_1)_* = (h_1)_K$$

be the unique transformation such that

(5.6)

$$\begin{array}{ccc}
(h_K)_1 \circ K_* & \xrightarrow{\ \eta_K \circ K_* \ } & (h_1)_K \circ K_* \\
\diagdown {\scriptstyle \varepsilon_K} & & {\scriptstyle Ker \circ \delta_*} \diagup \\
& Ker \circ h_* &
\end{array}$$

commutes. In a similar fashion starting from $Im : Comp\,\mathcal{C} \to \mathcal{C}$ we obtain ε_l and η_l. From the commutative diagram

(5.7)

$$\begin{array}{ccccc}
(h_1)_K \circ K_* & \xleftarrow{\ [(\mu \circ 1_{(h_1)_*}) \eta_l] \circ K_* \ } & (h_l)_1 \circ K_* & \xrightarrow{\ [\eta_K (\mu \circ 1_{h_*})_1] \circ K_* \ } & (h_1)_K \circ K_* \\
\diagdown {\scriptstyle (\mu \circ \delta_*)} & & {\scriptstyle \varepsilon_l}\Big\downarrow & & {\scriptstyle (\mu \circ \delta_*)} \diagup \\
& & Im \circ h_* & &
\end{array}$$

it is clear that the universality of ε_l implies that (5.4) commutes. Thus if η_K is epic then the right vertical arrow of (5.4) is an isomorphism and h_1 is exact.

COROLLARY 15. *If $(X \downarrow K)^{op}$ is filtered and $\mathcal{C}$ is a category in which filtered colimits commute with finite limits (e. g., an algebraic category such as abelian groups, rings, Lie algebras), then the exactness problem for Kan extensions is finished.*

The remainder of this Section follows from the direct and classical approach of describing the Kan extension of exactness explicitly for $\mathcal{C}$ the category Ens_* of pointed sets and under hypotheses on $(X \downarrow K)^{op}$ weaker than those given in the preceding paragraph. This description is then extended to algebraic categories $\mathcal{C}$ and applied to the problem of extending cohomology theories with values in Ens_* or in an algebraic category.

Let $T : I \to Ens_*$ be a covariant functor. To establish notation we recall that

$$\varinjlim T = (\coprod_{f \in |I|} T(f)) / E$$

where the coproduct $\coprod_{f \in |I|} T(f)$ is the set

$$\{ (a, f) \mid a \in T(f) \text{ and } f \in |I| \}$$

with basepoints under the inclusions identified, and E is the equivalence generated by

$$(a, f) \sim (T(\alpha)a, g) \quad \text{for } a \in T(f) \text{ and } \alpha : f \to g \text{ in } I.$$

An element of $\varinjlim T$ is denoted $[a, f]$.

We next describe the natural transformation

$$\eta_K : (h_K)_1 = (Ker \circ h_*)_1 \to (h_1)_K = Ker \circ (h_1)_*$$

for $h : \mathcal{P} \to Ens_*$ contravariant. Let

$$\underline{X} = X' \to X \to X''$$

be a weak cokernel sequence in $\mathcal{P}$, i.e. $\underline{X}$ is an object of the category $\mathcal{P}_*$. Then

$$(b_K)_1 \underline{X} = \underset{\longrightarrow}{lim}((\underline{X}\downarrow K_*) \xrightarrow{D_{\underline{X}}} \mathcal{P}_* \xrightarrow{b_*} Comp\, \mathcal{C} \xrightarrow{Ker} \mathcal{C}) =$$

(5.8)
$$\left\{ [A,F] \,\middle|\, \begin{array}{c} F = \begin{array}{ccccc} X' & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & X'' \\ \downarrow{F_0} & & \downarrow{F_1} & & \downarrow{F_2} \\ K\,Y' & \xrightarrow{Ku} & K\,Y & \xrightarrow{Ku'} & K\,Y'' \end{array} \quad in \; \mathcal{J}_* \\[2ex] with \; A \in ker\, b\, u: b\,Y \to b\,Y' \end{array} \right\} .$$

In this case $T = Ker \circ b_* \circ D_{\underline{X}}$ is regarded as a functor from $I = (\underline{X}\downarrow K_*)^{op}$ to Ens_*. Furthermore

$$(b_1)_K \underline{X} = Ker \circ (b_1)_* (X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'') =$$
$$= Ker\, b_1\, \alpha: (b_1 X \to b_1 X') =$$
$$= \{ [A, F_1] \,|\, F_1: X \to K Y, \quad A \in b\, Y \; and \; [A, F_1 \alpha] = 0 \} .$$

PROPOSITION 16. *The natural transformation* $\eta_K: (b_K)_1 \to (b_1)_K$ *is explicitly described by the formula*

$$(\eta_K)_{\underline{X}} [A, F] = [A, F_1] \quad for \; F = (F_0, F_1, F_2) \; in \; \underline{X}\downarrow K_* .$$

PROOF. The element $[A, F]$ of $(b_K)_1 \underline{X}$ is explicitly described in (5.8). Clearly $[A, F_1] \in b_1 X$ and furthermore

$$(b_1 \alpha) [A, F_1] = [A, F_1 \alpha] = [A, Ku \circ F_0] =$$
$$= [b(u)A, F_0] = [0, F_0] = 0 .$$

It is easy to show that this yields a well defined natural transformation η_K for which diagram (5.6) commutes. Hence by uniqueness in (5.6) this must be the η_K previously defined.

Suppose that we are in the general case again with D the relation in $\underset{f \in |I|}{\amalg} T(f)$ determined by

$$(a, f) \sim (b, g) \quad if \; there \; exist \; u: f \to b \; and \; v: g \to b$$
$$with \; T(u) a = T(v) b .$$

The equivalence relation E previously described is clearly the one generated by D.

If any diagram of the form

(5.9)
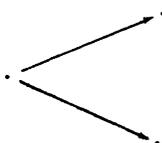
in I can be completed to
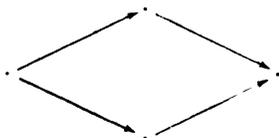
commuting in I , then we say that the *diamond condition holds for I*.

PROPOSITION 17. *If $T: I \to Ens_*$ and the diamond condition holds for I, then $D = E$. Furthermore $[a, f] = [0, g]$ in $\varinjlim T$ implies that there exists $u: f \to b$ with $T(u) a = 0$.*

THEOREM 18. *If the diamond condition holds for $I = (X \downarrow K)^{op}$, then the limit interchange map $\eta_K : (b_K)_1 \to (b_1)_K$ is an epimorphism for any exact functor $b : \mathscr{P} \to Ens_*$.*

PROOF. Let

$$[A, F_1] \in (b_1)_K \underline{X} = Ker \circ (b_1)_* (X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'') = ker b_1 \alpha$$

be given. Then

$$F_1 : X \to K Y, \quad A \in b Y \quad \text{and} \quad [A, F_1 \alpha] = 0.$$

By Proposition 17 there exist $F_0 : X' \to K Y'$ for some $Y' \in |\mathscr{P}|$ and a morphism $u: F_0 \to F_1 \alpha$ in $(X \downarrow K)$ with

$$b D_{X'}(u) A = b(u) A = 0.$$

We thus have a diagram

(5.10)

$$
\begin{array}{ccccc}
X' & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & X'' \\
\downarrow F_0 & & \downarrow F_1 & & \downarrow F_2 \\
K Y' & \xrightarrow{K u} & K Y & \xrightarrow{K v} & K Y''
\end{array}
$$
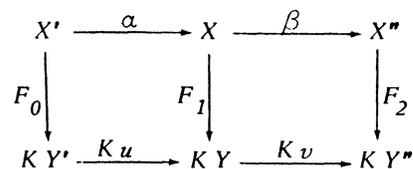
with commutative left square. Then let v be a weak coequalizer of u in $\mathscr{P}$ and then since β is a weak coequalizer of α in $\mathcal{I}$ there exists F_2 such then the right square commutes. Thus $F = (F_0, F_1, F_2)$ is in $\mathcal{I}_*$ since K preserves weak coequalizers and $[A, F]$ is an element of $(h_K)_1 \underline{X}$ with $(\eta_K)_{\underline{X}} [A, F] = [A, F_1]$.

The following result is now immediate from Theorem 14.

COROLLARY 19. *The Kan extension* $h_1 : \mathcal{I} \to Ens_*$ *of* $h : \mathscr{P} \to Ens_*$ *along* $K : \mathscr{P} \to \mathcal{I}$ *is exact provided that* h *is exact and that the diamond condition holds for* $I = (X{\downarrow}K)^{op}$.

If we assume that $h : \mathscr{P} \to Ens_*$ is a homotopy functor, then the next theorem shows that we can get exactness results under the assumption that $(X{\downarrow}\widetilde{K})^{op}$ satisfies the diamond condition, which is weaker than the same assumption for $(X{\downarrow}K)^{op}$ used in the preceding corollary. As in Section 3 we assume diagram (3.1) with $\widetilde{K}\,[\,] = [\,]\,K$ is given. Furthermore we shall assume that $\widetilde{\mathscr{P}}$ and $\widetilde{\mathcal{I}}$ have weak coequalizers and that $\widetilde{K} : \widetilde{\mathscr{P}} \to \widetilde{\mathcal{I}}$ preserves them. It will not be necessary to make coequalizer assumptions on $\mathscr{P}, \mathcal{I}$ or K directly.

THEOREM 20. *Let*

$$h = \widetilde{h}\,[\,] : \mathscr{P} \to \widetilde{\mathscr{P}} \to Ens_*$$

be a homotopy factorization and suppose that $\widetilde{h} : \widetilde{\mathscr{P}} \to Ens_*$ *satisfies the exactness axiom. If* $\widetilde{\mathcal{I}}$ *has weak local pullbacks relative to the functor*

$$L_X : (X{\downarrow}K) \to (X{\downarrow}\widetilde{K})$$

defined in Section 3, then the Kan extension $h_1 : \mathcal{I} \to Ens_*$ *of* h *along* K *is naturally equivalent to* $\widetilde{h}_1\,[\,]$ *and* $\widetilde{h}_1$ *is exact for* $\widetilde{h}_1 : \widetilde{\mathcal{I}} \to Ens_*$ *the Kan extension of* $\widetilde{h}$ *along* $\widetilde{K}$.

PROOF. Corollary 5 and Theorem 3 imply that $h_1 \overset{\sim}{=} \widetilde{h}_1\,[\,]$. The diamond condition for $I = (X{\downarrow}\widetilde{K})^{op}$ is equivalent to the statement that $\widetilde{\mathcal{I}}$ has weak local pullbacks relative to L_X (cf. [2]). Thus $\widetilde{h}_1$ is exact by Corollary 19.

We next show how to extend exactness results from Ens_* to alge-

braic categories over $Ens*$ such as the categories of groups, rings and Lie algebras. An *algebraic category* $\mathcal{C}$ over $Ens*$ is simply an equational class of algebras in the sense of MacLane ([5], page 120) together with a forgetful functor $U: \mathcal{C} \to \bar{E}ns*$.

The following theorem now brings us back to a concrete description of the result given in Corollary 15.

THEOREM 21. *If* $(X{\downarrow}\tilde{K})^{op}$ *is filtered and* $\mathcal{C}$ *is algebraic over* $Ens*$, *then the Kan extension* $\tilde{h}_1: \tilde{\mathfrak{J}} \to \mathcal{C}$ *of an exact functor* $\tilde{h}: \tilde{\mathfrak{P}} \to \mathcal{C}$ *along* $\tilde{K}$ *is exact.*

PROOF. We note that the validity of the diamond condition for $I = (X{\downarrow}\tilde{K})^{op}$ is part of the requirement for $(X{\downarrow}\tilde{K})^{op}$ to be filtered in the sense of Schubert [6]. The forgetful functor $U: \mathcal{C} \to Ens*$ preserves (and creates) filtered colimits. Thus

$$U\tilde{h}_1: \tilde{\mathfrak{J}} \longrightarrow \mathcal{C} \longrightarrow Ens*$$

is the Kan extension of $U\tilde{h}: \tilde{\mathfrak{P}} \to Ens*$ along $\tilde{K}$. By Corollary 19 it follows that $U\tilde{h}_1$ is exact, hence so is $\tilde{h}_1$.

We now conclude by applying the results obtained to the problem of extending cohomology theories.

Accordingly we let

$$(h, \sigma) = \{ h^n: J_0 \to \mathcal{C}, \quad \sigma^n: h^n \approx h^{n+1}\Sigma, \quad n \in Z \}$$

be a cohomology theory on an admissible topological category J_0 with values in a category $\mathcal{C}$, as defined in Section 1. We let $K: J_0 \to J_1$, [] and Σ be the inclusion, homotopy and suspension functors as described before Corollary 9.

From Corollary 11 it follows that if

$$h = \{ h^n: J_0 \to \mathcal{C} \}$$

is a cohomology theory and if

$$N_X = \tilde{M}_X L_X: (X{\downarrow}K) \to (X{\downarrow}\tilde{K}) \to (\Sigma X{\downarrow}\tilde{K})$$

is initial for each $X \in |J_1|$, then the Kan extension $h_1^n: J_1 \to \mathcal{C}$ of h^n along K satisfies the homotopy and suspension axioms (if h_1^n exists). We next

see how to bring in exactness as well.

THEOREM 22. *If $\tilde{J}_0$ has weak local pullbacks relative to*

$$L_X : (X \downarrow K) \to (X \downarrow \tilde{K})$$

and if $\tilde{M}_X : (X \downarrow \tilde{K}) \to (\Sigma X \downarrow \tilde{K})$ is initial, then the Kan extension of a coho-mology theory

$$(h, \sigma) = \{ h^n : J_0 \to Ens_*, \quad \sigma^n : h^n \overset{\approx}{\to} h^{n+1}\Sigma \}$$

is a cohomology theory

$$(h_1, \sigma_1) = \{ h_1^n : J_1 \to Ens_*, \ \sigma_1^n : h_1^n \to h_1^{n+1}\Sigma \} \ .$$

PROOF. The homotopy functor $[] : J_1 \to \tilde{J}_1$ takes each mapping cone sequence in J_1 to a weak coequalizer sequence in $\tilde{J}_1$. The existence of weak local pullbacks implies by Theorem 20 that $h_1^n \overset{\approx}{\to} \tilde{h}_1^n []$ and that $\tilde{h}_1^n$ satis-fies the exactness axiom for each $n \in Z$. Thus h_1^n takes mapping cone se-quences to exact sequences in Ens_*. The existence of weak local pullbacks implies L_X is initial. Thus $N_X = \tilde{M}_X L_X$ is initial and the suspension ax-iom holds for h_1.

THEOREM 23. *If $\tilde{J}_0$ has weak local pullbacks relative to*

$$L_X : (X \downarrow K) \to (X \downarrow \tilde{K}),$$

J_0 has finite products and if $\tilde{M}_X : (X \downarrow \tilde{K}) \to (\Sigma X \downarrow \tilde{K})$ is initial, then the Kan extension of a cohomology theory

$$h = \{ h^n : J_0 \to \mathcal{C}; \ c_*^n \}$$

is a cohomology theory

$$h_1 = \{ h_1^n : J_1 \to \mathcal{C}; \ \sigma_1^n \},$$

for $\mathcal{C}$ any algebraic category over Ens_*.

PROOF. The first two conditions imply that $(X \downarrow \tilde{K})^{op}$ is filtered by a result of Hilton [2], Corollary 2.13. Thus by Theorem 21 it follows that the Kan extension $\tilde{h}_1^n : \tilde{J}_1 \to \mathcal{C}$ of $\tilde{h}^n : \tilde{J}_0 \to \mathcal{C}$ along $\tilde{K} : \tilde{J}_0 \to \tilde{J}_1$ is exact for each n, where

$$h^n = \tilde{h}^n [] : J_0 \to \mathcal{C}$$

is a homotopy factorization of h^n. By Corollaries 3 and 5 the functor

$$b_1^n \approx \widehat{b}_1^n \, [\,] : J_1 \to \mathcal{C}$$

Thus b_1^n takes mapping cone sequences in J_1 to exact sequences in $\mathcal{C}_*$. As before the suspension axiom follows from the initiality of $N'_X = \widehat{M}_X L_X$ by Corollary 11.

COROLLARY 24. *If* $\widehat{J}_0$ *has weak local pullbacks relative to* L_X *and if* Σ *is locally right* $\widehat{J}_0$*-adjunctable, then the Kan extension of a cohomology theory*

$$\{ b^n : J_0 \to Ens_* ; \; \sigma^n \}$$

is a cohomology theory

$$\{ b_1^n : J_1 \to Ens_* ; \; \sigma_1^n \} .$$

Furthermore, if in addition J_0 *has finite products, then the same result holds if we replace* Ens_* *by an algebraic category* $\mathcal{C}$ *over* Ens_* .

PROOF. The local adjunctability condition implies that $\widehat{M}_X$ is initial as the paragraph preceding Corollary 9 clearly shows.

COROLLARY 25 (*Deleanu-Hilton* [2]). *If* J_0 *is* J_1*-adapted and* Σ *is locally right* $\widehat{J}_0$*-adjunctable, then the Kan extension of a cohomology theory* (b, σ) *is a cohomology theory* (b_1, σ_1) .

PROOF. This is just the result of Corollary 24 for $\mathcal{C}$ the category of abelian groups, since J_0 is J_1-adapted just means J_0 closed under finite products and $\widehat{J}_0$ has weak local pullbacks relative to L_X.

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