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DUALITY OF VECTOR SPACES ★

by Michael BARR

The notion that one can get a nice duality theory for vector spaces by introducing a topology into the dual is old - it apparently goes back to Lefschetz ([3] , pp. 78-83) - but is surprisingly little known. Briefly, if V is an infinite dimensional vector space and V^* its dual, then there are many more functionals on V^* than are induced by elements of V. Nonetheless those that can be identified are precisely the functionals which are continuous on V^* when it is topologized by pointwise convergence, the ground field being discrete.

In this paper we extend this duality to the category of topological vector spaces over a discrete field. The structure involved turns out to be rather well-behaved.

In a subsequent paper [(1)] I will extend many of these ideas to the category of (real or complex) Banach spaces. In a few cases in the present paper, the desire to smooth out this extension has led to a slightly more cumbersome exposition.

In accordance with the doctrine of Reyes («Don't scratch if you don't itch»), we leave aside all questions of coherence. The concreteness of the constructions guarantees that all necessary coherence conditions are satisfied.

★ N.D.R. Cet article et le suivant sont le développement des deux conférences données par M. BARR aux «Journées T.A.C. de Chantilly» (Septembre 1975). Ces conférences ont été résumées dans le volume XVI-4 (1975) des «Cahiers de Topologie et Géométrie Différentielle».

(1) In this same issue.

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1. Preliminaries.

Let K be an abstract (i. e. discrete) field. By a separated linearly topologized K-vector space we mean a vector space V equipped with a topology for which addition $V \times V \to V$ and scalar multiplication $K \times V \to V$ are continuous. In addition we require that for any $v \neq O$, there be an open subspace U with $v \notin U$. By a morphism of such spaces we mean a continuous linear map. The category so described we call $\mathfrak{V}(K)$ or simply $\mathfrak{V}$ if K is understood. The one dimensional space, equipped -necessarily-with the discrete topology, will also be denoted by K. A morphism $V \to K$ is called a (linear) functional on V.

The full subcategory of $\mathfrak{V}(K)$ whose objects are the discrete spaces is denoted by $\mathfrak{D}(K)$ or simply $\mathfrak{D}$. The inclusion $\mathfrak{D} \to \mathfrak{V}$ has a left adjoint $V \to |V|$, the same vector space with the discrete topology.

If $V \in \mathfrak{V}$, a subset $A \subset V$ is called a linear subvariety if $A = v + W$ where W is a linear subspace of V. We say that V is linearly compact (LC) if the family of closed linear subvarieties has the finite intersection property (see [1], [3], for details as well as proofs of the assertions below). The following assertions are easily proved in ways similar to their topological analogues.

PROPOSITION 1. *A closed subspace and a (separated) quotient space of an LC space is LC; the topological product of LC spaces is LC; a morphism whose domain is LC is closed and in particular a 1-1 map from an LC space onto another space is an isomorphism. A discrete space is LC iff it is finite dimensional.*

The full subcategory of LC spaces is denoted $\mathfrak{C}(K)$, or simply $\mathfrak{C}$.

The inclusion $\mathfrak{C} \to \mathfrak{V}$ has a right adjoint, $V \mapsto \beta V$. It is worth mentioning that when K is a finite field, linear compactness is exactly the same as topological compactness. Thus the concept of linear compactness can be viewed as the extension to an arbitrary field of notions easily described for finite ones.

It is well known that $\mathfrak{C}$ and $\mathfrak{D}$ are dual. If $V \in \mathfrak{D}$, then $V^* \in \mathfrak{C}$ is the set of all functionals of V topologized as a subset of the topological product K^V. Since K is LC (trivial), so is the product. For any two vectors $v_1, v_2 \in V$,

$$\{ \phi \in K^V \mid \phi(v_1 + v_2) - \phi(v_1) - \phi(v_2) = 0 \}$$

is the inverse image of the discrete space

$$K^{\{v_1+v_2\}} \times K^{\{v_1\}} \times K^{\{v_2\}}$$

and is hence closed. The intersection of these sets, for all v_1 and v_2, is closed and so the set of additive maps is closed. Similarly the set of all linear functionals on V is closed and is hence LC. The reverse equivalence is obtained by sending the LC space V to the discrete space of functionals on it.

Let $V, W \in \mathfrak{V}$. We will topologize the space $Hom(V, W)$ of continuous linear maps in two ways. We let (V, W) denote that space topologized by letting a subbasic subspace be

$$\{ f \colon V \to W \mid f(v) \in W_o \},$$

where W_o is an open subspace in W. In fact, we are topologizing (V, W) as a subspace of W^V. Since the latter is separated, so is (V, W). It is not a closed subspace in general unless V is discrete. In particular, if V is discrete and W is LC, (V, W) is also LC. The second topology is finer. A basic open subspace of $[V, W]$ is

$$\{ f \colon V \to W \mid f(V_o) \subset W_o \},$$

where $V_o \subset V$ is an LC subspace and $W_o \subset W$ is open. We call (V, W) and $[V, W]$ *the weak and strong internal hom* respectively. The fact that

$$[V, W] \to (V, W)$$

is continuous follows from the fact that the subspace spanned by finitely many vectors is LC.

We let V^* denote (V, K) and $V^{\wedge}$ denote $[V, K]$.

PROPOSITION 1.2. *The space K is a generator and a cogenerator in $\mathfrak{V}$.*

PROOF. That it is a generator is evident. To show it is a cogenerator we need only show that if $V \in \mathfrak{V}$ and $v \neq 0$ in V, then there is a $\phi \in V^*$ such that $\phi(v) \neq 0$. But first choose an open subspace $U \subset V$ with $v \notin U$. Then V/U is discrete and $v + U$ is not the zero class. The existence of the required functional now follows from standard linear theory.

PROPOSITION 1.3. *In the category $\mathfrak{V}$ monos are 1-1 maps; epis are maps with dense image; regular monos are inclusions of closed subspaces; regular epis are quotient maps and are open; the category is well- and co-well-powered.*

PROOF. This is routine and is left to the reader.

PROPOSITION 1.4. *The category $\mathfrak{V}$ is complete and cocomplete.*

PROOF. The completeness is routine. Direct sums are topologized by the finest topology rendering the injections continuous. The coequalizer of two maps is the cokernel of their difference. The cokernel of a map is the quotient modulo the closure of the image.

THEOREM. *The space K is projective and injective (with respect to the classes of regular epis and subspace inclusions, respectively).*

PROOF. The projectivity is trivial. Suppose $V \subset W$ is a subspace and suppose $\phi: V \to K$. As usual ϕ is uniformly continuous and K is discrete so that ϕ has a unique extension to the closure of V. Thus we may suppose V to be closed. Let $V_0 = \ker\phi$. Then V_0 is closed in V and hence in W. We have that $V/V_0 \to W/V_0$ is continuous and V/V_0 is LC so that the map is an isomorphism with the image, hence V/V_0 still has the subspace topology from W/V_0. Thus we may suppose that V is one dimensional, in fact that $V = Kv$. Now there is some functional defined on all of W which is nonzero on v and a suitable scalar multiple of it is $\phi(v)$.

2. The weak duality.

PROPOSITION 2.1. *For a fixed* $V \in \mathfrak{V}$, *the functor* $(V, \cdot)$ *commutes with inverse limits. Thus it has an adjoint* $\cdot \otimes V$.

PROOF. At the level of the underlying vector spaces, that is clear. Only the topology is at issue. But if $\{V_i\}$ is a family of spaces, $(V, \Pi V_i)$ is topologized as a subspace of $(\Pi V_i)^V \cong \Pi(V_i^V)$ and $\Pi(V, V_i)$ is topologized as a subspace of the latter. Similarly, if $V_1 \subset V_2$ with the subspace topology, (V, V_1) is topologized as a subspace of V_1^V, hence of V_2^V, hence of (V, V_2). The existence of the adjoint now follows from the special adjoint functor theorem.

PROPOSITION 2.2. *The* $\cdot \otimes \cdot$ *described above is symmetric.*

PROOF. Both maps $U \otimes V \to W$ and $V \otimes U \to W$ correspond to a certain set of bilinear maps $U \times V \to W$. It is only necessary to show that they each correspond to the same set. Let $F: U \times V \to W$ be a bilinear map and

$$F_1 : U \to Hom(V, W)$$

be the corresponding linear map. In order that F_1 map $|U|$ to (V, W), it is necessary and sufficient that for each $u \in U$, $F(u, \cdot)$ be continuous on V. In order that F_1 be continuous on U, it is necessary and sufficient that the inverse image of

$$\{f \mid f(v) \in W_o\}, \quad \text{where } v \in V, \ W_o \text{ is an open subspace of } W,$$

be open in U. But this is exactly the condition that for a fixed v, $F(\cdot, v)$ be continuous on U. Thus F corresponds to a map $U \to (V, W)$ iff it is separately continuous in U and V which is evidently the condition to correspond to a map $V \to (U, W)$.

COROLLARY 2.3. *The natural map* $|V| \to |V^{**}|$, *which assigns to* $v \in V$ *the functional on* V^* *evaluation at* v, *is realized by a map* $V \to V^{**}$.

PROOF. In view of the symmetry above we have the following sequence of maps, the first being the identity:

$$V^* \to V^* = (V, K), \quad V^* \otimes V \to K,$$

$$V \otimes V^* \to K, \quad V \to (V^*, K) = V^{**}.$$

When this natural map is an isomorphism we will say that V *is weak reflexive* or, if the context is clearly that of the weak *hom* , simply *reflexive.*

THEOREM 2.4. *The natural map* $V \to V^{**}$ *is always 1-1 and onto.*

PROOF. The hypothesis of separation is equivalent to its being 1-1. To show it is onto, observe that we have topologized V^* as a subspace of K^V. Thus, since K is injective, any functional $V^* \mapsto K$ extends to a functional $f: K^V \to K$. Continuity together with discreteness of K implies that this factors through the projection onto a finite number of factors. For

$$ker f = f^{-1}(0)$$

is open, and any open subset of the product is the inverse image of some (necessarily open) set under the projection to a finite product. So f has a factorization

$$K^V \to K^{\{v_1\}} \times \ldots \times K^{\{v_n\}} \to K.$$

The second, being a functional on a finite dimensional space, is a linear combination of evaluations, say

$$\phi \mapsto \alpha_1 \phi(v_1) + \ldots + \alpha_n \phi(v_n).$$

Applied to a $\phi \in V^*$, that is, to a linear map, this is the same as evaluation at $\alpha_1 v_1 + \ldots + \alpha_n v_n$.

Hence reflexiveness reduces to the purely topological question of whether or not $V \to V^{**}$ is closed (or open). For example, an immediate consequence is :

COROLLARY 2.5. *If* V *is LC,* V *is reflexive.*

COROLLARY 2.6. *For any* V, V^* *is reflexive.*

PROOF. We have $V \to V^{**}$ is onto and 1-1. Dualizing gives us $V^{***} \to V^*$, which is right inverse - as always - to the natural map $V^* \to V^{***}$. Since that is 1-1 and onto, they are inverse isomorphisms.

A space V is called *representationally discrete* (RD) if every linear map $|V| \to K$ is continuous. Evidently a discrete space is RD, but the converse fails. Clearly the condition is equivalent to every cofinite dimensional subspace being open and that, in turn, implies that every subspace - being the intersection of the hyperplanes containing it - is closed. The minimal RD (MRD) topology is the one in which the open subspaces are precisely the cofinite dimensional ones. That clearly defines a topology, which, just as clearly, is discrete only in the finite dimensional case.

THEOREM 2.7. *A space is RD iff its dual is LC; the dual of an LC space is MRD; an RD space is reflexive iff it is MRD.*

PROOF. If V is RD, then V^* is closed in K^V (see the discussion following 1-1 above) and hence is LC. If V is LC, then from the known duality $V \cong |V^*|^*$, whence the natural map $V^{**} \to |V^*|^*$ is 1-1 and onto. Thus V^* is RD. Suppose there is another RD topology on the same set of points as V^*. Call the resultant topology W. We can assume, by taking their *inf* (which can be immediately seen to give an RD topology) that the topology on W is coarser than that on V^*, which means that $V^* \to W$ is continuous. Both V^{**} and W^* are LC so that the map $W^* \to V^{**}$, which is clearly 1-1 and onto (they are both the set of all functionals on $|V^*| = |W|$), is an isomorphism. Thus $V^{***} \to W^{**}$ is also. The result can now be seen from the commutative square

$$\begin{tikzcd} V^* \arrow[r] \arrow[d] & W \arrow[d] \\ V^{***} \arrow[r] & W^{***} \end{tikzcd}$$

in which the bottom and left hand map are isomorphisms. If V^* is LC, V^{**} is RD and $V \to V^{**}$ continuous means V has a finer topology which is evidently RD. If V is RD, $V \to V^{**}$ with V^{**} MRD implies they are the same. If V is RD and reflexive, then $V^{**} \cong V$ is RD.

PROPOSITION 2.8. *If $W \subset V$ and V is reflexive, so is W.*

PROOF. Under the map $V^{**} \to V$, the points of W^{**} are carried into W. Since W has the subspace topology, that defines a continuous map $W^{**} \to W$ which is inverse to the natural map.

COROLLARY 2.9. *If V is reflexive, every RD subspace of V must be MRD.*

THEOREM 2.10. *The space V is reflexive iff every discrete subspace is finite dimensional.*

PROOF. The necessity of this condition is 2.9. Suppose now that V is not reflexive and let W be an open subspace of V which is not open in V^{**}. W cannot be cofinite dimensional, for then it would be the intersection of of the kernels of a finite number of functionals. But V and V^{**} have the same functionals, thus W is open in the latter. Now V/W is discrete. Discrete spaces are projective and so $V \to V/W$ splits, giving V an infinite dimensional discrete subspace.

3. The weak *hom*.

PROPOSITION 3.1. *There is a composition law $(V, W) \to ((U, V), (U, W))$ in the sense of [2], I.2.*

PROOF. The function is the obvious one. The only question is continuity and with the pointwise *hom* that is easy.

From this it follows that we have a closed monoidal category. We would like to show that the full subcategory of reflexives is likewise. First we need:

PROPOSITION 3.2. *For any V and any reflexive W, (V, W) is reflexive.*

PROOF.

$$(V, W) \cong (V, W^{**}) \cong (V, (W^*, K)) \cong (V \otimes W^*, K) = (V \otimes W^*)^*$$

and the result follows from 2.6.

We now let $\mathfrak{WR}(K)$ denote the full subcategory of $\mathfrak{V}(K)$ consisting of the weak reflexive spaces.

THEOREM 3.2. *The inclusion $\mathfrak{WR}(K) \subset \mathfrak{V}(K)$ has a left adjoint $V \mapsto V^{**}$.*

PROOF. If W is reflexive, a map $V \to W$ gives

$$V^{**} \to W^{**} \cong W$$

while a map $V^{**} \to W$ gives, up on composition with $V \to V^{**}$, a map $V \to W$.

THEOREM 3.3. *When equipped with $(\cdot, \cdot)$, $\mathfrak{WR}(K)$ is a closed monoidal category.*

PROOF. If U, V, W are reflexive spaces, we have

$$(U.(V, W)) \cong (U \otimes V, W) \cong ((U \otimes V)^{**}, W)$$

so that $(\cdot \otimes V)^{**}$ is the adjoint to $(V, \cdot)$. The same composition law as is used in $\mathfrak{B}(K)$ is of course intended here.

4. The strong dual.

We began this theory with the idea of better understanding vector spaces, including discrete spaces. Although the theory developed in the preceding two sections is attractive, it has the disadvantage of excluding the discrete spaces. It is true that the full subcategory of MRD spaces is equivalent to the category of discrete spaces, but nonetheless the theory leaves something to be desired. Additionally, experience tells us that pointwise convergence is the wrong topology for function spaces. Thus we are led to consider the strong dual and strong *hom*. Recall that $[V, W]$ is the space of homomorphisms which has, as a basis of open subspaces,

$$\{f: V \to W \mid f(V_0) \subset W_0\},$$

where V_0 is an LC subspace in V and W_0 is an open subspace of W. Also $V^{\wedge} = [V, K]$. A space V will be called *strong reflexive* - just reflexive if the context is clear - provided the natural map $|V| \to |V^{\wedge\wedge}|$ comes from an isomorphism $V \to V^{\wedge\wedge}$. As we will see in the next section, it will be necessary to restrict to a subcategory in order to make $[\cdot, \cdot]$ into an internal *hom*.

THEOREM 4.1. *The dual of an RD space is compact; the dual of a compact space is discrete.*

PROOF. The second statement is obvious. The first is clear as soon as we know that an RD space cannot have an infinite dimensional LC subspace. But an infinite dimensional LC space must have discontinuous functionals; otherwise the associated discrete space would be self dual for the discrete dual, which is known to be impossible.

PROPOSITION 4.2. *The natural map $|V| \to |V^{\wedge\wedge}|$ takes open sets in V to open sets in $V^{\wedge\wedge}$.*

PROOF. Let $U \subset V$ be open. Consider

$$0 \to U \to V \to V/U \to 0.$$

Thus V/U is discrete, so $(V/U)^{\wedge}$ is LC, and then $(V/U)^{\wedge\wedge}$ is discrete. The kernel of $V^{\wedge\wedge} \to (V/U)^{\wedge\wedge}$ is the image of U in $V^{\wedge\wedge}$. The kernel is the inverse image of the open set 0 in $(V/U)^{\wedge\wedge}$ and is thus open.

When $U \subset V$, we let $U^{\downarrow} \subset V^{\wedge}$ denote the annihilator of U.

THEOREM 4.3. *The natural map* $|V| \to |V^{\wedge\wedge}|$ *is 1-1 and onto.*

PROOF. When V is LC, the functionals on $V^{\wedge}$ are the same as those on V^* and hence the result follows from 2.4. In the general case a functional $\phi: V^{\wedge} \to K$ must, in order to be continuous, contain in its kernel a subspace of the form $U^{\downarrow}$ for some LC subspace $U \subset V$. But then ϕ induces a functional $\phi': U^{\wedge} \to K$ (continuity is not an issue since $U^{\wedge}$ is discrete) which is evaluation at a $u \in U$. Then so is ϕ.

The result of this and of 2.2 is that there is a natural map $V^{\wedge\wedge} \to V$ which is 1-1 and onto. This is of course not the natural direction for such a map. But it is the direction in which it is continuous and permits an easy proof of

COROLLARY 4.4. *For any* V, $V^{\wedge}$ *is reflexive.*

Since any map from an LC space is closed and any map to a discrete one is open, it follows that discrete and LC spaces are reflexive. A space is locally LC if it has an open LC subspace. In that case the quotient is discrete, hence projective, and thus the quotient map splits and the space is the direct sum of a discrete and an LC space. Such a space is clearly reflexive. To see that not every reflexive is locally LC, we consider the following

EXAMPLE 4.5. Let V be the direct sum of $\aleph_0$ copies of K topologized as a subspace of the direct product W. We have

$$|V| \to V \to W \quad \text{and} \quad W^{\wedge} \to V^{\wedge} \to |V|^{\wedge}.$$

Clearly

$$W^{\wedge} \cong |V|, \quad |V|^{\wedge} \cong W.$$

and I claim that under these isomorphisms, $V^{\wedge} \cong V$. For V is evidently dense in W, from which $W^{\wedge} \to V^{\wedge}$ is 1-1. It is also onto, since every continuous homomorphism between topological groups is uniformly continuous and hence has a unique extension to the closure. V has no infinite dimensional LC subspace, since V has countable dimension and any infinite dimensional LC subspace, being the dual of a discrete space, must have dimension at least $2^{\aleph_0}$. Thus $V^{\wedge}$ is topologized by pointwise convergence and since $|V|^{\wedge}$ is also, it follows that $V^{\wedge}$ has the subspace topology from $|V|^{\wedge}$. Thus $V \cong V^{\wedge}$ and $V^{\wedge\wedge} \to V$ is an isomorphism. Also V is not locally LC.

We say that a subspace $U \subset V$ *is representationally open* provided every functional on $|V|$ which vanishes on U is continuous on V. It is clear that U is representationally open iff V/U is RD.

THEOREM 4.6. *The space V is strong reflexive iff every representationally open subspace is open iff every RD quotient is discrete.*

PROOF. The equivalence of the two latter statements is evident. If V is strong reflexive, so is every quotient (just turn around the proof of 2.8). But a reflexive RD must be discrete, since its dual is compact and its second dual discrete. Conversely, if V is not reflexive, $V^{\wedge\wedge} \to V$ is not open, so there is a set $U \subset V$ whose image in $V^{\wedge\wedge}$ is open. Since $V^{\wedge} \cong V^{\wedge\wedge\wedge}$, V and $V^{\wedge\wedge}$ have the same functionals. Since U is open in $V^{\wedge\wedge}$, any functional on $|V^{\wedge\wedge}|$ which vanishes on U is continuous on $V^{\wedge\wedge}$, hence on V. Thus U is representationally open and not open.

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ADDED IN PROOF.

In a subsequent paper I will discuss at length the strong *hom* and the modifications necessary to make a closed monoidal category using it.

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