CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

VÁCLAV KOUBEK

On categories into which each concrete category can be embedded

Cahiers de topologie et géométrie différentielle catégoriques, tome 17, nº 1 (1976), p. 33-57

<http://www.numdam.org/item?id=CTGDC_1976__17_1_33_0>

CAHIERS DE TOPOLOGIE
ET GEOMETRIE DIFFERENTIELLE

Vol. XVII-1 (1976)

ON CATEGORIES INTO WHICH EACH CONCRETE CATEGORY CAN BE EMBEDDED

by Vaclav KOUBEK

Hedrlin and Kučera proved that under some set-theoretical assumptions (the non-existence of «too many» measurable cardinals) each concrete category is embeddable into the category of graphs. Therefore under these assumptions each concrete category is embeddable into every binding category, i.e. a category into which the category of graphs is embeddable. The aim of the present Note is to characterize the binding categories in a class of concrete categories, the categories $S(F)$, defined as follows: let F be a covariant functor from sets to sets; the objects of $S(F)$ are pairs (X, H) where X is a set, $H \subset FX$, and the morphisms from (X, H) to (Y, K) are mappings $f: X \to Y$ such that $Ff(H) \subset K$.

The categories $S(F)$, explicitely defined by Hedrlin, Pultr and Trnkova, are categories which play an important role in Topology, Algebra and other fields. They also describe a great number of concrete categories created by Bourbaki construction of structures. They are investigated in a lot of papers [1, 3, 4, 9, 10, 11].

The main result: $S(F)$ is binding if and only if F does not preserve unions of a set with a finite set; assuming the finite set-theory, $S(F)$ is binding for all functors F with the exception (up to natural equivalence) of $(C \times I) \vee K$, where C, K are constant functors and I is the identity functor.

I want to express my appreciations to J. Adamek and J. Reiterman with whom I discussed various parts of the manuscript.

1

CONVENTION. *Set* denotes the category of sets and mappings. A covariant functor from *Set* to *Set* is called a set functor.

DEFINITION. Let $(\mathfrak{K}, U)$, $(\mathfrak{L}, V)$ be concrete categories. A full embedding $\Phi:(\mathfrak{K}, U) \to (\mathfrak{L}, V)$ is said to be *strong* if there exists a set functor F, such that

$$\begin{array}{ccc} \mathfrak{K} & \xrightarrow{\ \Phi\ } & \mathfrak{L} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle V} \\ Set & \xrightarrow{\ F\ } & Set \end{array}$$

commutes. The functor F is said to carry Φ.

PROPOSITION 1.1. *Denote* $\mathfrak{R}$ *the category of graphs (relations* (X, R), $R \subset X \times X$ *) and compatible mappings*

$$(f:(X, R) \to (Y, S) \quad \text{with} \quad f \times f(R) \subset S)$$

and $\mathfrak{R}_\Delta$ *its full subcategory of undirected, antireflexive, connected graphs (symmetric antireflexive relations where each pair of vertices is connected by some path). There exists a strong embedding of* $\mathfrak{R}$ *into* $\mathfrak{R}_\Delta$.

Proof: see [12].

DEFINITION. An object o of a category is *rigid* if $\{1_o\} = Hom(o, o)$.

PROPOSITION 1.2. *For each infinite cardinal* α *(considered to be the set of all ordinals with type smaller than* α*) there exists a full subcategory* $\mathfrak{R}_\alpha$ *of* $\mathfrak{R}_\Delta$ *into which* $\mathfrak{R}$ *is strongly embeddable such that for each* (X, R) *in* $\mathfrak{R}_\alpha$, $\alpha \subset X$, *and, for each* $f:(X, R) \to (Y, S)$ *in* $\mathfrak{R}_\alpha$, $f/\alpha = 1_\alpha$.

PROOF. In [12] a strong embedding Φ of the following category $\mathfrak{R}_{222}$ into $\mathfrak{R}_\Delta$ is constructed: objects of $\mathfrak{R}_{222}$ are (X, R_1, R_2, R_3) with $R_i \subset X \times X$, morphisms $f:(X, R_1, R_2, R_3) \to (Y, S_1, S_2, S_3)$ are mappings

$$f: X \to Y \quad \text{with } f \times f(R_i) \subset S_i, \quad i = 1, 2, 3.$$

Furthermore it was proved in [16] that there exists a rigid graph (α, T).

Given a graph (X, R), put $(X^*, R_1, R_2, R_3) \in \mathfrak{R}_{222}$,

$$X^* = X \vee \alpha, \quad R_1 = R, \ R_2 = X \times X, \ R_3 = T.$$

Then for each morphism

$$f: (X^*, R_1, R_2, R_3) \to (Y^*, S_1, S_2, S_3)$$

there exists a compatible mapping

$$g: (X, R) \to (Y, S) \quad \text{with} \quad f = g \vee 1_\alpha .$$

In other words, a strong embedding $\psi_\alpha : \mathfrak{R} \to \mathfrak{R}_{222}$ is formed

$$(\psi_\alpha(X, R) = (X^*, R_1, R_2, R_3), \ \psi_\alpha g = g \vee 1_\alpha)$$

such that the image $\mathfrak{R}_\alpha$ of $\mathfrak{R}$ under $\phi\psi_\alpha$ has the required properties.

PROPOSITION 1.3. *If F is a subfunctor of a factorfunctor of G, then $S(F)$ is strongly embeddable into $S(G)$.*

Proof: see [11].

CONVENTION. All set functors F are supposed to be regular, i.e. each transformation from $C_{0,1}$ (where

$$C_{0,1} X = 1 \text{ if } X \neq \emptyset, \quad C_{0,1} \emptyset = \emptyset)$$

to F has a unique extension to a transformation of C_1 to F. In particular, if F is constant on the subcategory of all non-void sets and mappings, then $F = C_X$ for some X (which is the reason for this convention). For each set functor F we clearly have a regular functor F' coinciding with F on non-void sets and mappings; $S(F)$ is binding iff $S(F')$ is.

2

DEFINITION. Denote by $\mathfrak{J}$ the concrete category the objects of which, called *spaces*, are pairs $(X, \mathfrak{U})$ where X is a set and $\mathfrak{U} \subset \exp X$, and morphisms from $(X, \mathfrak{U})$ to $(Y, \mathfrak{V})$ are mappings $f: X \to Y$ such that

1° for each $A \in \mathfrak{U}$ there exists $B \in \mathfrak{V}$ with $B \subset f(A)$;

2° if f is one-to-one on $A \in \mathfrak{U}$, then $f(A) \in \mathfrak{V}$.

Furthermore, given a cardinal α, denote by $\mathfrak{J}_\alpha$ the full subcategory of $\mathfrak{J}$

over all $(X, \mathfrak{U})$ such that:

$$\text{if } A \in \mathfrak{U} \text{ then } card\, A = \alpha .$$

The spaces of $\mathcal{G}_\alpha$ are called α-spaces.

CONVENTION. Given $(X, \mathfrak{U}) \in \mathcal{G}$, $x \in X$, denote

$$st_{\mathfrak{U}}\, x = card \{ A \in \mathfrak{U} \mid x \in A \}.$$

LEMMA 2.1. *Let* $f: (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ *be a morphism in* $\mathcal{G}$. *If* f *is one-to-one then for each* $x \in X$, $st_{\mathfrak{U}}\, x \leqslant st_{\mathfrak{V}}\, f(x)$. *If moreover* $(X, \mathfrak{U}) = (Y, \mathfrak{V})$ *and* X *is finite, then* $st_{\mathfrak{U}}\, x = st_{\mathfrak{V}}\, f(x)$.

Proof is easy.

CONSTRUCTION 2.2. For each natural number $n \geqslant 3$ we are going to construct a rigid n-space

$$(X, \mathfrak{U}), \text{ where } X = \{0, 1, \dots, 2n\},$$

which has the following properties:

1° for each $a, b \in X$ there exist $T, S \in \mathfrak{U}$ with $a, b \in T$, $a \in S$, $b \in X - S$;

2° denote by m (or M) the minimum (the maximum, respectively) of all $st_{\mathfrak{U}}\, a$, $a \in X$; then $m + M < card\, \mathfrak{U}$ and there exists just one $y \in X$ with $st_{\mathfrak{U}}\, y = m$.

The construction is done by induction. The n-th space is denoted by $(X_n, \mathfrak{U}_n)$.

I. $n = 3$. $\mathfrak{U}_3$ contains the following set ($\{\ \}$ is omitted):

$$012,\ 024,\ 026,\ 036,\ 056,\ 134,\ 156,\ 235,\ 245,\ 246,\ 356.$$

Conditions 1 and 2 are easily verified. To prove that $(X_3, \mathfrak{U}_3)$ is rigid, the above Lemma 2.1 can be used. Any morphism $f: (X_3, \mathfrak{U}_3) \to (X_3, \mathfrak{U}_3)$ must be a bijection and routine reasoning concerning

$$st_{\mathfrak{U}_3}\, 0,\ st_{\mathfrak{U}_3}\, 1, \dots, st_{\mathfrak{U}_3}\, 6$$

shows that f must be the identity.

II. $n > 3$. Choose $x, y \in X_{n-1}$ with

$$st_{\mathfrak{U}_{n-1}}\, y = m_{n-1}, \quad st_{\mathfrak{U}_{n-1}}\, x = M_{n-1}$$

and choose

$$V \in \mathfrak{U}_{n-1} \text{ with } y \in V,\ x \in X_{n-1} - V.$$

Choose arbitrary n-point subsets Z_1, Z_2 of X_{n-1} with $Z_1 \cap Z_2 = \{ y \}$, and define $\mathfrak{U}_n$ as the following collection:

$W \cup \{ 2n-1 \}$ for all $W \in \mathfrak{U}_{n-1} - \{ V \}$;
$W \cup \{ 2n \}$ for all $W \in \mathfrak{U}_{n-1}$ with $y \in W$;
$V - \{ y \} \cup \{ 2n-1, 2n \}$; Z_1; Z_2.

The condition 1 is easy to verify. Let us check 2:

$$st_{\mathfrak{U}_n} 2n = st_{\mathfrak{U}_{n-1}} y + 1 < st_{\mathfrak{U}_{n-1}} y + st_{\mathfrak{U}_{n-1}} x < card\,\mathfrak{U}_{n-1} = st_{\mathfrak{U}_n} 2n-1$$

and because

$$\text{for each } a \in X_{n-1},\ st_{\mathfrak{U}_n} a > st_{\mathfrak{U}_{n-1}} y + 1,$$

we have $st_{\mathfrak{U}_n} 2n = m_n$ and $2n$ is the only element with $st_{\mathfrak{U}_n} = m_n$; further if $a \in X_{n-1} - \{ y \}$, then

$$st_{\mathfrak{U}_n} 2n-1 > st_{\mathfrak{U}_{n-1}} x + st_{\mathfrak{U}_{n-1}} y > st_{\mathfrak{U}_n} a$$

and

$$st_{\mathfrak{U}_n} y = 2(st_{\mathfrak{U}_{n-1}} y) + 1 \leqslant st_{\mathfrak{U}_{n-1}} x + st_{\mathfrak{U}_{n-1}} y < st_{\mathfrak{U}_n} 2n-1$$

and so $st_{\mathfrak{U}_n} 2n-1 = M_n$.

$$m_n + M_n = st_{\mathfrak{U}_n} 2n + st_{\mathfrak{U}_n} 2n-1 = st_{\mathfrak{U}_{n-1}} y + 1 + card\,\mathfrak{U}_{n-1} < card\,\mathfrak{U}_n.$$

The last thing to prove is that $(X_n, \mathfrak{U}_n)$ is rigid. Let

$$f: (X_n, \mathfrak{U}_n) \to (X_n, \mathfrak{U}_n);$$

then f is a bijection, due to 1, and

$$f(2n) = 2n,\ f(2n-1) = 2n-1$$

(as $2n-1$ is the only element with $st_{\mathfrak{U}_n} = M_n$). Therefore $f(X_{n-1}) = X_{n-1}$ and clearly the restriction of f is an endomorphism of $(X_{n-1}, \mathfrak{U}_{n-1})$. So, $f = 1_{X_n}$.

PROPOSITION 2.3. *Given the rigid n-space* $(X, \mathfrak{U})$ *as above, let* P *be an* *arbitrary* $(n-1)$*-point subset of* X *and let* $p \in P$. *Then* $(Y, \mathfrak{V})$ *is a rigid* n*-space, where* $Y = X \times \{0, 1\}$ *and*

$$\mathfrak{V} = (\mathfrak{U} \times \{0, 1\}) \cup \{S_0\}, \quad \textit{with} \quad S_0 = (P \times \{0\}) \cup \{(p, 1)\}.$$

PROOF. Let $f: (Y, \mathfrak{V}) \to (Y, \mathfrak{V})$. Clearly if $S \in \mathfrak{V}$ then f/S is one-to-one and so $f(S) \in \mathfrak{V}$. Let us show that

$$f(X \times \{0\}) = X \times \{i\}, \quad i = 0 \text{ or } 1.$$

If $f(a, 0) = (a_1, 1)$ for some a, a_1, then $f(X \times \{0\}) \subset X \times \{1\}$; if not, let $f(b, 0) = (b_1, 0)$, let $T \in \mathfrak{U}$ contain $\{a, b\}$ (see condition 1, above); we have $f(T \times \{0\}) \in \mathfrak{V}$ and so necessarily $f(T \times \{0\}) = S_0$, in particular $a_1 = p$ and $b_1 \in P$. Therefore, if $x \in X$ then

$$f(x, 0) = (x_1, 0) \quad \text{implies} \quad x_1 \in P$$

and

$$f(x, 0) = (x_1, 1) \quad \text{implies} \quad x_1 = p \text{ and } x = a$$

(if $x \neq a$, then

$$f(x, 0) = f(a, 0) = (p, 1),$$

but it follows from the condition 1 that f is one-to-one on $X \times \{0\}$ and on $X \times \{1\}$). Therefore

$$f((X - \{a\}) \times \{0\}) \subset P \times \{0\}$$

- a contradiction, as f is one-to-one on $X \times \{0\}$ and $\operatorname{card} P < \operatorname{card} X - 1$. So

$$f(X \times \{1\}) = X \times \{i\}, \quad i = 0 \text{ or } 1.$$

Analogously

$$f(X \times \{1\}) = X \times \{j\}, \quad j = 0 \text{ or } 1.$$

It follows that

$$f(x, 0) = (x, i), \quad f(x, 1) = (x, j)$$

(since $(X, \mathfrak{U})$ is rigid). As

$$f(S_0) = P \times \{i\} \cup \{(p, j)\} \in \mathfrak{V},$$

we have $i = 0$, $j = 1$.

CONSTRUCTION 2.4. *For each infinite cardinal* α *we shall construct a ri-*

gid α*-space* $(X, \mathfrak{U})$.

Put $X = \alpha \cup \{a, b\}$ (recall that α is the set of all ordinals with type less than α, assume $a, b \notin \alpha$, $a \neq b$).

$\mathfrak{U}$ con· ists of the following subsets of X :

$$E = \{x+2n\}, \quad \bar{O} = \{x+2n+1\}, \quad D = \{x+3n\},$$

where x runs over all limit ordinals in α and zero while n runs over all naturals ;

$$P_x = \{y \in E \mid y > x\} \cup \{x\} \quad \text{if } x \in \bar{O};$$

$$P_x = \{y \in \bar{O} \mid y > x+2\} \cup \{x\} \quad \text{if } x \in E;$$

$$V \cup \{a, x, y\}, \ x, y \in \bar{O}, \ x \neq y, \ V \subset E, \ card\, E - V = card\, V = \alpha;$$

$$V \cup \{b, x, y\}, \ x, y \in E, \ x \neq y, \ V \subset \bar{O}, \ card\, \bar{O} - V = card\, V = \alpha.$$

PROOF. Let $f: (X, \mathfrak{U}) \to (X, \mathfrak{U})$; we shall show that $f = 1_X$. As $E \in \mathfrak{U}$, $card\, f(E) = \alpha$, therefore there clearly exists $J_E \subset E$ such that:

a) $card\, J_E = \alpha$;

b) f is one-to-one on J_E ;

c) either $f(J_E) \subset E$ or $f(J_E) \subset \bar{O}$, $card\, E - f(J_E) = card\, \bar{O} - f(J_E) = \alpha$.

Analogously $J_{\bar{O}} \subset \bar{O}$.

1° $f(E) \subset E$ *or* $f(E) \subset \bar{O}$.

Assume that, on the contrary, either

$$f(\beta_1) \in E, \quad f(\beta_2) \in \bar{O} \quad \text{with } \beta_1, \beta_2 \in E,$$

or

$$f(\beta) \in \{a, b\} \quad \text{with } \beta \in E.$$

In the former case $J_{\bar{O}} \cup \{\beta_1, \beta_2, b\} \in \mathfrak{U}$ and so there exists

$$T \in \mathfrak{U}, \quad T \subset f(J_{\bar{O}} \cup \{\beta_1, \beta_2, b\}).$$

There follows

$$card(\bar{O} - T) = card(E - T) = \alpha$$

while

$$card(T \cap (E \cup \{a, b\})) \leq 2 \quad \text{or} \quad card(T \cap (\bar{O} \cup \{a, b\})) \leq 2;$$

clearly there is no such $T \in \mathfrak{U}$. In the latter case either there exists

$$\beta' \in E - \{\beta\} \quad \text{with } f(\beta') \in \{a, b\},$$

but then

$$card\, f(J_{\bar{O}} \cup \{\beta, \beta', b\}) \cap E \leqslant 1 \quad \text{or} \quad card\, f(J_{\bar{O}} \cup \{\beta, \beta', b\}) \cap \bar{O} \leqslant 1$$

and you get a contradiction in a similar way - or β is the only one. Choose distinct $\beta_1, \beta_2 \in E - \{\beta\}$; then, as

$$J_{\bar{O}} \cup \{\beta_1, \beta_2, b\} \in \mathfrak{U},$$

clearly $f(b) \in \{a, b\}$, but while $J_{\bar{O}} \cup \{\beta_1, \beta, b\} \in \mathfrak{U}$, this leads to a contradiction in the same way as above.

2° *$f(\bar{O}) \subset E$ or $f(\bar{O}) \subset \bar{O}$.* Analogous.

3° *f is one-to-one.*

a) f is one-to-one on $\bar{O}, E$. In fact, let

$$\beta_1, \beta_2 \in E \quad \text{with } J_{\bar{O}} \cup \{\beta_1, \beta_2, b\} \in \mathfrak{U};$$

then $f(\beta_1) \neq f(\beta_2)$ because else the meet of $f(J_{\bar{O}} \cup \{\beta_1, \beta_2, b\})$ with either E or $\bar{O}$ would have at most one element - a contradiction (analogous as above).

b) f is one-to-one on $\bar{O} \cup E$. Let

$$\beta \in \bar{O}, \ \gamma \in E \quad \text{with } f(\beta) = f(\gamma).$$

We may choose $\beta_1 \in \bar{O} - \{\beta\}$ such that f is one-to-one on $J_E \cup \{\beta_1, \beta, b\}$ - then $f(J_E \cup \{\beta_1, \beta, b\}) \in \mathfrak{U}$, again a contradiction.

c) f is one-to-one - clearly $f(\bar{O} \cup E) = \bar{O} \cup E$ and c easily follows.

Now we have $f(D) = D$ because D is just the element of $\mathfrak{U}$ with

$$card\, D \cap E = card\, D \cap \bar{O} = \alpha .$$

There follows $f(0) \neq 1$ and as either

$$card\, \bar{O} - f(P_0) \leqslant 1 \quad \text{or} \quad card\, E - f(P_0) \leqslant 1,$$

clearly $f(0) = 0$. Clearly then $f(P_0) = P_0$; furthermore

$$f(E) = E, \quad f(\bar{O}) = \bar{O} \quad \text{and} \quad f(a) = a, \quad f(b) = b.$$

Let us prove that $f = 1_X$. If not, we can choose the least ordinal γ , with $f(\gamma) \neq \gamma$; we have

$$\gamma > 0 \quad \text{and clearly } f(P_\gamma) = P_{f(\gamma)}.$$

If $f(\gamma) < \gamma$, then $\gamma \notin E$ because f is one-to-one while $P_{f(\gamma)}$ meets the set $\{\delta \mid \delta < \gamma\}$; analogously $\gamma \notin \overline{O}$. Therefore $f(\gamma) > \gamma$; if $\gamma \in \overline{O}$ then

$$card\, E - (P_{f(\gamma)} \cup \{\delta \mid \delta < \gamma\}) > 1$$

but as $f(E) = E$ and

$$card\, E - (P_{\gamma} \cup \{\delta \mid \delta < \gamma\}) = 1$$

this is a contradiction - analogously if $\gamma \in E$. That concludes the proof.

THEOREM 2.5. *For each cardinal* $\alpha > 1$ *there exists a strong embedding* $\phi: \mathfrak{R}_{\Delta} \to \mathfrak{I}_{\alpha}$ *carried by the sum of the identity functor and a constant functor.* ϕ *has the following property:*

given a morphism $f: (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ *in* $\mathfrak{I}_{\alpha}$ *which is an image of a morphism in* $\mathfrak{R}_{\Delta}$ *under* ϕ, *then* f *is one-to-one on each set* $A \in \mathfrak{U}$.

PROOF. *1° α is finite.*

As $\mathfrak{R}_{\Delta} = \mathfrak{I}_2$ we may assume $\alpha \geqslant 3$. Let $(X, \mathfrak{U})$ and $(Y, \mathfrak{V})$ be the rigid α-spaces from Construction 2.2 and Proposition 2.3. Let V be an $(\alpha - 2)$-point subset of X, disjoint from P (see 2.3). Define

$$\phi: \mathfrak{R}_{\Delta} \to \mathfrak{I}_{\alpha} \text{ by } \phi(Z, R) = (Z \vee Y, \mathfrak{V}_R),$$

where

$$\mathfrak{V}_R = \mathfrak{V} \cup \{\{x, y\} \cup V \times \{i\} \mid (x, y) \in R, \quad i = 0, 1\};$$

if $f: (Z_1, R_1) \to (Z_2, R_2)$, then

$$\phi f = f \text{ on } Z_1, \quad \phi f = 1_Y \text{ on } Y.$$

Clearly ϕ is a faithful functor.

Let us prove that ϕ is full. Let (M, R), (N, Q) be graphs of $\mathfrak{R}_{\Delta}$; let

$$f: (M \vee Y, \mathfrak{V}_R) \to (N \vee Y, \mathfrak{V}_Q)$$

be a morphism in $\mathfrak{I}_{\alpha}$. We shall show that $f(M) \subset N$ and f/M is a compatible mapping.

a) $f(Y) \subset Y$. If, on the contrary, $f(x, i) \in N$ for some $(x, i) \in Y$, choose $T \in \mathfrak{U}$ with $x \in T$; as $f(T \times \{i\}) \in \mathfrak{V}_Q$ necessarily

$$f(T \times \{i\}) = V \times \{j\} \cup \{\overline{x}, \overline{y}\}$$

and so for an arbitrary $v \in V$ there exists

$$y_1 \in T \quad \text{with} \quad f(y_1, i) = (v, j).$$

Choose

$$T' \in \mathfrak{U} \quad \text{with} \quad x \in T', \; y_1 \notin T'$$

and apply the same reasoning to T' - there exists

$$y_2 \in T' \quad \text{with} \quad f(y_2, i) = (v, j').$$

Choose $T'' \in \mathfrak{U}$ with $y_1, y_2 \in T''$; then $j = j'$ because else

$$f(T'' \times \{i\}) \cap (V \times \{k\}) \neq \emptyset, \; k = 0, 1.$$

Therefore $f(y_1, i) = f(y_2, i)$ - a contradiction with

$$card\, f(T'' \times \{i\}) = card(T'' \times \{i\}).$$

b) $f = 1_Y$ on Y - follows from the fact that $(Y, \mathfrak{V})$ is rigid.

c) $f(M) \subset N$. Assume on the contrary $f(z) \in Y$ with $z \in M$. Let

$$z_1 \in M \quad \text{with} \quad (z, z_1) \in R ;$$

then

$$\{z, z_1\} \cup (V \times \{i\}) \in \mathfrak{V}_R \quad \text{and} \quad V \cap P = \emptyset ;$$

we have $f(z) \in X \times \{i\}$ for both $i = 0, 1$ - a contradiction.

d) f is compatible. This follows easily from

$$\{z_1, z_2\} \cup V \times \{0\} \in \mathfrak{V}_R \quad \text{for all} \quad (z_1, z_2) \in R.$$

2^0 α *is infinite.*

Let $(X, \mathfrak{U})$ be the rigid α-space from Construction 2.4. Define

$$\Phi : \mathfrak{R}_\alpha \to \mathfrak{J}_\alpha \quad \text{by} \quad \Phi(Z, R) = (Z \vee X, \mathfrak{V}_R),$$

where

$$\mathfrak{V}_R = \mathfrak{U} \cup \{(D \cap E) \cup \{x, y\} \mid (x, y) \in R\} ;$$

if $f : (Z_1, R_1) \to (Z_2, R_2)$, then

$$\Phi f = f \;\text{ on }\; Z_1, \quad \Phi f = 1_X \;\text{ on }\; X.$$

Again Φ is clearly a faithful functor and we shall prove that it is full. To this end, let

$$f : (M \vee X, \mathfrak{V}_R) \to (N \vee X, \mathfrak{V}_Q)$$

be a morphism in $\mathfrak{J}_\alpha$. Then as $E \in \mathfrak{V}_R$, clearly $card\, f(E) = \alpha$ and so there exists $J_E \subset E$ such that $card J_E = \alpha$, f is one-to-one on J_E,

$f(J_E) \subset E$ or $f(J_E) \subset \bar{O}$ and $card E - f(J_E) = card \bar{O} - f(J_E) = \alpha$.

Analogously $J_{\bar{O}} \subset \bar{O}$.

a) $f(a), f(b) \in \{a, b\}$. Choose $\beta_1, \beta_2 \in J_E$; as

$$J_{\bar{O}} \cup \{\beta_1, \beta_2, b\} \in \mathfrak{O} \quad .$$

there is $A \in \mathfrak{O}_Q$ with

$$A \subset f(J_{\bar{O}} \cup \{\beta_1, \beta_2, b\} ;$$

clearly $f(J_{\bar{O}} \cup \{\beta_1, \beta_2, b\})$ meets $\{a, b\}$ - therefore $f(b) \in \{a, b\}$. Analogously $f(a) \in \{a, b\}$.

b) $f(X) \subset X$ (thus $f = 1_X$ on X). Let $\delta \in E$ with $f(\delta) \in N$; then f is one-to-one on $J_{\bar{O}} \cup \{\delta, \beta, b\}$ for some $\beta \in J_E$, but clearly

$$f(J_{\bar{O}} \cup \{\delta, \beta, b\}) \notin \mathfrak{O}_Q$$

- a contradiction. Analogously $\delta \in \bar{O}$ - therefore

$$f(E \cup \bar{O}) \subset X \quad \text{and so} \quad f(X) \subset X.$$

c) $f(M) \subset N$. Assume that, on the contrary, there exists $z \in M$ with $f(z) \in X$. Let $z_1 \in M$ with $(z, z_1) \in R$; then as

$$\{z, z_1\} \cup (D \cap E) \in \mathfrak{O}_R,$$

there is

$$T \in \mathfrak{O}_Q \text{ with } T \subset f(\{z, z_1\} \cup (D \cap E)) .$$

As $f(z) \in X$, clearly $T \in \mathfrak{U}$ - an evident contradiction.

d) f is compatible. This follows easily from the construction of $\mathfrak{O}_R, \mathfrak{O}_Q$. Thus we found a full embedding $\phi : \mathfrak{R}_\Delta \to \mathfrak{J}_\alpha$ for all $\alpha > 1$; a straightforward verification of the required properties of ϕ is left to the reader.

3

CONVENTION. Let $\mathfrak{F}$ be a filter on a set V. Put

$$P\mathfrak{F} = \bigcap_{A \in \mathfrak{F}} A, \quad |\mathfrak{F}| = \min_{A \in \mathfrak{F}} card(A - P\mathfrak{F}).$$

Given a mapping $f: V \to X$, let $f(\mathfrak{F})$ be the filter on X with

$$f(\mathfrak{F}) = \{ B \subset X \mid f(A) \subset B \quad \text{for some } A \in \mathfrak{F} \}.$$

For each set X put

$$\mathfrak{F}(X) = \{ i(\mathfrak{F}/Z) \mid Z \in \mathfrak{F}, \quad i: Z \to X \text{ is a one-to-one mapping} \},$$

where

$$\mathfrak{F}/Z = \{ A \cap Z \mid A \in \mathfrak{F} \}.$$

DEFINITION. Let $\mathfrak{G}$ be a filter on a set V. Denote by $\mathfrak{J}^{\mathfrak{G}}$ the concrete category whose objects are couples $(X, \mathfrak{U})$ where $\mathfrak{U} \subset \mathfrak{G}(X)$, and whose morphisms f from $(X, \mathfrak{U})$ to $(Y, \mathfrak{V})$ are mappings $f: X \to Y$ such that:

1° for each $\mathfrak{H} \in \mathfrak{U}$ there exists $\mathfrak{K} \in \mathfrak{V}$ with $f(\mathfrak{H}) \subset \mathfrak{K}$;

2° if f is one-to-one on some $A \in \mathfrak{H} \in \mathfrak{U}$, then $f(\mathfrak{H}) \in \mathfrak{V}$.

NOTE. $\mathfrak{J}^{\mathfrak{G}} = \mathfrak{J}_{\alpha}$ if $\mathfrak{G}$ is a filter with

$$P\mathfrak{G} \in \mathfrak{G} \quad \text{and} \quad card P\mathfrak{G} = \alpha .$$

Therefore if

$$P\mathfrak{G} \in \mathfrak{G} \quad \text{and} \quad card P\mathfrak{G} > 1$$

there is a strong embedding of $\mathfrak{R}_{\Delta}$ to $\mathfrak{J}^{\mathfrak{G}}$ - see Theorem 2.5.

THEOREM 3.1. *Let $\mathfrak{G}$ be a filter such that $card P\mathfrak{G} > 1$. Then there exists a strong embedding of $\mathfrak{R}$ into $\mathfrak{J}^{\mathfrak{G}}$.*

PROOF. To prove the theorem we shall construct a strong embedding ψ of $\mathfrak{R}_{\alpha}$ into $\mathfrak{J}^{\mathfrak{G}}$ (see Proposition 1.2) where $\alpha = |\mathfrak{G}|$. Let $\phi: \mathfrak{R}_{\Delta} \to \mathfrak{J}_{\beta}$ be the strong embedding constructed in Theorem 2.5,

$$\beta = card P\mathfrak{G}, \quad \phi(Z, R) = (\overline{Z}, \mathfrak{W}_R).$$

Put $\psi(Z, R) = (\overline{Z}, \mathfrak{U}_R)$, where

$$\mathfrak{U}_R = \{ \mathfrak{F} \in \mathfrak{G}(\overline{Z}) \mid P\mathfrak{F} \in \mathfrak{W}_R \quad \text{and} \quad (\alpha \cup P\mathfrak{F}) \in \mathfrak{F} \},$$

and if f is a morphism put $\psi f = \phi f$. Then ψ is easily seen to be a faithful functor. To prove that ψ is full, assume that

$$g: (\overline{Z}, \mathfrak{U}_R) \to (\overline{Z}', \mathfrak{U}_Q) \quad \text{in } \mathfrak{J}^{\mathfrak{G}};$$

we shall show that g is a morphism from $(\overline{Z}, \mathfrak{W}_R)$ to $(\overline{Z'}, \mathfrak{W}_Q)$ in $\mathfrak{I}_\beta$. Then ψ is full because ϕ is full and $\psi = \phi$ on morphisms.

Let $U \in \mathfrak{W}_R$; we have to show that $g(U) \in \mathfrak{W}_Q$. If $\alpha > \beta$ then there exists $Y \subset \alpha$ with $card\, Y = \alpha$ such that g is one-to-one on Y. Denote by T the underlying set of the filter $\mathfrak{G}$. Let $h : T \to \overline{Z}$ be a one-to-one mapping with

$$h(T) = U \cup Y \quad \text{and} \quad h(P\mathfrak{G}) = U.$$

Let $\mathfrak{H} \in \mathfrak{U}_R$ with

$$P\mathfrak{H} = U \quad \text{and} \quad U \cup Y \in \mathfrak{H}.$$

As there exists $\mathfrak{F} \in \mathfrak{U}_Q$ with

$$\mathfrak{F} \supset \{ V \subset \overline{Z'} \mid g(V_1) \subset V \text{ for some } V_1 \in \mathfrak{H} \},$$

clearly $g(U) \supset P\mathfrak{F} \in \mathfrak{W}_Q$. If $\alpha \leqslant \beta$ then we prove that again g is one-to-one on a set $Y \subset \alpha$ with $card\, Y = \alpha$ and proceed analogously as above. Assume the contrary. Then $card\, g(\alpha) < \alpha$. Let $E \subset X_\beta$ as in Construction 2.4. As $card(\alpha \cup E) \geqslant \beta$ (take $\mathfrak{H} \in \mathfrak{W}_R$ with

$$P\mathfrak{H} = E \quad \text{and} \quad E \cup \alpha \in \mathfrak{H}$$

and proceed with $g(\mathfrak{H})$ as above), there exists $S_1 \subset E$ with

$$card\, S_1 = card\, E - S_1 = \beta,$$

g is one-to-one on S_1 and either

$$g(S_1) \subset \alpha \quad \text{or} \quad g(S_1) \cap \alpha = \emptyset.$$

Clearly there exist $k_1, k_2 \in \overline{O}$ such that g is one-to-one on

$$C = S_1 \cup \{k_1, k_2, a\} \in \mathfrak{W}_R.$$

Let $\mathfrak{K} \in \mathfrak{U}_R$ with $P\mathfrak{K} = C$. Then there exists $\mathfrak{L} \in \mathfrak{U}_Q$ such that for each $B \in \mathfrak{K}$ we have $g(B) \in \mathfrak{L}$. That is clearly impossible if $g(S_1) \cap \alpha = \emptyset$. As $\mathfrak{L} \in \mathfrak{U}_Q$,

$$card\, U \cap (\overline{Z'} - \alpha) = \beta \geqslant \alpha \quad \text{for each } U \in \mathfrak{L}$$

- a contradiction.

NOTE 3.2. The embedding $\psi : \mathfrak{R}_\alpha \to \mathfrak{I}^{\mathfrak{G}}$ defined above has the property that, if $f : (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ is a morphism in $\mathfrak{I}^{\mathfrak{G}}$ which lies in the image of ψ;

then for each $\mathcal{H} \in \mathfrak{U}$ there exists $A \in \mathcal{H}$ such that f is one-to-one on A. This follows from the above proof.

We shall now investigate $\mathfrak{g}^{\mathcal{G}}$ where $\mathcal{G}$ is a filter on a set V such that

$$card\, P\mathcal{G} = 1, \quad card\, A = card\, V \quad \text{for each} \quad A \in \mathcal{G}.$$

Write

$$\mathcal{G}^* = \{A - P\mathcal{G} \mid A \in \mathcal{G}\}$$

and notice that

$$card\, \mathcal{G}(X) = card\, \mathcal{G}^*(X) = card\, \mathcal{G}(Y)$$

for arbitrary sets $X \subset Y$ with $card\, X = card\, Y$.

DEFINITION. A system $\mathfrak{A}$ of subsets of a set X is said to be α-*almost disjoint* if

$$card\, A = \alpha \quad \text{for each} \quad A \in \mathfrak{A},$$

while

$$card\, A \cap B < \alpha \quad \text{for each} \quad A, B \in \mathfrak{A},\ A \neq B.$$

THEOREM 3.4. *For each cardinal* α *there exists an* α*-almost disjoint system* $\mathfrak{A}$ *on a set* X *such that* $card\, \mathfrak{A} = card\, 2^X$.

PROOF. Define cardinals β_i, where i is an ordinal:

$$\beta_0 = \aleph_0, \quad \beta_{i+1} = 2^{\beta_i}, \quad \beta_i = \sup_{i>j} \beta_j \quad \text{if } i \text{ is a limit ordinal.}$$

Put

$$\mathfrak{B} = \{ U \subset \beta_\alpha \mid U \subset \delta \quad \text{for some} \quad \delta < \beta_\alpha \}.$$

Clearly $card\, \mathfrak{B} = \beta_\alpha$. It is easy to see that $(\beta_\alpha)^\alpha = 2^{\beta_\alpha}$ so that there exist 2^{β_α} subsets L of β_α whose power is α, i.e. 2^{β_α} monotone mappings

$$f: \alpha \to \beta_\alpha \quad (\text{put } f = f_L \text{ if } f(\alpha) = L\ ).$$

Put for each L :

$$T(L) = \{ f_L(\delta) \mid \delta < \alpha \} \subset \mathfrak{B}.$$

Then

$$\mathfrak{A} = \{ T(L) \mid L \subset \beta_\alpha,\ card\, L = \alpha \}$$

is an α-almost disjoint system : clearly $card\, T(L) = \alpha$ and if $L_1 \neq L_2$, then

$$card\,(T(L_1) \cap T(L_2)) < \alpha\ .$$

Clearly $card\, \mathfrak{A} = 2^{\beta_\alpha}$.

COROLLARY 3.5. *For each filter* $(V, \mathcal{G})$ *such that*

$$card\, V = card\, A = \alpha \geqslant \aleph_0 \qquad \text{for each} \quad A \in \mathcal{G},$$

there exists a set X *with* $card\, \mathcal{G}(X) = card\, 2^X$.

PROOF. Let X be a set with an α-almost disjoint system $\mathfrak{A}$ on X such that

$$card\, \mathfrak{A} = card\, 2^X\ ;$$

for each $T \in \mathfrak{A}$, let $f_T : V \to X$ be a one-to-one mapping with $f_T(V) = T$. Then clearly $T_1, T_2 \in \mathfrak{A}$, $T_1 \neq T_2$ implies $f_{T_1}(\mathcal{G}) \neq f_{T_2}(\mathcal{G})$.

CONSTRUCTION 3.6. We are going to construct for each filter $(V, \mathcal{G})$ a rigid object $(X, \mathfrak{O})$ of $\mathcal{J}^{\mathcal{G}}$. X is a set with

$$card\, \mathcal{G}(X) = card\, 2^X.$$

Put $\alpha = card\, V$. First of all we shall introduce the following notation (each cardinal is considered to be the well-ordered set of all ordinals of smaller type) :

$$\widetilde{H} = \{ Z \subset X \mid card\, Z = \alpha \}$$

- assume that $\widetilde{H}$ is well-ordered,

$$\widetilde{H} = \{ Z_i \mid i \in card\, \widetilde{H} \}.$$

Given $f: X \to X$, put

$$W(f) = \{ x \in X \mid f(x) \neq x \},$$

$$\mathfrak{T} = \{ f: X \to X \mid card\, W(f) \geqslant \alpha \}$$

- assume that $\mathfrak{T}$ is well-ordered,

$$\mathfrak{T} = \{ f_j \mid j \in card\, \mathfrak{T} \}.$$

Choose distinct $a, b \in X$ and put

$$\mathfrak{T} = \{ f : X \to X \mid f \notin \mathfrak{T} \quad \text{and } W(f) - \{a, b\} \neq \emptyset \}$$

- assume that $\mathfrak{T}$ is well-ordered,

$$\mathfrak{T} = \{ g_j \mid j \in card\,\mathfrak{T} \} .$$

We are going to define

$$\mathfrak{U}_\beta \subset \mathfrak{G}^*(X), \quad \mathfrak{V}_\beta \subset \mathfrak{G}(X)$$

by transfinite induction :

$$\mathfrak{V}_{-1} = \emptyset, \quad \mathfrak{U}_{-1} = \{ \mathfrak{H}_1^*, \mathfrak{H}_2^*, \mathfrak{H}_3^* \}$$

where

$$\mathfrak{H}_1^*, \mathfrak{H}_2^*, \mathfrak{H}_3^* \in \mathfrak{G}^*(X) \quad \text{are distinct.}$$

Now assume that $\mathfrak{U}_\gamma$, $\mathfrak{V}_\gamma$ are defined for all $\gamma < \beta$.

a) If $card\,\beta \leqslant card\,\mathfrak{H}$, then we may choose $\mathfrak{A}_\beta \in \mathfrak{G}(Z_\beta \cup \{a\})$ with

$$P\,\mathfrak{A}_\beta = \{a\} \quad \text{and} \quad \mathfrak{A}_\beta^* \notin \mathfrak{U}_\gamma \quad \text{for any } \gamma < \beta .$$

b) Either $card\,f_\beta(W(f_\beta)) < \alpha$ then choose $\mathfrak{B}_\beta \in \mathfrak{G}(W(f_\beta) \cup \{b\})$ with

$$P\,\mathfrak{B}_\beta = \{b\} \quad \text{and} \quad \mathfrak{B}_\beta^* \notin \bigcup_{\gamma<\beta} \mathfrak{U}_\gamma \cup \{ \mathfrak{A}_\beta^* \} ,$$

or $card\,f_\beta(W(f_\beta)) \geqslant \alpha$ then use the theorem on mappings [6,7] to obtain a decomposition $X = X_0 \cup X_1 \cup X_2 \cup X_3$ with

$$X_0 = X - W(f_\beta) \quad \text{and} \quad f_\beta(X_t) \cap X_t = \emptyset, \quad t = 1, 2, 3 .$$

Choose t with $card\,f_\beta(X_t) \geqslant \alpha$. Then there is

$$Y \subset X_t \quad \text{with} \quad card\,Y = \alpha$$

such that f is one-to-one on Y . Choose

$$\mathfrak{B}_\beta \in \mathfrak{G}(Y \cup \{b\}) \quad \text{with} \quad P\,\mathfrak{B}_\beta = \{b\} \text{ and}$$

$$\mathfrak{B}_\beta^* \notin \bigcup_{\gamma<\beta} \mathfrak{U}_\gamma \cup \{ \mathfrak{A}_\beta^* \} \quad \text{and} \quad f_\beta(\mathfrak{B}_\beta)^* \notin \bigcup_{\gamma<\beta} \mathfrak{U}_\gamma \cup \{ \mathfrak{A}_\beta^* \} .$$

c) If $card\,\beta \leqslant card\,\mathfrak{T}$, choose

$$\mathfrak{C}_\beta \in \mathfrak{G}((X - W(f_\beta)) \cup \{t\}), \quad P\,\mathfrak{C}_\beta = \{t\},$$

where $t \in W(f_\beta) - \{a, b\}$, such that

$$\mathfrak{C}_\beta^* \notin \bigcup_{\gamma<\beta} \mathfrak{U}_\gamma \cup \{ \mathfrak{A}_\beta^*, \mathfrak{B}_\beta^*, f_\beta(\mathfrak{B}_\beta)^* \} .$$

Put

$$\mathfrak{U}_\beta = \bigcup_{\gamma<\beta} \mathfrak{U}_\gamma \cup \{ \mathfrak{A}^*_\beta, \mathfrak{B}^*_\beta, \mathfrak{C}^*_\beta, f_\beta(\mathfrak{B}_\beta)^* \} ,$$

$$\mathfrak{V}_\beta = \bigcup_{\gamma<\beta} \mathfrak{V}_\gamma \cup \{ \mathfrak{A}_\beta, \mathfrak{B}_\beta, \mathfrak{C}_\beta \}$$

(if $\mathfrak{A}_\beta$ was not chosen, then the definition of $\mathfrak{U}_\beta$, $\mathfrak{V}_\beta$ is the same, only without $\mathfrak{A}_\beta$, analogously $\mathfrak{C}_\beta$). The object we construct is

$$(X, \mathfrak{V}), \text{ with } \mathfrak{V} = \bigcup_{\beta \in card\, 2^X} \mathfrak{V}_\beta .$$

We are going to show that $(X, \mathfrak{V})$ is rigid. Let $f: (X, \mathfrak{V}) \to (X, \mathfrak{V})$.

- If $f \in \mathfrak{T}$, $f = f_i$, then, since clearly for each

$$A \in \mathfrak{B}_i , \quad card f_i(A) \geqslant \alpha ,$$

f is one-to-one on some set which belongs to $\mathfrak{B}_i$, therefore $f(\mathfrak{B}_i) \in \mathfrak{V}$. It is quite evident from the construction that $f_i(\mathfrak{B}_i) \notin \mathfrak{V}$.

- If $f \in \mathfrak{T}$, $f = g_i$, then $f = 1$ on some $A - P\,\mathfrak{C}_i$, $A \in \mathfrak{C}_i$. Then f is one-to-one on some $A' \in \mathfrak{C}_i$, therefore $f(\mathfrak{C}_i) \in \mathfrak{V}$. This is a contradiction with:

$$f(P\,\mathfrak{C}_i) \neq P\,\mathfrak{C}_i \quad \text{and} \quad f(\mathfrak{C}_i^*) = \mathfrak{C}_i^* .$$

- Finally if $f \notin \mathfrak{T} \cup \mathfrak{T}$, then $W(f) \subset \{a, b\}$. Let $a \in W(f)$, let

$$\mathfrak{H} \in \mathfrak{V} \text{ with } P\mathfrak{H} = \{a\}.$$

Then $f(\mathfrak{H}) \in \mathfrak{V}$ and we get a contradiction as above. Analogously if $b \in W(f)$. Therefore $W(f) = \emptyset$, i.e. $f = 1_X$.

CONSTRUCTION 3.7. Let $(X, \mathfrak{V})$ be the object of $\mathfrak{J}^{\mathfrak{G}}$ defined above. Put: $T = (X \cup \{c, d\})$. Define objects of $\mathfrak{J}^{\mathfrak{G}}$, $(T, \mathfrak{W})$ and $(T, \mathfrak{W}')$: choose filters $\mathfrak{F}_1$, $\mathfrak{F}_2$ on T with

$$\mathfrak{F}_1^* = \mathfrak{H}_1^* , \quad \mathfrak{F}_2^* = \mathfrak{H}_2^* \text{ (see } \mathfrak{U}_{-1} \text{)}, \; P\mathfrak{F}_1 = \{c\}, \; P\mathfrak{F}_2 = \{d\} ;$$

put $\mathfrak{W} = \mathfrak{V} \cup \{\mathfrak{F}_1, \mathfrak{F}_2\}$; choose a filter $\mathfrak{F}_3$ on T with

$$\mathfrak{F}_3^* = \mathfrak{H}_3^* \text{ and } P\mathfrak{F}_3 = \{a\} ;$$

put $\mathfrak{W}' = \mathfrak{W} \cup \{\mathfrak{F}_3\}$. Analogously as above we can prove that

1° $(T, \mathfrak{W})$ and $(T, \mathfrak{W}')$ are rigid;

2° there is no morphism from $(T, \mathfrak{W}')$ to $(T, \mathfrak{W})$.

THEOREM 3.8. *There exists a strong embedding from $\mathfrak{R}$ into $\mathcal{J}^{\mathfrak{G}}$.*

PROOF. Given a graph (H, R), put $\widetilde{H} = H \vee (X \times H \times H)$ and let $(\widetilde{H}, \widetilde{R})$ be an object of $\mathcal{J}^{\mathfrak{G}}$ such that

$$\widetilde{R} = \mathfrak{W} \quad \text{on} \quad \{k_1, k_2\} \vee (X \times \{(k_1, k_2)\}) \quad \text{where } (k_1, k_2) \in H \times H - R$$

and

$$\widetilde{R} = \mathfrak{W}' \text{ on this set if } (k_1, k_2) \in R$$

(more precisely, for $(k_1, k_2) \in H \times H$ denote $\phi_{k_1, k_2} : T \to \widetilde{H}$,

$$\phi_{k_1, k_2}(c) = k_1, \quad \phi_{k_1, k_2}(d) = k_2$$

$$\text{and} \quad \phi_{k_1, k_2}(x) = (x, k_1, k_2) \quad \text{if} \quad x \in X,$$

then

$$\widetilde{R} = \{\phi_r(\mathfrak{K}) \mid r \in H \times H, \ \mathfrak{K} \in \mathfrak{W}\} \cup \{\phi_r(\mathfrak{F}_3) \mid r \in R\}).$$

Given a morphism $f : (H, R) \to (K, S)$ in $\mathfrak{R}$, let

$$\widetilde{f} : (\widetilde{H}, \widetilde{R}) \to (\widetilde{K}, \widetilde{S}), \quad \widetilde{f} = f \vee (1_X \times f \times f).$$

We shall prove that this defines a strong embedding from $\mathfrak{R}$ to $\mathcal{J}^{\mathfrak{G}}$. The only fact whose verification is not routine is that this is a full functor. Let $g : (\widetilde{H}, \widetilde{R}) \to (\widetilde{K}, \widetilde{S})$ be a morphism in $\mathcal{J}^{\mathfrak{G}}$. To prove that $g = \widetilde{f}$ for some f, it is enough to show that for each $(h_1, h_2) \in H \times H$ there exists

$$(k_1, k_2) \in K \times K \quad \text{with}$$

$$g((X \times \{(h_1, h_2)\}) \cup \{h_1, h_2\}) \subset (X \times \{(k_1, k_2)\}) \cup \{k_1, k_2\}$$

- then the existence of f follows from the properties of $(T, \mathfrak{W})$ and $(T, \mathfrak{W}')$. Assume that on the contrary there exists $(h_1, h_2) \in H \times H$ such that for no $(k_1, k_2) \in K \times K$,

$$g((X \times \{(h_1, h_2)\}) \cup \{h_1, h_2\}) \subset (X \times \{(k_1, k_2)\}) \cup \{k_1, k_2\}.$$

a) $g(a, h_1, h_2) \in X \times K \times K$. Denote $(x, k_1, k_2) = g(a, h_1, h_2)$. Let us show that $card\, A < \alpha$ where

$$A = X \times \{(h_1, h_2)\} - g^{-1}(X \times \{(k_1, k_2)\});$$

if not, $card\, f(A) \geqslant \alpha$ as we can choose a filter

$$\mathfrak{F} \in \widetilde{R} \quad \text{with} \quad A \cup \{(a, h_1, h_2)\} \in \mathfrak{F} ;$$

therefore there exists a set A_1 with $card\, A_1 = \alpha$ such that f is one-to-one on A_1 ; choose a filter

$$\mathfrak{F}_1 \in \widetilde{R} \quad \text{with} \quad A_1 \cup \{(a, h_1, h_2)\} \in \mathfrak{F}_1 .$$

As $f(\mathfrak{F}_1) \in \widetilde{S}$ we have a contradiction. Therefore $card\, A < \alpha$. There exists

$$y \in X \quad \text{with} \quad g(y, h_1, h_2) \notin X \times \{(k_1, k_2)\} .$$

Choose $\mathfrak{G} \in \mathfrak{O}$ with $P\mathfrak{G} = \{y\}$; then $\mathfrak{G} \times \{(h_1, h_2)\} \in \widetilde{R}$ and so there exists

$$\mathfrak{D} \in \widetilde{S} \quad \text{with} \quad \mathfrak{D} \subset g(\mathfrak{G} \times \{(h_1, h_2)\}).$$

Therefore there exists

$$\mathfrak{G}' \in \mathfrak{O} \quad \text{with} \quad \mathfrak{D} = \mathfrak{G}' \times \{(k_1, k_2)\}.$$

Let $g^*: X \to X$,

$$g^* \neq 1_X \quad \text{and if} \quad g(x, h_1, h_2) = (x', k_1, k_2) \text{ then } g^*(x) = x' ;$$

then $g^*: (X, \mathfrak{O}) \to (X, \mathfrak{O})$, a contradiction with $g^* \neq 1_X$. Analogously

$$g(\{h_1, h_2\}) \subset \{k_1, k_2\}.$$

b) $g(a, h_1, h_2) \in K$. Let

$$\mathfrak{G} \in \widetilde{R} \quad \text{with} \quad P\mathfrak{G} = \{(a, h_1, h_2)\} .$$

There exists $\mathfrak{G}_1 \in \widetilde{S}$ with $\mathfrak{G}_1 \subset g(\mathfrak{G})$, in particular there exists

$$X_1 \subset X \quad \text{with} \quad card\, X_1 = \alpha$$

and such that g is one-to-one on $X_1 \times \{(h_1, h_2)\}$. Then $X_1 \subset \widetilde{\Pi}$ (see the beginning of Construction 3.6) and so there exists $\mathfrak{G}' \in \widetilde{R}$ which contains $X_1 \times \{(h_1, h_2)\}$ and with $P\mathfrak{G}' = \{a, h_1, h_2\}$. We have $g(\mathfrak{G}') \in \widetilde{S}$. Let $X_2 \subset X_1$ with

$$g(X_2 \times \{(h_1, h_2)\}) \subset X \times \{(k_1, k_2)\} \quad \text{for some } k_1, k_2 \in K.$$

Denote $k = g(a, h_1, h_2)$; then $k \in \{k_1, k_2\}$; as at most two filters in $\widetilde{S}$ contain $\{k\} \cup g(X_2 \times \{(h_1, h_2)\})$, there clearly exists a set

$$Z \subset g(X_2 \times \{(b_1, b_2)\}) \quad \text{with} \quad card Z = \alpha$$

and such that no filter in $\tilde{\mathfrak{S}}$ contains $Z \cup \{k\}$ and has $\{k\}$ for its meet. Put $X_3 = X_2 \cap g^{-1}(Z)$; then $card X_3 = \alpha$ and g is one-to-one on X_3. Then there exists $\mathfrak{F} \in \tilde{\mathfrak{R}}$ which contains $(X_3 \cup \{a\}) \times \{(b_1, b_2)\}$ and with

$$P\mathfrak{F} = \{(a, b_1, b_2)\} .$$

But then

$$g(\mathfrak{F}) \in \tilde{\mathfrak{S}}, \quad P g(\mathfrak{F}) = \{k\} \quad \text{and} \quad Z \cup \{k\} \in g(\mathfrak{F})$$

- a contradiction.

NOTE 3.9. The embedding $\psi : \mathfrak{R} \to \mathcal{J}^{\mathfrak{G}}$ defined above has the property that, if $f: (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ is a morphism in $\mathcal{J}^{\mathfrak{G}}$ which lies in the image of ψ, then for each $\mathcal{H} \in \mathfrak{U}$ there exists $A \in \mathcal{H}$ such that f is one-to-one on A. This follows from the above proof.

4

Let F be a set functor. Denote by $S(F)$ the category whose objects are

$$(X, H) \text{ where } X \text{ is a set, } H \subset FX ,$$

and whose morphisms $f: (X, H) \to (Y, K)$ are mappings

$$f : X \to Y \quad \text{with} \quad Ff(H) \subset K.$$

DEFINITION. For each set functor F and each $x \in FX$, $X \neq \emptyset$, denote by $\mathfrak{F}_F^X(x)$ the filter on X of all sets $A \subset X$ such that $x \in Fj(FA)$, where $j: A \to X$ is the inclusion. ($exp X$ is a trivial filter on X.) See [14,15].

LEMMA 4.1. *For any set functor* F *and any* $f: X \to Y$, $x \in FX$,

$$f(\mathfrak{F}_F^X(x)) \subset \mathfrak{F}_F^Y(Ff(x))$$

and if f *is one-to-one on some* $A \in \mathfrak{F}_F^X(x)$, *then*

$$f(\mathfrak{F}_F^X(x)) = \mathfrak{F}_F^Y(Ff(x)).$$

Proof: see [8].

Denote by $\mathcal{S}$ a fixed full subcategory of $\mathcal{J}^{\mathcal{G}}$ with the property that, if $f:(X,\mathfrak{U})\to(Y,\mathfrak{V})$ is one of its morphisms, then for each $\mathcal{H}\in\mathfrak{U}$, f is one-to-one on some $Z\in\mathcal{H}$.

THEOREME 4.2. *For each functor F such that there exists $x\in FX$ for which $\mathfrak{F}_F^X(x)$ is neither a free filter nor an ultrafilter there exists a strong embedding from $\mathcal{R}$ into $S(F)$.*

PROOF. Define for each $(X,\mathfrak{U})\in\mathcal{S}$:

$$\bar{U}\subset FX,\quad \bar{U}=\{x\in FX \mid \mathfrak{F}_F^X(x)\in\mathfrak{U}\};$$

then the strong embedding from $\mathcal{S}$ to $S(F)$ is

$$(X,\mathfrak{U})\to(X,\bar{U}),\quad f\mapsto f;$$

this follows from the property of $\mathcal{S}$ and from Lemma 4.1.

NOTE 4.3. Let F be a set functor. If $\mathfrak{F}_F^X(x_0)$ is a fixed ultrafilter for some $x_0\in FX$, then it is a fixed ultrafilter for each $Ff(x_0)\in FY$, $f:X\to Y$.

THEOREM 4.4. *If F is such a set functor that each $\mathfrak{F}_F^X(x)$ is either a free filter or an ultrafilter, then $S(F)$ does not contain a rigid object whose underlying set has power bigger than $card\,2^{F1}$. In particular, $S(F)$ does not contain more than $card\,2^{F(2^{F1})}$ rigid objects and so it is not binding.*

PROOF. In fact, no object (X,R) with $card X > card(exp\,F1)$ is rigid. Really, put, for each $x\in X$,

$$p_x: 1\to X \text{ with } p_x(0)=x;$$

then as $card X > card(exp\,F1)$, there exist distinct

$$x_1,x_2\in X \text{ with } (Fp_{x_1})^{-1}(R)=(Fp_{x_2})^{-1}(R);$$

we shall prove that the transposition of x_1 and x_2 is a morphism

$$f:(X,R)\to(X,R).$$

Let $v\in R$. If $P\mathfrak{F}_F^X(v)$ contains neither x_1 nor x_2, then

$$X-\{x_1,x_2\}\in\mathfrak{F}_F^X(v) \text{ and so } Ff(v)=v.$$

If $P\mathfrak{F}_F^X(v)=\{x_1\}$, then there exists

$$u \in (Fp_{x_1})^{-1}(R) \quad \text{with} \quad Fp_{x_1}(u) = v.$$

Then $Fp_{x_2}(u) \in R$ and so $f \circ p_{x_1} = p_{x_2}$; we have $Ff(v) \in R$. Analogously for $P\mathfrak{F}_F^X(v) = \{x_2\}$. That proves the theorem.

A set functor F is said *to preserve unions of pairs* if

$$F(X \cup Y) = Fj_X(X) \cup Fj_Y(Y)$$

for arbitrary sets X, Y where

$$j_X: X \to X \cup Y, \quad j_Y: Y \to X \cup Y$$

are the inclusions. F is said *to preserve unions of a set with a finite set* if

$$F(X \cup Y) = Fj_X(X) \cup Fj_Y(Y)$$

for arbitrary sets X, Y one of which is finite. Denote by K_M the functor

$$K_M X = X \times M, \quad K_M f = f \times 1_M.$$

A transposition pair (r, f) on a set X is a transposition $r: X \to X$ (i. e.

$$r(a) = b, \quad r(b) = a \quad \text{and} \quad r(x) = x \text{ if } x \neq a, b\ )$$

and a mapping $f: X \to X$ with

$$f(x) = x \quad \text{iff} \quad x = a \text{ or } x = b.$$

MAIN THEOREM 4.5. *Given a set functor F, $S(F)$ is binding if and only if F does not preserve unions of a set with a finite set.*

PROOF. We proved above that $S(F)$ is binding iff some $\mathfrak{F}_F^X(x)$ is neither a free filter nor an ultrafilter. Now if

$$x \in F(A \cup B) - (Fj_A(FA) \cup Fj_B(FB))$$

where A is finite and

$$j_A: A \to A \cup B, \quad j_B: B \to A \cup B$$

are inclusions, then $\mathfrak{F}_F^X(x)$ ($X = A \cup B$) is not an ultrafilter since

$$A \notin \mathfrak{F}_F^X(x), \quad B \notin \mathfrak{F}_F^X(x),$$

and $\mathfrak{F}_F^X(x)$ is not free since $A \cup B \in \mathfrak{F}_F^X(x)$, A finite, would else imply $B \in \mathfrak{F}_F^X(x)$. If, conversely, $\mathfrak{F}_F^X(x)$ is not free (i.e. $P\mathfrak{F}_F^X(x) \neq \emptyset$) and it is not an ultrafilter, then we choose $a \in P\,\mathfrak{F}_F^X(x)$ and put

$$A = \{a\} \text{ and } B = X \cdot \{a\}:$$

$x \in F(A \cup B)$ while $x \notin Fj_A(FA) \cup Fj_B(FB)$.

COROLLARY 4.6. *The following conditions on a set functor* F *are equivalent:*

1° a) $\mathfrak{R}$ *is strongly embeddable into* $S(F)$*;*

b) $S(F)$ *is binding;*

c) $S(F)$ *contains more than* $card\, 2^{F(2^{F1})}$ *rigid objects;*

d) $S(F)$ *contains a rigid object on a set with power* $> card\, 2^{F1}$.

2° a) F *does not preserve unions of a set with a finite set;*

b) F *does not preserve unions of a set with a one-point set;*

c) $\mathfrak{F}_F^X(x)$ *is neither an ultrafilter nor a free filter for some* $x \in FX$*;*

d) There exists a transposition pair (r, f) *on a set* X *such that for some* $z \in FX$ *both* $Ff(z) \neq z$ *and* $Fr(z) \neq z$*;*

e) There exists a cardinal α *such that, for each transposition pair* (r, f) *on a set* X *with power at least* α*, there exists* $z \in FX$ *with both*

$$Fr(z) \neq z \quad and \quad Ff(z) \neq z.$$

PROOF. The equivalence of conditions *1a, 1b, 1c, 1d, 2a, 2c* follows from above.

2a $\Longleftrightarrow$ *2b* is easy.

2c $\Longrightarrow$ *2e*. Let $x \in FX$ such that $\mathfrak{F}_F^X(x)$ is neither free nor an ultrafilter. Let $card\, Y \geq card\, X$, let $r: Y \to Y$ be a transposition of $a, b \in Y$ and

$$f: Y \to Y \text{ with } f(t) \neq t \text{ iff } t \neq a, b.$$

There exists $Y' \subset Y$ with

$$card\, Y' = card\, Y \text{ and } f(Y') \cap Y' = \emptyset$$

(see [6,7]). Let $\pi: X \to Y$ be a one-to-one mapping,

$$\pi(X) = Y' \cup \{a\} \text{ such that } a \in \pi(P\,\mathfrak{F}_F^X(x)).$$

Then

$$Ff(F\pi(x)) \neq F\pi(x) \quad \text{and} \quad Fr(F\pi(x)) \neq F\pi(X),$$

which follows easily from the properties of $\mathcal{F}_F^X(\cdot)$.

2e $\Longrightarrow$ *2d* is easy.

2d $\Longrightarrow$ *2a*. Let (r, f) be a transposition pair on a set X,

$$Fr(z) \neq z \neq Ff(z),$$

where r is a transposition of $a, b \in X$. Put $Y = \{a, b\}$. Then

$$Fj_Y(FY) \cup Fj_{X-Y}(F(X-Y))$$

does not contain $z \in FX = F(Y \cup (X-Y))$: if on the contrary $z \in Fj_Y(FY)$, we have

$$f \circ j_Y = j_Y \quad \text{and so} \quad Ff(z) = z,$$

which is not true, and if $z \in Fj_{X-Y}(F(X-Y))$ we have

$$r \circ j_{X-Y} = j_{X-Y} \quad \text{and so} \quad Fr(z) = z.$$

COROLLARY 4.7. *In the finite set-theory the following conditions on a set functor F are equivalent:*

1° S(F) is binding;

2° a) F does not preserve unions;

b) F is not naturally equivalent to $K_M \vee C$ *for any set M and any constant functor C.*

PROOF. The equivalence of *2a* and *2b* is proved in [13,15].

REFERENCES

1. N. BOURBAKI, *Théorie des Ensembles*, Herman, Paris.

2. Z. HEDRLIN, A. PULTR, On full embeddings of categories of algebras, *Ill. J. of Math.* 10 (1966), 392-406.

3. Z. HEDRLIN, A. PULTR, On categorical embeddings of topological structures into algebraic ones, *Comment. Math. Univ. Carolinae* 7 (1966), 377-400.

4. Z. HEDRLIN, Extension of structures and full embeddings of categories, *Actes du Congrès international des Math. 1970*, Tome 1, 319-322.

5. J. R. ISBELL, Subobjects, adequacy, completeness and categories of algebras, *Rozprawy Matem.* XXXV, Warszawa, 1964.

6. M. KATETOV, A theorem on mappings, *Comment. Math. Univ. Carolinae* 8 (1967), 431-434.

7. H. KENYON, Partition of a domain,... , *Amer. Math. Monthly* 71 (1964), 219.

8. V. KOUBEK, Set functors, *Comment. Math. Univ. Carolinae* 12 (1971), 175-195.

9. L. KUČERA, A. PULTR, On a mechanism of defining morphisms in concrete categories, *Cahiers Topo. et Géo. Diff.* 13-4 (1973), 397-410.

10. A. PULTR, On selecting of morphisms among all mappings between underlying sets of objects..., *Comment. Math. Univ. Carolinae* 8 (1967), 53-83.

11. A. PULTR, Limits of functors and realizations of categories, *Comment. Math. Univ. Carolinae* 8 (1967), 663-683.

12. A. PULTR, On full embeddings of concrete categories with respect to a forgetful functor, *Comment. Math. Univ. Carolinae* 9 (1968), 281-307.

13. V. TRNKOVA, P. GORALCIK, On products in generalized algebraic categories, *Comment. Math. Univ. Carolinae* 10 (1969), 49-89.

14. V. TRNKOVA, On some properties of set functors, *Comment. Math. Univ. Carolinae* 10 (1969), 323-352.

15. V. TRNKOVA, On descriptive classification of set functors, *Comment. Math. Univ. Carolinae* 12 (1971): Part I, 143-174 ; Part II, 345-357.

16. P. VOPENKA, A. PULTR, Z. HEDRLIN, A rigid relation exists on any set, *Comment. Math. Univ. Carolinae* 6 (1965), 149-155.

Katedra Zakladnich Matematickych struktur
Matematicko-fysikalni Fakulta, Karlova Universita,
Sokolovska 83
18600 PRAHA 8 - Karlin
TCHECOSLOVAQUIE