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A NOTE ON GENERATORS IN CATEGORIES

by Syed A. HUQ

The aim of this note is to give a new definition of generators in categories and to show that in certain cases this is the usual definition. Obviously our definition is not necessarily equivalent to the usual one. The author thanks Max Kelly for valuable suggestions for the generalization of the author's original ideas, which was done for «one generator» instead of a generating set. For notations and other useful definitions we refer to [1], [2].

1. Preliminaries.

Suppose $\mathcal{C}$ is a category with coproducts. Given any set $\mathcal{G}$ of objects, there exists for any object $C \in \mathcal{C}$, a canonical map

$$\varepsilon_C : \Sigma_{G \in \mathcal{G}} Hom(G, C) \otimes G \longrightarrow C,$$

where for a set X, $X \otimes G$ denotes the coproduct of X copies of G. We notice that the (G, f)-component, where $G \in \mathcal{G}$ and $f \in Hom(G, C)$ is to be f itself.

Next we define a functor H from $\mathcal{C}$ to $\mathcal{S}$, the category of sets by $HC = \Sigma_{G \in \mathcal{G}} Hom(G, C)$.

LEMMA. *If the set $\mathcal{G}$ reduces to a single object A, then H is the representable Hom functor, denoted by H^A.*

2. Reflecting definition of a generator.

DEFINITION 2.1 (Classical Definition). If $\mathcal{E}$ is any class of morphisms of $\mathcal{C}$, we say the set $\mathcal{G}$ is *generating with respect to* $\mathcal{E}$, if for each C, ε_C is in $\mathcal{E}$.

DEFINITION 2.2 (New Definition). The set $\mathcal{G}$ is said to be *generating with respect to* $\mathcal{E}$ if, for any f, $f \in \mathcal{E}$ whenever Hf is a regular epimorphism, i.e. H reflects regular epimorphisms to $\mathcal{E}$. Note that in the category of sets, every epimorphism is regular, being a retraction.

The advantage of the new definition is, that it *does not require* the existence of coproducts in the original category $\mathcal{C}$, even. However if the category has coproducts, we have

PROPOSITION 2.1. *Every generating set in the sense of Definition 2.2 is in fact a generating set in the sense of Definition 2.1.*

Proof. We notice that, in sets, $H\varepsilon_C$ is an epimorphism as such a retraction, thus by Definition 2.2, $\varepsilon_C \in \mathfrak{S}$.

In the classical situation, when $\mathfrak{S}$ consits of all epimorphisms, and $\mathcal{G}$ consists of a single object A, and even the category does not have coproducts, we have

PROPOSITION 2.2. *In categories with equalizers the following conditions are equivalent:*

(i) A is a generator;

(ii) $H^A : \mathcal{C} \longrightarrow \mathcal{S}$ is faithful;

(iii) H^A reflects epimorphisms.

Proof. Since (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is trivial, we prove (iii) $\Longrightarrow$ (i): if $D \overset{f}{\underset{g}{\rightrightarrows}} B$ is a parallel pair of morphisms, let (C, θ) be their equalizer.

Then for all $\mu : A \to D$, such that $f\mu = g\mu$, μ must be of the form $\theta\mu'$ for some μ'; i.e. $H^A(\theta) : Hom[A, C] \longrightarrow Hom[A, D]$ is an epimorphism, hence by (iii) θ is an epimorphism, so $f = g$.

Next we answer the question when is the new definition exactly equivalent to the original Definition 2.1.

PROPOSITION 2.3.(*) *If the class $\mathfrak{S}$ satisfies the conditions:*

a) if $f \in \mathfrak{S}$, and g is a retraction, then $fg \in \mathfrak{S}$,

b) if $fg \in \mathfrak{S}$, $g \in \mathfrak{S}$, then $f \in \mathfrak{S}$,

the new definition is the same as the original one.

Since the new definition implies the original one, we prove the converse.

(*) The author thanks Professor G.M. Kelly for suggesting this improvement of Proposition 2.3. Originally this was known to the author only for regular epimorphisms.

So let each $\varepsilon_C \in \mathfrak{E}$. Let $f: C \to D$ be such that Hf is regular, that is $\Sigma_G Hom(G, f)$ is epimorphism. Then each $Hom(G, f)$ is an epimorphism in sets and as such a retraction.

Therefore in the commutative diagram

$$\begin{array}{ccc} \Sigma_G Hom(G, C) \otimes G & \xrightarrow{\Sigma_G Hom(G, f) \otimes G} & \Sigma_G Hom(G, D) \otimes G \\ \downarrow \varepsilon_C & & \downarrow \varepsilon_D \\ C & \xrightarrow[f]{} & D \end{array}$$

the top line is a retraction. Thus $\varepsilon_D(\Sigma_G Hom(G, f) \otimes G) \in \mathfrak{E}$ by (a), i.e. $f \varepsilon_C \in \mathfrak{E}$ since the diagram was commutative by naturality; hence, $f \in \mathfrak{E}$ by (b). Q.E.D.

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